Penalty Methods for Continuous-Time Portfolio Selection with Proportional Transaction Costs

Min Dai∗ Yifei Zhong†

Abstract

This paper is concerned with numerical solutions to a singular stochastic control problem arising from the continuous-time portfolio selection with proportional transaction costs. The associated value function is governed by a variational inequality with gradient constraints. We propose a penalty method to deal with the gradient constraints and employ a finite difference discretization. Convergence analysis is presented. We also show that the standard penalty method can be applied in the case of a single risky asset where the problem can be reduced to a standard variational inequality. Numerical results are given to demonstrate the efficiency of the methods and to examine the behaviour of the optimal trading strategy.

1 Introduction

Merton (1971) initiated the study of continuous-time portfolio selection problems. In the absence of transaction costs, he showed that the optimal strategy of a constant relative risk aversion (CRRA) investor is to keep a constant fraction of total wealth in each asset and to consume at a constant rate. Such a strategy leads to incessant trading, which is impracticable in a real market with transaction costs. Magill and Constantinides (1976) introduced proportional transaction costs into Merton’s model and provided a fundamental insight that there exists a no-trading region. Mathematically, the portfolio selection problem with proportional transaction costs can be described as a singular stochastic control problem and the associated value function is governed by a variational inequality with gradient constraints. The problem gives rise to two free boundaries, known as the optimal buying and selling boundaries. To study optimal trading strategies, one only needs to examine the behaviour of the two boundaries.

Most theoretical analyses of the optimal buying and selling boundaries are confined to a market that consists of a single risky asset and a bank account. For example, Davis and Norman (1990) first provided a theoretical analysis for an infinite horizon investment and consumption decision with transaction costs. In terms of the notion of viscosity solutions, Shreve and Soner (1994) conducted a thorough and entire analysis on the optimal trading strategies in the infinite horizon case. Janecek and Shreve (2004) derived an asymptotic expansion of the associated value function and obtained some asymptotic results on the optimal buying and selling boundaries. However, these papers cannot handle the finite horizon scenario. Liu and Loewenstein (2002) first studied the finite horizon optimal investment problem by virtue of a sequence of approximate analytical solutions. Dai and Yi (2009) considered the same problem and obtained an equivalent standard variational inequality by which the behaviour of the optimal buying and selling boundaries was completely characterized. It is worthwhile pointing out that Dai and Yi (2009) essentially established a connection between optimal stopping and singular control problems, which, though well-known [cf. Karatzas and Shreve (1984) or Soner and Shreve (1991)], had never been revealed for the present problem. The idea of Dai and Yi (2009) was further extended by Dai et al. (2009) and Dai, Xu and Zhou (2009) to deal with the consumption case and the continuous-time mean-variance framework, respectively.

∗Department of Mathematics, National University of Singapore (NUS), Singapore. Dai is also an affiliated member of Risk Management Institute, NUS. This work was partially supported by Singapore MOE AcRF grant (No. R-146-000-096-112) and NUS RMI grant (R-146-000-117-720/646). Email:<matdm@nus.edu.sg>. Dai would like to thank his students Joline A.V. Uichanco and Kaiyun Chong for parts of numerical implementation. He also thanks Yue-Kuen Kwok and Xingye Yue for helpful comments and an anonymous referee for valuable suggestions.

†Department of Mathematics, National University of Singapore, Singapore.
Due to lack of analytical solutions, it is natural to seek numerical solutions to determine the optimal buying and selling boundaries. Previous work along these lines has been extensive. For the single risky asset case, Davis and Norman (1990) employed a numerical scheme for a system of ordinary differential equations reduced from the infinite horizon problem. Gennotte and Jung (1994) came up with a dynamic programming method for the finite horizon problem. Muthuraman (2006) considered the infinite horizon problem and provided a computational scheme that transforms the resulting free boundary problem to a sequence of fixed boundary problems. Assuming that stock returns are uncorrelated, Akian, Menaldi and Sulem (1996) numerically solved the multiple risky-asset case by use of policy iteration together with the multigrid method. Muthuraman and Kumar (2006) extended the approach of Muthuraman (2006) to the case of multiple risky assets.

In this paper, we will propose a penalty method combined with a finite difference discretization to solve the variational inequality that the value function satisfies. The advantages of the penalty method are abundant [cf. Forsyth and Vetzal (2002)]: it can be used for any type of discretization, in any dimension, and on an unstructured mesh; standard sparse matrix software can be used to solve the Jacobian matrix; no prior knowledge of free boundaries is required; the Newton iteration linearizing the penalty terms can handle other nonlinear terms as well.\footnote{For example, the nonlinearity caused by the consumption in the present paper can be readily dealt with in this way.}

The standard penalty method, first proposed by Forsyth and Vetzal (2002) for pricing American vanilla options, has demonstrated its efficiency. Forsyth and Vetzal (2002) also provided convergence analysis of the method. An extension to the jump-diffusion model was made by D’Halluin, Forsyth and Labahn (2005). Dai, Kwok and You (2007) established a linkage between the intensity-based framework and the (standard) penalty method of optimal stopping problems. However, it should be emphasized that the variational inequality of the American option pricing model is different from that arising from a singular stochastic control problem because the latter gets gradient constraints involved. Fortunately, Dai, Kwok and Zong (2008) demonstrated numerically the efficiency of the penalty method for another singular control problem arising from the pricing of guaranteed minimum withdrawal benefits. In this paper, we will show that the method is still efficient for the present problem after a series of changes of variables. Theoretical analysis will be provided as well. Moreover, we will show that the standard penalty method can work in the case of a single risky asset in which there is a linkage between the singular stochastic control problem and the optimal stopping problem.

Throughout the paper, we assume the investors to be of CRRA and then focus on log or power utility function. We will confine ourselves to the finite horizon problem, and it is straightforward to extend to the infinite horizon case. We believe that the penalty method for variational inequality with gradient constraints is also feasible when a general utility function is considered.

The rest of the paper is organized as follows. In the next section we present the problem formulation. In section 3, we reduce the problem dimension by changes of variables. In section 4, we present a penalty method combined with a finite difference discretization. We also point out that the corresponding penalty approximation is associated with the original control problem restricted to a class of admissible strategies. Section 5 is devoted to convergence analysis. In section 6, we show that the standard penalty method can be employed in the case of a single risky asset. Numerical results are given in section 7. We conclude the paper in section 8.

2 Model formulation

Suppose that there are $N + 1$ assets available for investment: a riskfree asset (bank account) and $N$ risky assets (stocks). Their prices, denoted by $S_0(t)$ and $S_i(t)$, $i = 1, 2, \ldots, N$, respectively, evolve according to the following equations:

\begin{align*}
\frac{dS_0}{S_0} &= r dt, \\
\frac{dS_i}{S_i} &= \left(\alpha_i dt + \sigma_i dB_i\right),
\end{align*}

where $r > 0$ is the constant riskfree rate, $\alpha_i > r$ and $\sigma_i > 0$ are constant expected rate of return and volatility, respectively, of the $i^{th}$ risky asset. The processes $\{B_i(t); t > 0\}$ are standard Brownian motions.
on a filtered probability space \( (\mathcal{S}, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) with \( \mathcal{B}_t(0) = 0 \) almost surely and constant coefficients of correlation \( \rho_{ij} \), namely, \( \mathbb{E}(dB_t dB_j) = \rho_{ij} dt \). We assume that the filtration \( \{\mathcal{F}_t\}_{t \geq 0} \) is right-continuous and each \( \mathcal{F}_t \) contains all \( \mathbb{P} \)-null sets of \( \mathcal{F}_\infty \).

Assume that an investor holds a portfolio \( X_t = (X_0(t), X_1(t), \ldots, X_N(t)) \), where \( X_0(t) \) and \( X_i(t) \) are respectively the dollar values in bank and in the \( i^{th} \) risky asset at time \( t \). In the presence of transaction costs, the equations describing their evolution are

\[
dX_0 = (rX_0 - \kappa C(t))dt - \sum_{i=1}^N (1 + \lambda_i) dL_i + \sum_{i=1}^N (1 - \mu_i) dM_i, \tag{2.1}
\]

\[
dX_i = \alpha_i X_idt + \sigma_i X_i dB_i + dL_i - dM_i, \tag{2.2}
\]

Here \( C(t) \geq 0 \) is the consumption rate, and \( \kappa \) is taken to be either 1 or 0 subject to whether there are consumptions or not. \( L_i(t) \) and \( M_i(t) \) are right-continuous (with left hand limits), nonnegative, and nondecreasing \( \{\mathcal{F}_t\}_{t \geq 0} \)-adapted processes with \( L_i(0) = M_i(0) = 0 \), representing cumulative dollar values for the purpose of buying and selling the \( i^{th} \) stock, respectively. The constants \( \lambda_i \in [0, \infty) \) and \( \mu_i \in [0, 1) \), \( i = 1, 2, \ldots, N \), appearing in these equations account for proportional transaction costs incurred on purchase and sale of the \( i^{th} \) stock, respectively. We will always assume \( \lambda_i + \mu_i > 0 \), \( i = 1, 2, \ldots, N \).

