In Search of Demiregular Tilings

Helmer Aslaksen
Department of Mathematics
National University of Singapore
Singapore 117543
Singapore
aslaksen@math.nus.edu.sg
www.math.nus.edu.sg/aslaksen/

Abstract

Many books on mathematics and art discuss a topic called demiregular tilings and claim that there are 14 such tilings. However, many of them give different lists of 14 tilings! In this paper we will compare the lists from some standard references that give a total of 18 such tilings. We will also show that unless we add further restrictions, there will in fact be infinitely many such tilings. The "fact" that there are 14 demiregular tilings has been repeated by many authors. The goal of this paper is to put an end to the concept of demiregular tilings.

1 Introduction

Several authors, including Ghyka (1946, [5]), Critchlow (1969, [2]), Williams (1979, [13]) and Lundy (2001, [9]), introduce a concept called demiregular tilings They define an edge-to-edge tiling by regular polygons to be demiregular if it has more than one type of vertex, and claim that there are only 14 such tilings. However, they all give different lists of demiregular tilings! In addition, some of them cite Steinhaus (1937, [11]) who says: "[demiregular tilings] are perhaps even more beautiful ... their number is unlimited. (Why?)". Even though he does not explain why there are infinitely many such tilings and just leaves it as an exercise, it is still amazing that some cite him while still claiming that there are only 14 such tilings!

2 *N*-uniform Tilings

In order to understand Steinhaus's statement that there are infinitely many such tilings, we need to have a precise definition of "type of vertex". We can either say that two vertices are the same if they look the same locally, or we can require that they must be the same globally in the sense that there is a symmetry of the tiling that maps one vertex to the other.

If we consider a tilings consisting of rows of regular triangles, as in the regular tiling of type (3^6), and rows of squares, as in the regular tiling of type (4^4), then there will be vertices of types (4^4) where rows of squares meet, type (3^6) where rows of triangles meet, and type $(3^3 4^2)$ where rows of triangles and squares meet. However, by varying the pattern of rows of triangles and squares, we can get tilings where vertices of the same local type do not belong to the same transitivity class of symmetries, and are therefore not of the same global type.

If we define demiregular to mean more than one local type, then we can easily construct infinitely many different tilings of type $(3^6; 3^3 4^2; 4^4)$, i.e., tilings where all vertices are of these three types, but where there are no mappings taking one tiling onto another. This explains why Steinhaus says there are infinitely many demiregular tilings.

We will define a tiling to be *n*-Archimedean if it has *n* different types of vertices, where the type of a vertex refers to the type and order of polygons surrounding the vertex. We will define a tiling to be *n*-uniform if the group of symmetries of the tiling divide the vertices into *n* transitivity classes.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	Total
Ghyka	D	D	D	P		D	P	D		D	D	P	D	D	P			D	14
Critchlow	P	P	P	P	P	P	P	P	P	P	P	P	P	P					14
Steinhaus		P						P								P	P		5

Table 1: Eighteen demiregular tilings pictured or described in the literature

It follows from the classification of Archimedean tilings that there are no n-Archimedean tilings for $n > 14$ ([10]), but for $n = 2, \ldots, 14$ it follows from the arguments above that there are infinitely many n-Archimedean tilings.

It is know that the number of n-uniform tilings is equal to $11, 20, 61, 151, 332, 673$ for $n = 1, \ldots, 6$. This was proved for $n = 2$ by Krötenheerdt ([8]) in 1969, for $n = 3$ by Chavey ([1]) in 1984 and for $n = 4, 5, 6$ by Galebach ([3]) in 2002 and 2003.

However, if we consider n-Archimedean, n-uniform tilings, known as Krötenheerdt tilings ([8]), we do get a finite family of tilings. The number of Krötenheerdt tilings is $11, 20, 39, 33, 15, 10, 7$ for $n = 1, \ldots, 7$ and 0 for $n \geq 8$.

This confusion was briefly alluded to by Grünbaum and Shephard in [6]. We hope that the discussion above will put an end to the concept of demiregular tilings. However, we would like to engage in some mathematical archaeology and look at the lists given by Critchlow ([2]), Ghyka ([5]) and Steinhaus ([11]).