Due to transaction costs, the investor’s net wealth in monetary terms is \( X_0 + \sum_{i=1}^N [(1 - \mu_i)X_i^+ - (1 + \lambda_i)X_i^-] \). With the requirement that the net wealth at any time always be positive, the solvency region \( \mathcal{S} \) is defined as

\[
\mathcal{S} = \left\{ x = (x_0, x_1, \ldots, x_N) \in \mathbb{R}^{N+1} : x_0 + \sum_{i=1}^N [(1 - \mu_i)x_i^+ - (1 + \lambda_i)x_i^-] > 0 \right\}.
\]

Assume that the investor is given an initial position \( x^0 \in \mathcal{S} \) at time 0. An investment and consumption strategy \( \{(L_i), \{M_i\}, C\} \) is admissible for a position \( x \) from time \( s \in [0, T] \) if \( X_t \) given by (2.1)-(2.2) with \( X_s = x \) is in \( \mathcal{S} \). We denote by \( \mathcal{A}_s(x) \) the set of all admissible investment strategies for \( x \) from time \( s \). The investor aims to choose an admissible strategy so as to maximize the discounted expected utility of consumption and terminal wealth, that is,

\[
\sup_{\{(L_i), \{M_i\}, C\} \in \mathcal{A}_s(x)} \mathbb{E}_0^x \left[ \int_0^T e^{-\beta s} u(C(s)) ds + e^{-\beta T} u(W_T) \right], \tag{2.3}
\]

where \( u(\cdot) \) is a utility function and \( \beta > 0 \) is the discount rate. We will only consider CRRA investors whose utility function takes the following form:

\[
u(W) = \begin{cases} \frac{W^\gamma}{\gamma} & \text{if } \gamma \neq 0, \gamma < 1, \\ \log W & \text{if } \gamma = 0. \end{cases}
\]

Define the value function by

\[
V(x, t) = \sup_{\{(L_i), \{M_i\}, C\} \in \mathcal{A}_s(x)} \mathbb{E}_t^x \left[ \int_t^T e^{-\beta s} u(C(s)) ds + e^{-\beta T} u(W_T) \right],
\]

for \( x \in \mathcal{S}, t \in [0, T) \). The problem is indeed a singular control problem for the displacement of the state variables \( X_i \) due to control effort might be discontinuous. It turns out that the value function satisfies the following HJB equation [cf. Shreve and Soner (1994), Akian, Menaldi and Sulem (1996), or Fleming and Soner (2006)]:

\[
\max \left\{ \frac{\partial V}{\partial t} + \mathcal{L}_0 V + \kappa u^* \left( \frac{\partial V}{\partial x_0} \right), \max_{1 \leq i \leq N} \mathcal{L}_0 V, \max_{1 \leq i \leq N} \mathcal{M}_0 V \right\} = 0, \quad x \in \mathcal{S}, \ t \in [0, T), \tag{2.4}
\]

with the terminal condition

\[
V(x, T) = u \left( x_0 + \sum_{i=1}^N [(1 - \mu_i)x_i^+ - (1 + \lambda_i)x_i^-] \right), \tag{2.5}
\]
where

\[
\mathcal{L}_0 V = \frac{1}{2} \sum_{i,j=1}^{N} \rho_{ij} \sigma_i \sigma_j x_i x_j \frac{\partial^2 V}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \alpha_i x_i \frac{\partial V}{\partial x_i} + r x_0 \frac{\partial V}{\partial x_0} - \beta V,
\]

\[
\mathcal{L}_0 V = -(1 + \lambda_i) \frac{\partial V}{\partial x_i} + \gamma \frac{\partial V}{\partial x_i}, \quad \mathcal{M}_0 V = (1 - \mu_i) \frac{\partial V}{\partial x_i} - \frac{\partial V}{\partial x_i},
\]

\[
u^*(\nu) = \max_{c \geq 0} (-c \nu + u(c)) = \begin{cases} (\frac{1}{\gamma} - 1) \nu \frac{\gamma}{\sigma} & \text{if } \gamma \neq 0, \gamma < 1, \\ -\log \nu - 1 & \text{if } \gamma = 0. \end{cases}
\]

### 3 Change of variables

Due to the homotheticity of the utility function, it follows that for any positive constant \( \rho \),

\[
V(\rho x, t) = \begin{cases} \rho^\gamma V(x, t) & \text{if } \gamma \neq 0, \gamma < 1, \\ g(t) \log \rho + V(x, t) & \text{if } \gamma = 0, \end{cases}
\]

where \( g(t) = \frac{\kappa(1-e^{-\beta(T-t)})}{\beta} + e^{-\beta(T-t)} \). This enables us to adopt the wealth fractions as state variables so as to reduce the dimension of the problem. Indeed, we take

\[
\rho = \frac{1}{\sum_{i=0}^{N} x_i}
\]

and denote \( y = (y_1, y_2, \ldots, y_N) \), \( \pi = \left( 1 - \sum_{i=1}^{N} y_i, y_1, y_2, \ldots, y_N \right) \), and \( \varphi(y, t) = V(\pi, t) \), then

\[
V(x, t) = \begin{cases} \rho^\gamma \varphi(y, t) & \text{if } \gamma \neq 0, \gamma < 1, \\ g(t) \log \rho + \varphi(y, t) & \text{if } \gamma = 0. \end{cases}
\]

(3.6)

It is easy to verify that for \( \gamma \neq 0 \) and \( \gamma < 1 \), (2.4)-(2.5) reduce to

\[
\varphi(y, T) = \left( \frac{1 - \sum_{i=1}^{N} (\mu_i y_i^+ + \lambda_i y_i^-)}{\gamma} \right)^\gamma, \quad y \in \Omega^N, \quad t \in [0, T),
\]

(3.7)

where \( \Omega^N = \left\{ y = (y_1, y_2, \ldots, y_N) \in \mathbb{R}^N : 1 - \sum_{i=1}^{N} (\mu_i y_i^+ + \lambda_i y_i^-) > 0 \right\} \).

\[
\mathcal{L}_1 \varphi = \sum_{k,l=1}^{N} a_{k,l} \frac{\partial^2 \varphi}{\partial y_k \partial y_l} + \sum_{k=1}^{N} b_k \frac{\partial \varphi}{\partial y_k} - \theta \gamma \varphi,
\]

\[
\mathcal{L}_{11} \varphi = \sum_{k=1}^{N} (\delta_{ik} + \lambda_i y_k) \frac{\partial \varphi}{\partial y_k} - \lambda_i \gamma \varphi, \quad \mathcal{M}_{11} \varphi = \sum_{k=1}^{N} (-\delta_{ik} + \mu_i y_k) \frac{\partial \varphi}{\partial y_k} - \mu_i \gamma \varphi
\]

with

\[
a_{k,l} = y_k y_l \sum_{i,j=1}^{N} \frac{1}{2} \rho_{ij} \sigma_i \sigma_j (\delta_{il} - y_i)(\delta_{jk} - y_j),
\]

\[
b_k = y_k \sum_{i=1}^{N} (\delta_{ik} - y_i) \left[ (\alpha_i - r) + \sum_{j=1}^{N} (\gamma - 1) \rho_{ij} \sigma_i \sigma_j y_j \right],
\]

\[
\theta = \frac{\beta}{\gamma} - \left( r + \sum_{i=1}^{N} y_i \left( \alpha_i - r - \frac{1 - \gamma}{2} \sum_{j=1}^{N} \rho_{ij} \sigma_i \sigma_j y_j \right) \right).
\]
Here $\delta_{ij}$ represents Kronecker index, i.e., $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise.

The above change of variables is well-known and has been widely adopted, see Davis and Norman (1990) for $N = 1$ (in a slightly different form), and Akian, Menaldi and Sulem (1996) and Muthuraman and Kuman (2006) for $N = 2$. These authors then considered numerical implementation based on (3.7). However, applying the penalty method directly to (3.7) would result in the following penalty approximation:

$$
\frac{\partial \varphi}{\partial t} + \mathcal{L}_1 \varphi + \kappa (\frac{1}{\gamma} - 1) \left( \gamma \varphi - \sum_{i=1}^{N} y_i \frac{\partial \varphi}{\partial y_i} \right) \frac{\tau}{\gamma} + K \sum_{i=1}^{N} \left[ (\mathcal{L}_i \varphi)^+ + (\mathcal{M}_i \varphi)^+ \right] = 0,
$$

which may cause numerical oscillation for $\gamma < 0$ and $K$ large enough since the term $K \sum_{i=1}^{N} \left[ (\mathcal{L}_i \varphi)^+ + (\mathcal{M}_i \varphi)^+ \right]$ contains $-K \lambda_i \gamma \varphi$ and $-K \mu_i \gamma \varphi$. To cure the problem, we further make the following transformation originally done by Dai and Yi (2009) in the case of a single risky asset:

$$
W(y, t) = \frac{\log(\gamma \varphi)}{\gamma}. \tag{3.8}
$$

It follows

$$
\begin{align*}
\max \{ \frac{\partial W}{\partial t} + \mathcal{L} W + \kappa f(W), \max_{1 \leq i \leq N} \mathcal{L}_i W, \max_{1 \leq i \leq N} \mathcal{M}_i W \} &= 0, \\
W(y, T) &= \log \left( 1 - \sum_{i=1}^{N} (\mu_i y_i^+ + \lambda_i y_i^-) \right), \quad y \in \Omega^N, \ t \in [0, T),
\end{align*}
$$

where

$$
\mathcal{L} W = \sum_{k, l=1}^{N} a_{k, l} \left( \frac{\partial^2 W}{\partial y_k \partial y_l} + \gamma \frac{\partial W}{\partial y_k} \frac{\partial W}{\partial y_l} \right) + \sum_{k=1}^{N} b_k \frac{\partial W}{\partial y_k} - \theta,
$$

$$
\mathcal{L}_i W = \sum_{k=1}^{N} (\delta_{ik} + \lambda_i y_k) \frac{\partial W}{\partial y_k} - \lambda_i, \quad \mathcal{M}_i W = \sum_{k=1}^{N} (-\delta_{ik} + \mu_i y_k) \frac{\partial W}{\partial y_k} - \mu_i,
$$

$$
f(W) = \left( \frac{1}{\gamma} - 1 \right) e^{\frac{\tau}{\gamma} W} \left( 1 - \sum_{i=1}^{N} y_i \frac{\partial W}{\partial y_i} \right)^{\frac{\tau}{\gamma}}.
$$

Another advantage of transformation (3.8) is that a slight modification of (3.9) applies to the case of log utility. Indeed, for $\gamma = 0$, let

$$
W(y, t) = \frac{\varphi(y, t)}{g(t)}, \tag{3.10}
$$

then it can be verified that $W(y, t)$ satisfies (3.9) with

$$
f(W) = -\frac{1 + \log g(t) + \log \left( 1 - \sum_{i=1}^{N} y_i \frac{\partial W}{\partial y_i} \right)}{g(t)} + W,
$$

$$
\theta = -r - \sum_{i=1}^{N} y_i \left( \alpha_i - r - \frac{1}{2} \sum_{j=1}^{N} \rho_{ij} \sigma_i \sigma_j y_j \right).
$$

As a consequence, we can provide a unified framework to deal with both the power utility case and the log utility case.

For later use, we define

$$
\begin{align*}
\text{BR}_i &= \{(y, t) \in \Omega^N \times [0, T) : \mathcal{L}_i W = 0\}, \quad \text{SR}_i = \{(y, t) \in \Omega^N \times [0, T) : \mathcal{M}_i W = 0\}, \\
\text{NTR}_i &= \Omega^N \times [0, T) \setminus (\text{BR}_i \cup \text{SR}_i), \quad \text{NTR} = \bigcap_{i=1}^{N} \text{NTR}_i.
\end{align*}
$$

Here NTR represents the no-trading region, BR$_i$, SR$_i$ and NTR$_i$ represent the buy region, sell region and no-trading region with regard to the $i$th risky asset, respectively.
4 The penalty method for gradient constraints

In what follows, we consider the penalty approximation to (3.9):

\[
\begin{align*}
\left\{ \begin{array}{l}
-\frac{\partial W}{\partial t} - \mathcal{L}W - \kappa f(W) & = K \sum_{i=1}^{N} \left[ (\mathcal{L}_{i}W)^{+} + (\mathcal{M}_{i}W)^{+} \right], \quad y \in \Omega^{N}, \ t \in [0,T), \\
W(y,T) & = \log \left( 1 - \sum_{i=1}^{N} (\mu_{i}y_{i}^{+} + \lambda_{i}y_{i}^{-}) \right),
\end{array} \right.
\end{align*}
\]  

(4.1)

where \( K \) is a positive constant. (4.1) is expected to converge to (3.9) as \( K \) goes to infinity.

4.1 The control problem associated with (4.1).

The approximation (4.1) corresponds to the original problem (2.3) restricted to a class of admissible policies: \( L_{i}(t) \) and \( M_{i}(t) \) are absolutely continuous with bounded derivatives, i.e.,

\[
L_{i}(t) = \int_{0}^{t} l_{i}(s)ds, \quad M_{i}(t) = \int_{0}^{t} m_{i}(s)ds, \quad 0 \leq l_{i}(s) \leq K, \quad 0 \leq m_{i}(s) \leq K, \ \text{for } i = 1,2,\ldots,N.
\]

Indeed, it is easy to see that the associated value function, denoted by \( \mathcal{V}(x,t) \), satisfies (taking \( \gamma \neq 0 \) as an example)

\[
\max_{(i,m,C)} \left\{ \frac{\partial \mathcal{V}}{\partial t} + \mathcal{L}_{0}\mathcal{V} + \kappa \left( \frac{C}{\gamma} - C \frac{\partial \mathcal{V}}{\partial x_{0}} \right) + \sum_{i=1}^{N} (l_{i}\mathcal{L}_{0}\mathcal{V} + m_{i}\mathcal{M}_{0}\mathcal{V}) \right\} = 0 \text{ in } \mathcal{S} \times [0,T).
\]

The optimal strategies are

\[
C = \left( \frac{\partial \mathcal{V}}{\partial x_{0}} \right)^{+}, \quad l_{i} = \begin{cases} K & \text{if } \mathcal{L}_{0}\mathcal{V} \geq 0, \\ 0 & \text{otherwise}, \end{cases} \quad m_{i} = \begin{cases} K & \text{if } \mathcal{M}_{0}\mathcal{V} \geq 0, \\ 0 & \text{otherwise}, \end{cases}
\]

which yields

\[
\frac{\partial \mathcal{V}}{\partial t} + \mathcal{L}_{0}\mathcal{V} + \kappa u^{*} \left( \frac{\partial \mathcal{V}}{\partial x_{0}} \right) + K \sum_{i=1}^{N} \left[ (\mathcal{L}_{0}\mathcal{V})^{+} + (\mathcal{M}_{0}\mathcal{V})^{+} \right] = 0 \text{ in } \mathcal{S} \times [0,T).
\]

(4.2)

Applying the transformations (3.6), (3.8) and (3.10), (4.2) with terminal condition is reduced to the penalty approximation (4.1).

We point out that the above derivation was first presented by Davis and Norman (1990) when \( N = 1 \). In the PDE theory, penalty approximations have been widely used to show the existence of solution to variational inequality [cf. Evans (1979) and Friedman (1982)].

4.2 Computation domain and boundary conditions

We are most interested in the NTR which is generally much smaller than the solvency region. Then, we confine ourselves to a truncated domain\(^2\)

\[
D^{N} = [y_{1m},y_{1m}] \times \ldots \times [y_{Nm},y_{Nm}] \subset \Omega^{N}.
\]

Since we are considering a CRRA investor, it is natural to impose the boundary conditions as follows:

\[
\mathcal{L}_{i}W = 0 \text{ at } y_{i} = y_{im}, \ i = 1,2,\ldots,N, \quad (4.3)
\]

\[
\mathcal{M}_{i}W = 0 \text{ at } y_{i} = y_{im}, \ i = 1,2,\ldots,N, \quad (4.4)
\]

which imply buying the \( i^{th} \) risky asset at \( y_{im} \) (the wealth fraction in the \( i^{th} \) asset being low enough) and selling the \( i^{th} \) risky asset at \( y_{im} \) (the wealth fraction being high enough), respectively. Figure 1 shows the truncated computation domain when \( N = 2 \) and \( \lambda_{1} = \lambda_{2} = \mu_{1} = \mu_{2} = 1% \), where the diamond refers to the solvency region, the rectangle inside the diamond is the truncated computation domain, and the circle stands for the no-trading region \( NT \).

\(^2\)We have tested that increasing the size of computation domain does not affect the solution in areas of interest.
4.3 Finite difference discretization

Let $\Delta t$ be the time step, and $t_n = n\Delta t$. Assume that we have a uniform grid for $D_N$, denoted by $D_N^h$. Let $h_i$ and $e_i$ respectively be the mesh size and the associated unit vector in $y_i$ direction. For illustration, let us perform discretization at $(y, t_n)$ with $y \in D_N^h$ and denote $W(y, t_n) = W^n(y)$.

The first order term $\frac{\partial W}{\partial y}$ is discretized by the upwind scheme. For example,

$$b_i \frac{\partial W}{\partial y_i} \sim \begin{cases} b_i \frac{W^n(y+h_i e_i) - W^n(y)}{h_i} & \text{if } b_i > 0, \\ b_i \frac{W^n(y) - W^n(h_i e_i)}{h_i} & \text{if } b_i < 0. \end{cases}$$

Since the upwind scheme is only of the first order, we use the fully implicit approximation to the temporal term:

$$\frac{\partial W}{\partial t} \sim \frac{W^{n+1}(y) - W^n(y)}{\Delta t}.$$

The term $\frac{\partial^2 W}{\partial y_i^2}$ is discretized as usual:

$$\frac{\partial^2 W}{\partial y_i^2} \sim \frac{W^n(y+h_i e_i) - 2W^n(y) + W^n(y-h_i e_i)}{h_i^2}.$$

As in Oksendal and Sulem (2005) and Clift and Forsyth (2008), we discretize the cross derivative term $\frac{\partial^2 W}{\partial y_i \partial y_j}$ as follows:

$$a_{ij} \frac{\partial^2 W}{\partial y_i \partial y_j} \sim \begin{cases} \frac{a_{ij}}{2h_i h_j} \left[ W^n(y + h_i e_i + h_j e_j) + W^n(y - h_i e_i - h_j e_j) + 2W^n(y) - W^n(y + h_i e_i) - W^n(y - h_i e_i) - W^n(y + h_j e_j) - W^n(y - h_j e_j) \right] & \text{if } a_{ij} > 0, \ i \neq j, \\ \frac{a_{ij}}{2h_i h_j} \left[ W^n(y + h_i e_i - h_j e_j) + W^n(y - h_i e_i + h_j e_j) + 2W^n(y) - W^n(y + h_i e_i) - W^n(y - h_i e_i) - W^n(y + h_j e_j) - W^n(y - h_j e_j) \right] & \text{if } a_{ij} < 0, \ i \neq j. \end{cases}$$