3 Lists of Tilings

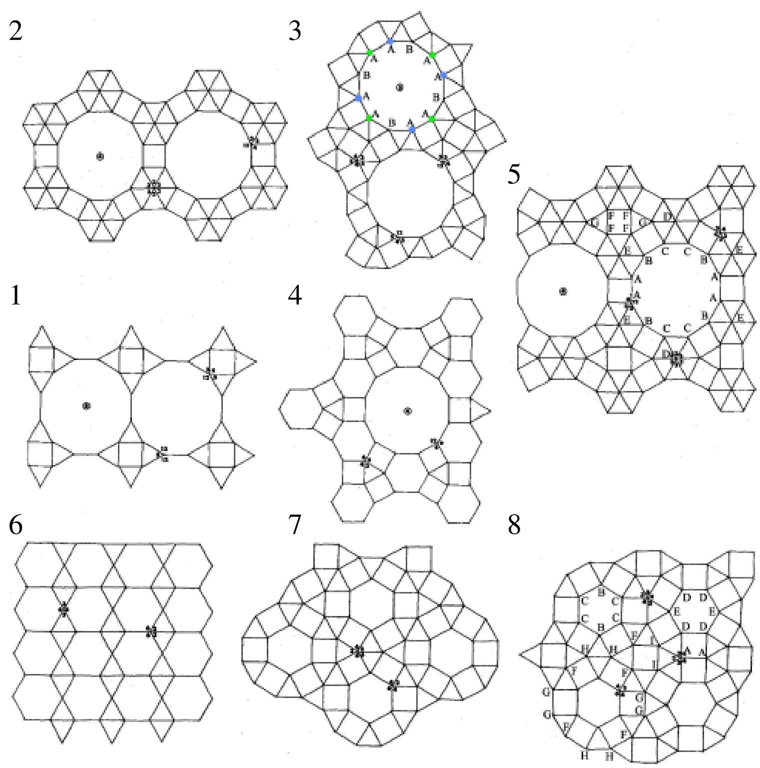

Figure 1: Seventeen demiregular tilings pictured in the literature, part 1

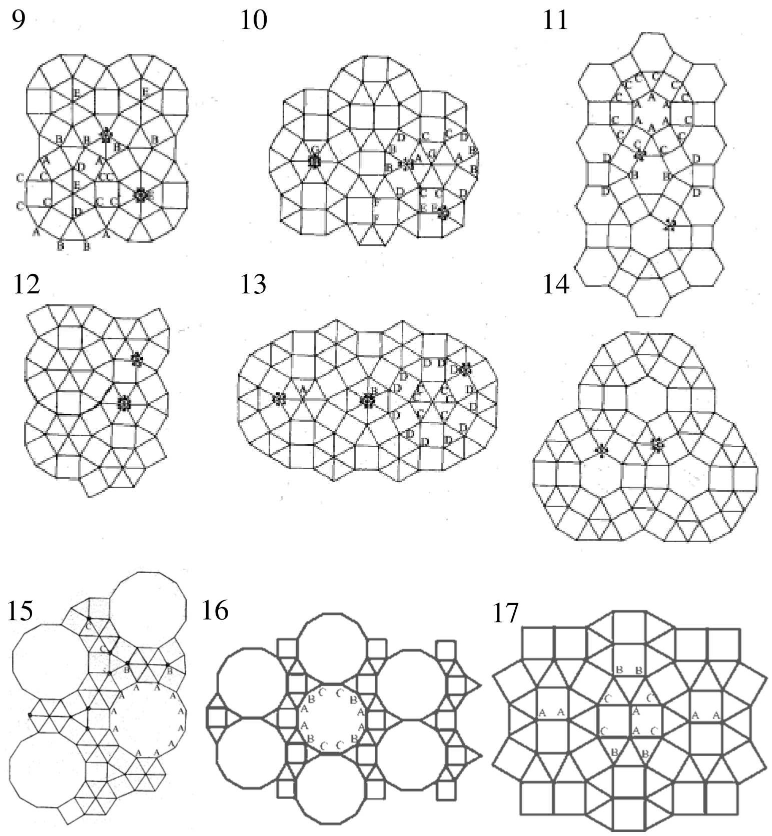

Figure 2: Seventeen demiregular tilings pictured in the literature, part 2

Table 1 lists the 18 tilings described in the three books. Ghyka lists 22 tilings, and claim that seven of them are “regular polymorph” (Archimedean), and 15 are demiregular. He obviously means that there are eight Archimedean and 14 demiregular. He only shows pictures of four of them, but he describes 10 others in a table. Nine of them are of the same type as the tilings in Critchlow that we have listed them above, but number 13 in his list does not correspond to any of Critchlow’s pictures and we we have included it as Tiling 18 in Table 1, although it is not in Figures 1 or 2. In Table 1 a P denotes a tiling that is pictured, while a D denotes a tiling that is just described in the list.

Tiling 15 from Ghyka is similar to Tiling 4. The only difference is that the hexagon is broken up into six triangles. Critchlow does not include Tiling 15, but instead adds Tilings 5 and 9 to Ghyka’s list.

In various printings of Lundy ([9]), there have been different lists. In one printing the list was identical to Critchlow except that Tilings 8 was replaced by a slightly altered version. In another printing, the heading talks about 14 demiregular tilings, the text says “more than twenty” and the picture shows 24. That’s about par for the “theory” of demiregular tilings!

In an early version of [12], the claim about 14 demiregular tilings was repeated, but that has been corrected, and the web page now gives a brief explanation of the problem.

References

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