It is worthwhile pointing out that the above discretization will result in an $M$-matrix under the condition [cf. Oksendal and Sulem (2005), p. 142]

$$0 \leq \frac{a_{ii}(y)}{h_i} - \sum_{j \neq i} \frac{|a_{ij}(y)|}{h_j}, \text{ for all } y \in D_N, \ i = 1, \ldots, N. \quad (4.5)$$
4.4 Newton iteration for nonlinear terms

(4.1) contains several nonlinear terms: the penalty terms, the nonlinear terms in \( \mathcal{L}W \) due to transformation (3.8) (if \( \gamma \neq 0 \)), and the consumption term \( f(W) \) (if \( \kappa = 1 \)). All these terms can be linearized by Newton iteration. Especially, owing to the non-smoothness of the penalty terms, we use the so-called non-smooth Newton iteration [cf. Forsyth and Vetzal (2002)]. Let us take \( K (\mathcal{L}W)^+ \) for example. Assume that \( W^l \) be the \( l \)th estimate for \( W \). Then we linearize \( K (\mathcal{L}W)^+ \) as

\[
\begin{cases}
K \mathcal{L}_i W^{l+1} & \text{if } \mathcal{L}_i W^l \geq 0, \\
0 & \text{if } \mathcal{L}_i W^l < 0,
\end{cases}
\]

where we emphasize that the upwind scheme should be made for discretizing the first order terms in \( \mathcal{L}_i W \).

5 Convergence analysis

In this section we will restrict attention to the case of a single risky asset with log utility and without consumption, namely, \( N = 1, \kappa = 0 \) and \( \gamma = 0 \). We point out that our analysis only requires the resulting matrix to be an \( M \)-matrix and can be extended to the case of multiple risky assets under the condition (4.5).

For ease of presentation, we omit some subscripts when no confusion will arise, for example, \( y = y_1 \), \( h = h_1 \), \( a = a_{11} \), \( b = b_1 \). Let \( \bar{n} = T/\Delta t \), and the subscript \( k \) below means \( y = k h \) (assume \( y_m = mh \) and \( y_{\bar{m}} = \bar{m}h \)). Let us first assume the convergence of nonlinear iteration. Then, the discrete scheme can be written as follows:

\[
\begin{align*}
(FW^n)_k &= -W^{n+1}_k + W^n_k + (AW^n)_k + \theta_k \Delta t = P^n_{1k} \left( \frac{E^+W^n}_k - \left( \frac{\lambda}{1 + \lambda y} \right)_k \right) + P^n_{2k} \left( \frac{\lambda}{1 + \lambda y} \right)_k + \Delta t b_k I_{\{b_k < 0\}} (E^-W^n)_k, \\
W^n_k &= \log(1 - (\mu y^+ + \lambda y^-))_k, \quad \text{for } k = m + 1, ..., \bar{m} - 1, \quad n = 0, ..., \pi - 1,
\end{align*}
\]

where

\[
(\text{AW}^n)_k = -\Delta t a_k \left( (E^+W^n)_k - (E^-W^n)_k \right) + \frac{\Delta t}{h} b_k I_{\{b_k < 0\}} (E^+W^n)_k + \frac{\Delta t}{h} b_k I_{\{b_k > 0\}} (E^-W^n)_k,
\]

\[
(E^+W^n)_k = W^n_{k+1} - W^n_k \text{ and } (E^-W^n)_k = W^n_{k} - W^n_{k-1},
\]

\[
P^n_{1k} = \begin{cases} K \Delta t, & \text{if } \frac{(E^+W^n)_k}{h} > \left( \frac{\lambda}{1 + \lambda y} \right)_k, \\
0, & \text{otherwise}, \end{cases}
\]

\[
P^n_{2k} = \begin{cases} K \Delta t, & \text{if } \frac{(E^-W^n)_k}{h} < \left( \frac{-\mu}{1 - \mu y} \right)_k, \\
0, & \text{otherwise}. \end{cases}
\]

with \( a_k = \frac{1}{2} \sigma^2 (y^2(1 - y)^2)_k \), \( b_k = ((\alpha - r - \sigma^2 y)(1 - y))_k \), \( \theta_k = -\left( r + (\alpha - r)y - \frac{\sigma^2}{2} y^2 \right)_k \), and the first and last rows of \( A \) will have to be modified to allow for the boundary conditions:

\[
\frac{(E^+W^n)_k}{h} = \left( \frac{\lambda}{1 + \lambda y} \right)_k \text{ at } k = m, \quad \frac{(E^-W^n)_k}{h} = \left( \frac{-\mu}{1 - \mu y} \right)_k \text{ at } k = \bar{m}.
\]

To begin with, we present a stability result.

Proposition 5.1 (Stability) Let \( W^n_k \) be the solution to (5.1). Then

\[
|W^n_k| \leq \|\theta\|_\infty T + \|W(T, y)\|_\infty, \text{ for all } n, k,
\]

where \( \| \cdot \|_\infty \) refers to the \( L_\infty \)-norm.
Proof: It is easy to see that $Z^n_k = -\|\theta\|_\infty(T - t_n) - \|W(T, y)\|_\infty$ satisfies
\begin{equation}
-Z^{n+1}_k + Z^n_k + (AZ^n)_k + \theta_k \Delta t \leq 0, \text{ and } Z^n_k \leq W^n_k \text{ for all } k, \ n. \tag{5.5}
\end{equation}
Note that $A$ is an $M$-matrix. Due to the discrete maximum principle, we get $Z^n_k \leq W^n_k$ for all $k, \ n$.

It remains to prove $W^n_k \leq \|\theta\|_\infty T + \|W(T, y)\|_\infty$ for all $k, \ n$. First, we prove \begin{equation}
W^n_k \leq U^n_k \text{ for all } k \text{ and } n \tag{5.6}
\end{equation}
provided that $U$ satisfies
\begin{align}
(FU^n)_k &\geq 0, \tag{5.7} \\
\frac{(E^+U^n)_k}{h} &\leq \left(\frac{\lambda}{1 + \lambda y}\right)_k \quad \text{and} \quad \frac{(E^-U^n)_k}{h} \geq \left(\frac{-\mu}{1 - \mu y}\right)_k, \tag{5.8} \\
U^n_\pi &\geq W^n_\pi \tag{5.9}
\end{align}
for all $k$ and $n$. Assume contrary: there exists a node $(k_0, n_0)$ such that $W^n_{k_0} - U^n_{k_0} > 0$. Without loss of generality, we assume $W^n_{k_0} - U^n_{k_0} = \max_{(k, n)} \{W^n_k - U^n_k\} > 0$.

Moreover, we can assume $n_0$ is the maximum index of the nodes, if there are more than one maximum point. Since $W^n_{k_0} - U^n_{k_0} \geq W^n_{k_0 - 1} - U^n_{k_0 - 1}$ and $W^n_{k_0} - U^n_{k_0} \geq W^n_{k_0} - U^n_{k_0 - 1}$, we deduce that
\begin{align}
\frac{(E^+W^n)_{k_0}}{h} &= \frac{W^n_{k_0 + 1} - W^n_{k_0}}{h} \leq \frac{U^n_{k_0 + 1} - U^n_{k_0}}{h} \leq \left(\frac{\lambda}{1 + \lambda y}\right)_{k_0}, \tag{5.10} \\
\frac{(E^-W^n)_{k_0}}{h} &= \frac{W^n_{k_0} - W^n_{k_0 - 1}}{h} \geq \frac{U^n_{k_0} - U^n_{k_0 - 1}}{h} \geq \left(\frac{-\mu}{1 - \mu y}\right)_{k_0}. \tag{5.11}
\end{align}

Then, according to the terminal and boundary conditions, we are able to choose an interior node $(k_0, n_0)$, i.e. $m < k_0 < n_\pi$, and $0 \leq n_0 < n_\pi$.

Subtracting (5.7) from (5.1) at the node $(k_0, n_0)$, we have
\begin{align*}
(W^n_{n_0} - U^n_{n_0})_{k_0} + (A(W^n_{n_0} - U^n_{n_0}))_{k_0} &\leq (W^n_{n_0 + 1} - U^n_{n_0 + 1})_{k_0} + P^n_{1k_0} \left(\frac{(E^+W^n)_{k_0}}{h} - \left(\frac{\lambda}{1 + \lambda y}\right)_{k_0}\right) \\
&\quad + P^n_{2k_0} \left(\left(\frac{-\mu}{1 - \mu y}\right)_{k_0} - \frac{(E^-W^n)_{k_0}}{h}\right) \\
&= (W^n_{n_0 + 1} - U^n_{n_0 + 1})_{k_0}, \tag{5.12}
\end{align*}
where the equality is due to (5.10)-(5.11) and the definition of $P^n_{1k_0}$ and $P^n_{2k_0}$. Since $A$ is an $M$-matrix, $(A(W^n_{n_0} - U^n_{n_0}))_{k_0} \geq 0$. Hence,
\begin{equation}
(W^n_{n_0} - U^n_{n_0})_{k_0} \leq (W^n_{n_0 + 1} - U^n_{n_0 + 1})_{k_0},
\end{equation}
which is in contradiction with the selection of $n_0$. So, (5.6) follows.

It is easy to verify that $U^n_k = \|\theta\|_\infty(T - t_n) + \|W(T, y)\|_\infty$ for all $k, \ n$, satisfies (5.7)-(5.9). This completes the proof.

Thanks to the stability result, it is not hard to show that if $\frac{\Delta t}{h} < \text{const.}$, then\footnote{See Zhong’s thesis [27] for details.}
\begin{align*}
P^n_{1k} \left(\frac{(E^+W^n)_k}{h} - \left(\frac{\lambda}{1 + \lambda y}\right)_k\right) &\leq C, \\
P^n_{2k} \left(\left(\frac{-\mu}{1 - \mu y}\right)_k - \frac{(E^-W^n)_k}{h}\right) &\leq C,
\end{align*}
where constant $C$ is independent of $K, \Delta t, h$. It follows that the penalty method (5.1) solves

\[
\begin{cases}
(FW^n)_k \geq 0 \\
\left(\frac{1}{1+\lambda y}\right)_{k} - \left(\frac{E+W^n}{h}\right)_k \geq -\frac{C}{K\Delta t} \\
\left(\frac{1}{1-\mu y}\right)_{k} - \left(\frac{-\mu}{1-\mu y}\right)_k \geq -\frac{C}{K\Delta t} \\
(W^n)_{k} = \log(1 - (\mu y^+ + \lambda y^-))_k.
\end{cases}
\]

Here the notation $(\cdot) \lor (\cdot) \lor (\cdot)$ means that at least one holds. As a result, (5.1) converges to the following discrete formulation of (3.9) as $K \to +\infty$:

\[
\begin{cases}
(FW^n)_k \geq 0 \\
\left(\frac{1}{1+\lambda y}\right)_{k} - \left(\frac{E+W^n}{h}\right)_k \geq 0 \\
\left(\frac{1}{1-\mu y}\right)_{k} - \left(\frac{-\mu}{1-\mu y}\right)_k \geq 0 \\
((FW^n)_k = 0) \lor \left(\frac{1}{1+\lambda y}\right)_{k} = \left(\frac{E+W^n}{h}\right)_k \lor \left(\frac{E-W^n}{h}\right)_k = \left(\frac{-\mu}{1-\mu y}\right)_k
\end{cases}
\]

\[W^n_k = \log(1 - (\mu y^+ + \lambda y^-))_k.\]

Remark 5.2 We would like to point out that the lower bound $-\frac{C}{K\Delta t}$ in Eq. (5.13) is suboptimal. We conjecture that the optimal estimate of the lower bound should be independent of $\Delta t$.

Now we analyze the convergence of the nonlinear iteration caused by the penalty terms (assuming no consumption). Let $W^{n,l}$ be the $l^{th}$ estimate for $W^n$, and $W^{n,0} = W^{n,1}$. For notational convenience, we define

\[P^{n,l}_1 = P^{n}_1(W^{n,l}), \quad P^{n,l}_2 = P^{n}_2(W^{n,l}).\]

The iteration process for the nonlinear system (5.1) can be written as follows:

For $l = 0, 1, \ldots$ until convergence

\[[(I + A) - \frac{1}{h}P^{n,l}_1 E^+ + \frac{1}{h}P^{n,l}_2 E^+]W^{n,l+1} = W^{n,1} - \theta \Delta t - P^{n,l}_1 \frac{\lambda}{1+\lambda y} - P^{n,l}_2 \frac{\mu}{1-\mu y}.\]

If $\frac{\|W^{n,l+1} - W^{n,l}\|_\infty}{\max(1,\|W^{n,l}\|_\infty)} < tol$, quit.

Proposition 5.3 (Convergence of the nonlinear iteration) Assume no consumption term (i.e., $\kappa = 0$). The algorithm for the nonlinear iteration has the following properties.

i) The iteration converges monotonically, i.e., $W^{n,l} \leq W^{n,l+1}$ for $l \geq 1$.

ii) Each iterate is bounded independent of $l$. More precisely, $W^{n,l} \leq \|W^{n,1}\|_\infty + \|\theta\|_\infty \Delta t$, for all $l \geq 1$.

iii) The nonlinear iteration (5.14) converges to the unique solution to equation (5.1), for any initial iterate value $W^{n,0}$.

The proof is placed in Appendix, which is similar to that in Forsyth and Vetzal (2002). In contrast to Forsyth and Vetzal (2002), we are unable to prove the so-called “finite termination of iteration” due to gradient constraints. However, our algorithm still converges for a given tolerance owing to the monotone convergence and the boundedness of iteration sequence.
6 The standard penalty method

In some cases a singular stochastic control problem has a connection with an optimal stopping problem [cf. Karatzas and Shreve (1984)]. In other words, the variational inequality with gradient constraints arising from a singular stochastic control problem can be reduced to a standard variational inequality (i.e., complimentary problem or obstacle problem) in some cases, which enables us to make use of the standard penalty method proposed by Forsyth and Vetzal (2002) and Dai, Kwok and You (2007). In fact, for the present problem, Dai and Yi (2009) and Dai et al. (2009) have proved such a reduction when \( N = 1 \): let

\[
v = w_y,
\]

which is shown to satisfy the following double obstacle problem:4

\[
\begin{align*}
\min \left\{ \max \left\{ -v_t - \mathcal{T}v - \kappa f(w), v - \frac{\lambda}{1 + \lambda y} \right\}, v + \frac{\mu}{1 - \mu y} \right\} &= 0, \\
v(y, T) &= -\frac{\mu}{1 - \mu y} \text{ in } y \in [0, \frac{1}{\mu}], t \in [0, T).
\end{align*}
\]

Here

\[
\begin{align*}
\mathcal{T}v &= \frac{1}{2} \sigma^2 y^2 (1 - y)^2 v_{yy} + (\alpha - r + (\gamma - 1)\sigma^2 y + (1 - 2y)\sigma^2) y(1 - y)v_y \\
+ ((\alpha - r)(1 - 2y) + (\gamma - 1)\sigma^2 y(2 - 3y)) v + (\alpha - r + (\gamma - 1)\sigma^2 y) v_y \\
+ \gamma \sigma^2 y (1 - y)v(1 - 2y)v + y(1 - y)v_y \\
f(w) &= \begin{cases} 
\frac{w_y}{\rho(t)(1 - y^2)(v^2 + v_y)} & \text{if } \gamma \neq 0, \gamma < 1, \\
\frac{w_y}{\mu(t)(1 - y^2)(v^2 + v_y)} & \text{if } \gamma = 0
\end{cases}
\end{align*}
\]

\( -\frac{\mu}{1 - \mu y} \) and \( \frac{\lambda}{1 + \lambda y} \) are called the lower obstacle and the upper obstacle, respectively. Note that we only need to consider \( y \geq 0 \) because it can be shown that short selling is suboptimal in the case of \( \alpha > r \) and \( N = 1 \) [see for example, Shreve and Soner (1994)]. Since the lower obstacle \( -\frac{\mu}{1 - \mu y} \) tends to infinity as \( y \to \frac{1}{\mu} \), we impose a boundary condition

\[
v(y, t) = -\frac{\mu}{1 - \mu y} \text{ at } y = \frac{1}{\mu} - \epsilon \text{ with } 0 < \epsilon < 1.
\]

The condition is without loss of generality because Dai and Yi (2009) and Dai et al. (2009) have proved that the selling boundary never hits \( y = \frac{1}{\mu} \).

It is seen that at \( y = 0 \), (6.2) reduces to

\[
\begin{align*}
\min \left\{ \max \left\{ -v_t - (\alpha - r)v - (\alpha - r), v - \lambda \right\}, v + \mu \right\} &= 0, \\
v(0, T) &= -\mu \text{ in } t \in [0, T),
\end{align*}
\]

solving which we obtain a boundary condition at \( y = 0 \):

\[
v(0, t) = v(0, t) = \min \{ (1 - \mu)e^{(\alpha - r)(T - t)} - 1, \lambda \} \text{ for all } t.
\]

Hence, we will use the following penalty approximation:

\[
\begin{align*}
-v_t - \mathcal{T}v - \kappa f(w) &= -K(v - \frac{\lambda}{1 + \lambda y})^+ + K(-\frac{\mu}{1 - \mu y} - v)^+ \\
v(y, T) &= -\frac{\mu}{1 - \mu y} \text{ in } y \in (0, \frac{1}{\mu} - \epsilon), t \in [0, T)
\end{align*}
\]

with the boundary conditions (6.3)-(6.4). The discretization is similar to that in Forsyth and Vetzal (2002) or Dai, Kwok and You (2007). We highlight that the Crank-Nicolson scheme can be used to improve the accuracy because the current penalty terms do not involve any first order terms.

In the following we restrict attention to the implementation of numerical methods respectively for \( \kappa = 0 \) and \( \kappa = 1 \). We consider only the case \( \gamma \neq 0, \gamma < 1, \) and the case \( \gamma = 0 \) is similar.5

4In Dai and Yi (2009) and Dai et al. (2009), they used a different state variable and the resulting double obstacle problem is slightly different.

5When consumption is involved, the case of \( \gamma = 0 \) is simpler because the double obstacle problem (6.2) becomes a self-contained system.
6.1 Without consumption (κ = 0)

In this case, at y = 1 (6.2) reduces to

\[
\begin{aligned}
\min \left\{ \max \left\{ -v_t + (\alpha - r - (1 - \gamma)\sigma^2)v - (\alpha - r - (1 - \gamma)\sigma^2), v - \frac{\lambda}{1 + \lambda} \right\}, v + \frac{\mu}{1 - \mu} \right\} = 0,
\end{aligned}
\]

which yields

\[
v(1, t) = \max \left\{ \min \left\{ 1 - \frac{1}{1 - \mu} e^{-(\alpha - r - (1 - \gamma)\sigma^2)(T - t)}, \frac{\lambda}{1 + \lambda} \right\}, -\frac{\mu}{1 - \mu} \right\}. \tag{6.6}
\]

In terms of (6.6), we can solve (6.5) separately in \(0 < y < 1\) and \(1 < y < \frac{1}{\mu} - \epsilon\), which significantly reduces the size of computations.

6.2 With consumption (κ = 1)

In this case, there is no explicit solution at y = 1 due to the presence of \(\overline{f}(w)\). As a consequence, we have to solve the problem in \(0 < y < \frac{1}{\mu} - \epsilon\). Moreover, (6.2) is not a self-contained system since \(w\) is involved. Fortunately, it has been shown in Dai et al. (2009) that there exists \(y_s(t)\) such that \(SR = \{(y, t) \in \Omega^1 \times [0, T] : y \geq y_s(t)\}\). In terms of (6.1), we then obtain a relation between \(w\) and \(v\):

\[
w(y, t) = B(t) + \log(1 - \mu y_s(t)) - \int_y^{y_s(t)} v(\xi, t)d\xi, \tag{6.7}
\]

where \(B(t)\) is to be determined. Clearly \(B(T) = 0\). It is shown in Dai et al. (2009) that

\[
B(t) = \frac{1 - \gamma}{\gamma} \log \left( e^{-\gamma T} \int_t^T h(y_s(\xi))d\xi \left( 1 + \int_t^T e^{-\gamma T} \int_t^T h(y_s(\xi))d\xi d\tau \right) \right).
\]

Let \(v^{n,l}(\cdot)\) and \(w^{n,l}(\cdot)\) be the \(l^{th}\) discrete solutions at time \(t_n\). In terms of (6.7), we can have an iterative algorithm as follows.

Step 1: At time step \(t = t_n\), start off with an initial guess of \(w^n\), denoted by \(w^{n,0}\).
Step 2: Find \(v^{n,l+1}\) by virtue of the penalty method for (6.5) with \(w = w^{n,l}\).
Step 3: Compute the corresponding boundary \(y^{l+1}_{s}(t_n) = \min \left\{ y \in (0, \frac{1}{\mu} - \epsilon) : v^{n,l+1}(y, t_n) \leq -\frac{\mu}{1 - \mu} \right\} \).
Step 4: Update \(B(t_n)\) by (6.8), then compute \(w^{n,l+1}\) by (6.7).
Step 5: Stop if \(\frac{\|w^{n,l+1} - w^{n,l}\|}{\max(1, \|w^{n,l}\|)} < tol\). Otherwise set \(l = l + 1\) and go back to Step 2.

7 Numerical results

The objectives of this section are twofold. First, we test the efficiency of the penalty methods. Second, we examine the behaviour of optimal trading strategies.

To begin with, let us look at the convergence as the penalty parameter \(K\) goes to infinity. Table 1 presents the values of \(v(y_M, 0)\) and \(v(y_M, 0)\) against varying \(K\), computed from the standard penalty method with \(N = 1\). Here \(y_M\) refers to the “Merton line” in the absence of transaction costs. For the purpose of comparison, we also provide the benchmark values computed from the projected SOR method with the same grid for the standard variational inequality (6.2) [See Section 9.4, Wilmott, Dewyne, and Howison (1995) for the projected SOR method]. It can be seen that the values clearly converge as \(K\) goes to infinity. Similar convergence for \(N = 2\) can be observed from Table 2, where the penalty method for variational inequality with gradient constraints (3.7) is adopted. Note that when \(N = 2\), no standard variational inequality is available and we have to work with (3.7). Fortunately, the projected SOR method can still
work and provide benchmark values, because the discrete formulations of the gradient constraints lead to upper-/lower-triangle matrices (due to the use of the upwind scheme) which can be readily reduced to the constraints on the solution itself.

Next, we examine the order of convergence of the penalty methods. There are two sources of errors for the penalty methods: one from the difference approximation, and the other from the penalty approximation. By (5.13), we can choose $K \Delta t$ to be big enough such that the error due to the penalty approximation can be neglected (we take $K \Delta t = 10^4$ in Table 3-4). Then, the order of convergence will be primarily determined by the difference approximation. In Table 3, we list the numerical results for $N = 1$ obtained from the standard penalty method with the Crank-Nicolson scheme. When there is no consumption ($\kappa = 0$), the second order of convergence can be observed. When consumption is involved ($\kappa = 1$), the rate of convergence is however slower than the expected rate due to the upwind treatment of the consumption term. In Table 4, we list the numerical results for $N = 2$ obtained from the penalty method with the fully implicit scheme. The apparent first order of convergence is revealed.

We now examine the properties of the optimal trading strategy when $N = 1$. Figure 2 presents the shape of the BR, the SR and the NTR in $y$-$t$ plane for both the consumption case and the no-consumption case. It turns out that there are two time-dependent boundaries, one being the optimal buying boundary (the lower) and the other being the optimal selling boundary (the upper), such that the BR is below the buying boundary, the SR is above the selling boundary and the NTR is between them. This indicates that a risk averse investor prefers to buy low and sell high. The solid and dashed lines represent the boundaries in the consumption case and in the no consumption case, respectively. Observe that the selling (buying) boundary in the consumption case is lower than that in the no consumption case. The intuition behind is that in the consumption case, the investor has to keep a larger fraction of wealth in the bank account to maintain consumption. In addition, Figure 2 reveals that it is never optimal to buy the risky asset provided that the time is greater than a threshold value no matter whether consumption is involved. Such a phenomenon, called “no-buying near maturity”, was first proved by Liu and Loewenstein (2002) for the no-consumption case and by Dai et al. (2009) for the consumption case. The theoretical threshold value turns out to be $T - \frac{1}{\alpha r} \log \left( \frac{1 + \lambda}{1 - \lambda} \right) = 2.818$ for the given example.

Dai and Yi (2009) proved that both the optimal buying and selling boundaries are monotonically decreasing with time $t$ in the no-consumption case, which is consistent with the conventional wisdom that the younger investor should allocate more weight to risky assets than the older investor [cf. Liu and Loewenstein (2002)]. However, it may not be true in the consumption case if the discount factor $\beta$ is big enough. Figure 3 presents an example with $\beta = 7$ where the optimal buying boundary in the consumption case is apparently not monotone.

In Figure 4, we plot the optimal buying and selling boundaries with varying $\alpha$. Observe that the buying (selling) boundary is increasing with $\alpha$ in both the consumption case and the no-consumption case, which means that the bigger the return rate of the risky asset $\alpha$, the larger the fraction of wealth in the risky asset. If $\alpha = 0.18$, then $\alpha - r < (1 - \gamma) \sigma^2$ and the NTR is contained in the region $\{y < 1\}$, which implies that leverage is always suboptimal. If $\alpha = 0.3$, then $\alpha - r > (1 - \gamma) \sigma^2$ and the NTR contains part of $\{y > 1\}$, which indicates that leverage is likely needed. If $\alpha = 0.25$, then $\alpha - r = (1 - \gamma) \sigma^2$ and the selling boundary is exactly $y = 1$. All these results are consistent with the theoretical analysis in Liu and Loewenstein (2002), Dai and Yi (2009) and Dai et al. (2009).

We now investigate the effects of risk aversion and transaction costs on the optimal strategy. Let us only take the consumption case for illustration. By Figure 5, we can see that both the optimal buying and selling boundaries are increasing with $\gamma$, or equivalently, decreasing with the index of risk aversion $1 - \gamma$. This is because a more risk averse investor would like to keep larger fraction of wealth in the bank account. Figure 6 shows that as transaction costs increase, the NTR expands, which means that the investor tends to decrease the trading frequency to save transaction costs. In addition, similar to Liu and Loewenstein (2002), we can observe that the optimal buying boundary is more sensitive to transaction costs than the optimal selling boundary.

Let us move to the case of $N = 2$. A time snapshot of the SR, BR, NTR, $i = 1, 2$, and NTR is depicted in Figure 7. As in the case of a single risky asset, the optimal trading strategy is to keep the wealth fractions $(y_1, y_2)$ in the NTR by selling high and buying low. In what follows, we only focus on the NTR. Figure 8 presents the time snapshots of the NTR at different times. It can be observed that as
time approaches to maturity, the bottom and left-hand sides of the NTR match \( \{ y_1 = 0 \} \) and \( \{ y_2 = 0 \} \), respectively, which confirms the phenomenon of “no-buying near maturity”. In Figure 9, we compare the consumption case against the no-consumption case. Similar to the single risky asset case, it can be seen that larger fractions of wealth are invested in risky assets when there is no consumption.

At last, we investigate the effect of the correlation between two risky assets on the optimal buying and selling boundaries. Figure 10 shows that the NTR elongates along the direction \((1, -1)\) and shrinks along the direction \((1, 1)\) with the increase in positive correlation. In contrast, Figure 11 shows that the NTR elongates along the direction \((1, 1)\) and shrinks along the direction \((1, -1)\) with the increase in negative correlation. These are the same as what Muthuraman and Kuman (2006) have observed for the infinite horizon problem. Further, we follow Muthuraman and Kuman (2006) and keep the Merton line fixed so that this effect can be displayed more clearly (see Figure 12).

8 Conclusion

We provide a general framework of applying the penalty methods to numerically solve the continuous-time portfolio selection with transaction costs. The problem is described as a singular stochastic control problem and the associated value function satisfies an HJB equation. In terms of a series of transformations, we obtain a unified variational inequality with gradient constraints for both power utility and log utility. It is straightforward to apply the penalty methods to the variational inequality. Since the upwind scheme has to be employed to discretize the penalty terms, we adopt a fully implicit finite difference discretization. Convergence analysis is provided as well.

It is worthwhile pointing out that the associated penalty approximation arises from the original control problem restricted to a class of policies being absolutely continuous and bounded. This is in contrast to the relation between the penalty approximation and the intensity framework for an optimal stopping problem [see Dai, Kwok and You (2007)].

When there is only one risky asset, Dai and Yi (2009) and Dai et al. (2009) showed that the problem can be reduced to a standard variational inequality (obstacle problem). In this case, we can make use of the standard penalty method as in Forsyth and Vetzal (2002) and Dai, Kwok and You (2007), which allows us to adopt the Crank-Nicolson scheme. Then a better order of convergence can be achieved.

In addition, we carry out a comprehensive numerical analysis on the behaviour of the optimal selling and buying boundaries. The effects of parameter values on the optimal strategy are investigated. In the case of a single risky asset, numerical results demonstrate the theoretical analysis in Liu and Loewenstein (2002), Dai and Yi (2009) and Dai et al. (2009). Moreover, we offer an example that the optimal buying boundary may not be monotone when consumption is involved. In the case of multiple risky assets, we find that one should never buy any risky assets when time is close to maturity. Such a phenomenon has been proved by Liu and Loewenstein (2002), Dai and Yi (2009) and Dai et al. (2009) for the single risky asset case, but has never been revealed when multiple risky assets are involved.

A Appendix: Proof of Proposition 5.3

Writing (5.14) for \((l - 1)\)th iteration yields

\[
[(I + A) - \frac{1}{h} P_1^{n,l-1} E^+] + \frac{1}{h} P_2^{n,l-1} E^- W^{n,l} = W^{n+1} - \theta \Delta t - P_1^{n,l-1} \frac{\lambda}{1 + \lambda y} - P_2^{n,l-1} \frac{\mu}{1 - \mu y}.
\]  

(A.1)

Note that equation (A.1) always has a solution, since \([(I + A) - \frac{1}{h} P_1^{n,l-1} E^+] + \frac{1}{h} P_2^{n,l-1} E^-] \) is an M-matrix.

Subtracting (A.1) from (5.14), we have

\[
[(I + A) - \frac{1}{h} P_1^{n,l} E^+] + \frac{1}{h} P_2^{n,l} E^- (W^{n,l+1} - W^{n,l})
\]

\[
= (P_1^{n,l} - P_1^{n,l-1}) \left( \frac{E^+ W^{n,l}}{h} - \frac{\lambda}{1 + \lambda y} \right) + (P_2^{n,l} - P_2^{n,l-1}) \left( - \frac{\mu}{1 - \mu y} - \frac{E^- W^{n,l}}{h} \right).
\]  

(A.2)
Now we examine each of the components of the right hand side of (A.2). Observe
\[(P_1^{n,l} - P_1^{n,l-1})(\frac{1}{h} E^+ W^{n,l} - \frac{\lambda}{1 + \lambda y}) \geq 0\]
and
\[(P_2^{n,l} - P_2^{n,l-1})(-\frac{\mu}{1 - \mu y} - \frac{1}{h} E^- W^{n,l}) \geq 0.\]
Therefore, we infer that
\[\[(I + A) - \frac{1}{h} P_1^{n,l} E^+ + \frac{1}{h} P_2^{n,l} E^-](W^{n,l+1} - W^{n,l}) \geq 0.\]
Since \[(I + A) - \frac{1}{h} P_1^{n,l} E^+ + \frac{1}{h} P_2^{n,l} E^-\] is an \(M\)-matrix, it follows that \(W^{n,l+1} - W^{n,l} \geq 0\). The monotonicity of iteration process is proved.

Now let us prove part ii). It is easy to see that
\[|W^{n+1}|_\infty + ||\theta||_\infty \Delta t \text{ satisfies}\]
\[\[(I + A) - \frac{1}{h} P_1^{n,l-1} E^+ + \frac{1}{h} P_2^{n,l-1} E^-]U = |W^{n+1}|_\infty + ||\theta||_\infty \Delta t.\]
(A.3)
Subtracting (A.1) from (A.3), we obtain
\[\[(I + A) - \frac{1}{h} P_1^{n,l-1} E^+ + \frac{1}{h} P_2^{n,l-1} E^-](U - W^{n,l}) \geq 0,\]
which yields the desired result by virtue of the discrete maximum principle.

It remains to show that the solution obtained by the penalty iteration is unique. Suppose that there are two solutions \(W\) and \(\bar{W}\) to the penalized equation (5.1). Then
\[\[(I + A) - \frac{1}{h} P_1 E^+ + \frac{1}{h} P_2 E^-]W = W^{n+1} - \theta \Delta t - P_1\frac{\lambda}{1 + \lambda y} - P_2\frac{\mu}{1 - \mu y},\]
(A.4)
\[\[(I + A) - \frac{1}{h} P_1 E^+ + \frac{1}{h} P_2 E^-]W = W^{n+1} - \theta \Delta t - P_1\frac{\lambda}{1 + \lambda y} - P_2\frac{\mu}{1 - \mu y}.\]
(A.5)
Subtracting (A.5) from (A.4) gives
\[\[(I + A) - \frac{1}{h} P_1 E^+ + \frac{1}{h} P_2 E^-](W - \bar{W}) = (P_1 - \bar{P}_1)(\frac{1}{h} E^+ \bar{W} - \frac{\lambda}{1 + \lambda y}) + (P_2 - \bar{P}_2)(-\frac{\mu}{1 - \mu y} - \frac{1}{h} E^- \bar{W}).\]
Using a similar argument as in proving monotone iteration, we obtain \(W - \bar{W} \leq 0\). In the same way we have \(\bar{W} - W \geq 0\), and hence \(W = \bar{W}\).

B Appendix: Derivation of (6.8)

As shown in Dai, et al. (2009), \(v(\cdot, t) \in C^1\) and \(w(\cdot, t) \in C^2\), and
\[w_t + \mathcal{L}w|_{y=y_s(t)} = 0.\]
(B.1)
Thus,
\[w_y|_{y=y_s(t)} = -\frac{\mu}{1 - \mu y_s(t)};\]
\[w_{yy}|_{y=y_s(t)} = -\frac{\mu^2}{(1 - \mu y_s(t))^2}.
\]
Substituting into (B.1) gives
\[-B'(t) = -w_t(y_s(t), t) = \mathcal{L}w|_{y=y_s(t)} = (\frac{1}{\gamma} - 1)e^{-\frac{t}{\gamma}} B(t) + h(y_s(t)).\]
(B.2)
Solving (B.2) with \(B(T) = 0\), we obtain (6.8).
References


Table 1: Numerical solutions of the (standard) penalty method against the penalty parameter $K$ ($N = 1$)

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\varphi(y_{M,0})$</th>
<th>$v(y_{M,0})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>-7.12355</td>
<td>-0.009888</td>
</tr>
<tr>
<td>10^2</td>
<td>-7.123531</td>
<td>-0.009182</td>
</tr>
<tr>
<td>10^3</td>
<td>-7.123726</td>
<td>-0.009140</td>
</tr>
<tr>
<td>10^4</td>
<td>-7.123784</td>
<td>-0.009137</td>
</tr>
<tr>
<td>10^5</td>
<td>-7.123796</td>
<td>-0.009136</td>
</tr>
<tr>
<td>10^6</td>
<td>-7.123798</td>
<td>-0.009136</td>
</tr>
<tr>
<td>benchmark</td>
<td>-7.123798</td>
<td>-0.009136</td>
</tr>
</tbody>
</table>

Default parameter values: $\alpha = 0.15$, $\sigma = 0.4$, $r = 0.07$, $\beta = 0.1$, $\gamma = -1$, $\lambda = \mu = 0.01$, $T = 2$, $\kappa = 1$, $y_{m} = 0$, $y_{m\tau} = 1$, $\Delta t = 5 \times 10^{-4}$, $h = 10^{-3}$, $\varphi(\cdot, \cdot)$ and $v(\cdot, \cdot)$ are the numerical solutions to (3.7) and (6.2), respectively. $y_M = \frac{\alpha - r}{(1 - \gamma) \sigma^2}$ refers to the “Merton line” in the absence of transaction costs. The benchmark value is computed from the projected SOR method with the same grid.

Table 2: Numerical solutions of the penalty method against the penalty parameter $K$ ($N = 2$)

<table>
<thead>
<tr>
<th>$K$</th>
<th>$\varphi(y_{M1,0}, y_{M2,0})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-7.104291</td>
</tr>
<tr>
<td>0.5</td>
<td>-7.097533</td>
</tr>
<tr>
<td>1</td>
<td>-7.097456</td>
</tr>
<tr>
<td>10</td>
<td>-7.097435</td>
</tr>
<tr>
<td>10^2</td>
<td>-7.097435</td>
</tr>
<tr>
<td>10^6</td>
<td>-7.097434</td>
</tr>
<tr>
<td>benchmark</td>
<td>-7.097434</td>
</tr>
</tbody>
</table>

Default parameter values: $\alpha_1 = 0.15$, $\sigma_1 = 0.4$, $\alpha_2 = 0.12$, $\sigma_2 = 0.3$, $r = 0.07$, $\beta = 0.1$, $\gamma = -1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 1$, $y_{m1} = y_{m2} = 0$, $y_{m\tau1} = y_{m\tau2} = 0.4$, $\Delta t = 5 \times 10^{-4}$, $h_1 = h_2 = 2 \times 10^{-3}$, $\varphi(\cdot, \cdot, \cdot)$ is the numerical solution to (3.7). $y_{M,i} = \frac{1}{1-r^2} \left( \frac{\alpha_i - r}{(1 - \gamma) \sigma_i^2} - \rho \frac{\alpha_j - r}{(1 - \gamma) \sigma_j^2} \right)$ refers to the “Merton line” of the $i^{th}$ risky asset in the absence of transaction costs, $i, j \in \{1, 2\}, i \neq j$. The benchmark value is computed from the projected SOR method with the same grid.

Table 3: The convergence rate of the standard penalty method with Crank-Nicolson scheme ($N = 1$)

<table>
<thead>
<tr>
<th>$\bar{n}$</th>
<th>$N_y$</th>
<th>$|\epsilon|_\infty, \kappa = 0$</th>
<th>Ratio</th>
<th>$|\epsilon|_\infty, \kappa = 1$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>1200</td>
<td>200</td>
<td>1.29e-05</td>
<td>-</td>
<td>3.12e-05</td>
<td>-</td>
</tr>
<tr>
<td>2400</td>
<td>400</td>
<td>3.46e-06</td>
<td>2.6</td>
<td>2.70e-06</td>
<td>11.6</td>
</tr>
<tr>
<td>4800</td>
<td>800</td>
<td>8.70e-07</td>
<td>4.0</td>
<td>1.11e-06</td>
<td>2.4</td>
</tr>
<tr>
<td>9600</td>
<td>1600</td>
<td>2.02e-07</td>
<td>4.3</td>
<td>3.57e-07</td>
<td>3.1</td>
</tr>
</tbody>
</table>

Parameter default values: $\alpha = 0.15$, $\sigma = 0.4$, $r = 0.07$, $\beta = 0.1$, $\gamma = -1$, $\lambda = \mu = 0.01$, $T = 0.6$, $y_{m} = 0$, $y_{m\tau} = 1$, $K = \frac{1000}{\Delta t}$, $\Delta t = \frac{T}{\bar{n}}$, $h = \frac{y_{m\tau} - y_{m}}{N_y}$. Here, $\|\epsilon\|_\infty$ is the error of solution in $L_\infty$-norm, where the benchmark values are computed from the projected SOR method with the grid $\bar{n} = 38400$, $N_y = 6400$. Ratio is the ratio of the changes $\|\epsilon\|_\infty$ on the successive grids.
Table 4: The convergence rate of the penalty method with fully implicit scheme ($N = 2$)

<table>
<thead>
<tr>
<th>$\bar{n}$</th>
<th>$N_{y1}$</th>
<th>$N_{y2}$</th>
<th>$| \epsilon |_{\infty}$</th>
<th>Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>10</td>
<td>10</td>
<td>3.07e-03</td>
<td>-</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>20</td>
<td>1.50e-03</td>
<td>2.05</td>
</tr>
<tr>
<td>200</td>
<td>40</td>
<td>40</td>
<td>6.99e-04</td>
<td>2.14</td>
</tr>
<tr>
<td>400</td>
<td>80</td>
<td>80</td>
<td>2.99e-04</td>
<td>2.34</td>
</tr>
</tbody>
</table>

Parameter default values: $\alpha_1 = 0.15$, $\sigma_1 = 0.4$, $\alpha_2 = 0.12$, $\sigma_2 = 0.3$, $\rho = 0.2$, $r = 0.07$, $\beta = 0.1$, $\gamma = -1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 0.6$, $\kappa = 1$, $y_{1m} = y_{2m} = 0$, $y_{1m'} = y_{2m'} = 0.4$, $K = \frac{1000}{\Delta t}$, $\Delta t = \frac{T}{\pi}$, $h_i = \frac{y_{im} - y_{im'}}{N_{y_i}}$. The definitions of $\| \epsilon \|_{\infty}$ and Ratio are the same as in Table 3. The grid of the projected SOR method for benchmark values is $\bar{n} = 1600$, $N_{y1} = 320$, $N_{y2} = 320$.

Figure 2: Shape of BR, SR and NTR, and comparison of the buying and selling boundaries between the consumption case and the no consumption case ($N = 1$). Default parameter values: $\alpha = 0.18$, $r = 0.07$, $\sigma = 0.3$, $\gamma = -1$, $\beta = 0.1$, $\lambda = \mu = 0.01$, $T = 3$.

Figure 3: An example of non-monotone buying boundary in the consumption case ($N = 1$). Default parameter values: $\alpha = 0.18$, $r = 0.07$, $\sigma = 0.3$, $\gamma = -1$, $\beta = 7$, $\lambda = \mu = 0.01$, $T = 3$. 
Figure 4: The effect of the risky asset return on the optimal strategy ($N = 1$). Default parameter values: $r = 0.07, \sigma = 0.3, \gamma = -1, \beta = 0.1, \lambda = \mu = 0.01, T = 3$.

Figure 5: The effect of risk aversion on the optimal strategy ($N = 1$). Default parameter values: $\alpha = 0.15, r = 0.07, \sigma = 0.3, \beta = 0.1, \lambda = \mu = 0.01, T = 3, \kappa = 1$. 
Figure 6: The effect of transaction costs on the optimal strategy ($N = 1$). Default parameter values: $\alpha = 0.15$, $r = 0.07$, $\sigma = 0.3$, $\gamma = -1$, $\beta = 0.1$, $\lambda = \mu$, $T = 3$, $\kappa = 1$.

Figure 7: The time snapshot of NTR, BR$_i$, SR$_i$ and NTR$_i$, $i = 1, 2$, and $N = 2$. Default parameter values: $r = 0.07$, $\beta = 0.10$, $\alpha_1 = 0.15$, $\alpha_2 = 0.12$, $\sigma_1 = 0.4$, $\sigma_2 = 0.35$, $\rho = 0.20$, $\gamma = -1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 1$. 
Figure 8: The different time snapshots of NTR ($N = 2$). Default parameter values: $\alpha_1 = 0.15$, $\alpha_2 = 0.12$, $r = 0.07$, $\sigma_1 = 0.4$, $\sigma_2 = 0.35$, $\rho = 0.2$, $\gamma = -1$, $\beta = 0.1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 1$.

Figure 9: The comparison of the NTR between the consumption case and the no consumption case ($N = 2$). Default parameter values: $\alpha_1 = 0.15$, $\alpha_2 = 0.11$, $r = 0.07$, $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $\rho = 0.2$, $\gamma = -1$, $\beta = 0.1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 4$. 
Figure 10: The effect of positive correlation on the NTR at $t = 0$ ($N = 2$). Parameter default values: $\alpha_1 = 0.14$, $\alpha_2 = 0.11$, $r = 0.07$, $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $\gamma = -1$, $\beta = 0.1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 0$.

Figure 11: The effect of negative correlation on the NTR at $t = 0$ ($N = 2$). Default parameter values: $\alpha_1 = 0.14$, $\alpha_2 = 0.11$, $r = 0.07$, $\sigma_1 = 0.4$, $\sigma_2 = 0.3$, $\gamma = -1$, $\beta = 0.1$, $\lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01$, $T = 2$, $\kappa = 0$. 
Figure 12: The effect of positive correlation on the NTR at $t = 0$ when the “Merton line” is fixed ($N = 2$). Default parameter values: $\alpha_1 = 0.15, \alpha_2 = 0.15, r = 0.07, \sigma_1^2 = \sigma_2^2 = (0.4 - \eta)^2 + \eta^2, \rho = \frac{2\eta(0.4-\eta)}{(0.4-\eta)^2 + \eta^2}, \gamma = -1, \beta = 0.1, \lambda_1 = \mu_1 = \lambda_2 = \mu_2 = 0.01, T = 2, \kappa = 0$. Given the parameter values, the “Merton line” as defined in Table 2 is constant. The positive correlation is measured by the parameter $\eta$ [cf. Muthuraman and Kuman (2006)].