

# Extending $\pi$ -systems to bases of root systems

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## Abstract

Let  $R$  be an indecomposable root system. It is well known that any root is part of a basis  $B$  of  $R$ . But when can you extend a set of two or more roots to a basis  $B$  of  $R$ ? A  $\pi$ -system is a linearly independent set of roots,  $C$ , such that if  $\alpha$  and  $\beta$  are in  $C$ , then  $\alpha - \beta$  is not a root. We will use results of Dynkin and Bourbaki to show that with two exceptions,  $A_3 \subset B_n$  and  $A_7 \subset E_8$ , an indecomposable  $\pi$ -system whose Dynkin diagram is a subdiagram of the Dynkin diagram of  $R$  can always be extended to a basis of  $R$ .

## 1 Introduction

Let  $R$  be an indecomposable root system in a Euclidean space  $V$ . A subset  $B$  of  $R$  is called a basis of  $R$  if  $B$  is a vector space basis of  $V$  and each root of  $R$  can be written as a linear combination of roots in  $B$  with integral coefficients that

are all nonnegative or all nonpositive. It is well known that any root is part of a basis  $B$  of  $R$ . But when can you extend a set of two or more roots to a basis  $B$  of  $R$ ? A  $\pi$ -system ([Dyn, Oni-Vin]) is a linearly independent set of roots,  $C$ , such that if  $\alpha$  and  $\beta$  are in  $C$ , then  $\alpha - \beta$  is not a root. (It is not assumed to be linearly independent in [Oni-Vin].) A subset of a basis will be a  $\pi$ -system, and a  $\pi$ -system will be a basis of a root subsystem. We can associate a Dynkin diagram to a  $\pi$ -system, and in order to extend to a basis of  $R$ , the Dynkin diagram of the  $\pi$ -system must be a subdiagram of the Dynkin diagram of  $R$ . By a subdiagram we mean a diagram obtained by deleting some nodes and their corresponding links. We will assume that the nodes corresponding to short roots are marked, and that the subdiagram preserves the marking. Hence two orthogonal short roots do not form a subdiagram of  $B_n$ , while two orthogonal long roots do. We will use results of Dynkin ([Dyn]) and Bourbaki ([Bou]) to show that with two exceptions,  $A_3 \subset B_n$  and  $A_7 \subset E_8$ , an indecomposable  $\pi$ -system whose Dynkin diagram is a subdiagram of the Dynkin diagram of  $R$  can always be extended to a basis of  $R$ . Our techniques can easily handle the decomposable case, too, but the results become more tedious to state, and we feel that it would distract from the main ideas of the paper.

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## 2 Results from Dynkin and Bourbaki

If  $C$  is a set of roots, then  $[C]$  denotes the set of all roots in  $R$  that are linear combinations of the roots in  $C$  with integer coefficients. Let  $\Pi$  and  $\Pi'$  be  $\pi$ -systems. We will say that  $\Pi'$  is obtained from  $\Pi$  by an *elementary transformation* if  $\Pi'$  is obtained by adjoining the lowest root to one of the indecomposable components of  $\Pi$  and then removing one root from that component.

We will first state three results due to Dynkin ([Dyn, Theo. 5.1, 5.2 and 5.3]).

**Proposition 1.** *Let  $C$  be a  $\pi$ -system in a root system,  $R$ . Then  $[C]$  is a root subsystem of  $R$  with basis  $C$ .*

**Proposition 2.** *Let  $C$  be a  $\pi$ -system in an indecomposable root system,  $R$ , of rank  $n$ . Then  $C$  can be extended to a  $\pi$ -system,  $D$ , with  $n$  elements.*

**Proposition 3.** *Let  $D$  be a  $\pi$ -system with  $n$  elements in an indecomposable root system,  $R$ , of rank  $n$ . Then  $D$  can be obtained by a sequence of elementary transformations of a basis of  $R$ .*

This shows that extending  $C$  will not always give us a basis of  $R$ , but that the extension can be obtained by a sequence of elementary transformations of a basis of  $R$ .

The next three propositions are due to Bourbaki ([Bou, Coro. to Prop. 4, Coro. 4 to Prop. 20 and Prop. 24 of Chap. VI, §1]).

**Proposition 4.** *Let  $V'$  be a subspace of  $V$  and let  $V''$  be the subspace spanned by  $R' = R \cap V'$ . Then  $R'$  is a root system in  $V''$ .*

Notice that  $V'$  may intersect  $R$  so that  $\dim V'' < \dim V'$ .

**Proposition 5.** *If  $C$  a subset of a basis  $B$  of  $R$  and  $V'$  the subspace of  $V$  spanned by  $C$ , then  $C$  is a basis of the root system  $R' = R \cap V'$ .*

Notice that  $[C]$  uses integer coefficients, while  $R' = R \cap V'$  includes rational coefficients. This proposition says that if  $C$  is a subset of a basis, then  $[C] = R'$ .

**Proposition 6.** *Let  $C$  be a basis of  $R' = R \cap V'$ , where  $V'$  is a subspace of  $V$ . Then  $C$  can be extended to a basis  $B$  of  $R$  and  $R'$  is the set of roots in  $R$  that are linear combinations of elements of  $C$ .*

This says that if  $[C] = R'$ , then  $C$  can be extended to a basis of  $R$ .

Then next result is a simple combination of Propositions 1 to 6.

**Theorem 7.** *Let  $C$  be a  $\pi$ -system in  $R$  and let  $V'$  be the subspace of  $V$  spanned by  $C$ . Then  $[C]$  is a root subsystem of  $R' = R \cap V'$ ,  $C$  can be obtained from a basis of  $R'$  by a sequence of elementary transformations, and  $C$  can be extended to a basis  $B$  of  $R$  if and only if  $C$  is a basis of  $R'$ , i.e.,  $[C] = R'$ .*

*Proof.* It follows from Proposition 1 that  $[C]$  is a root subsystem of  $R' = R \cap V'$  with basis  $C$ . In general  $[C] \neq R'$ , so we can only use Proposition 3 to say that  $C$  is obtained from a basis of  $R'$  by a sequence of elementary transformations. If  $[C] = R'$ , then it follows from Proposition 6 that  $C$  can be extended to a basis of  $R$ . The converse follows from Proposition 5.  $\square$

### 3 Extension results

It follows from Theorem 7 that  $C$  will extend to a basis of  $R$  unless there is a root system  $R'$  such that

$$[C] \subset R' \subset R,$$

where  $\text{rank}[C] = \text{rank } R' < \text{rank } R$ ,  $[C] \neq R'$  and  $C$  can be obtained from a basis of  $R'$  by a sequence of elementary transformations. We are for simplicity assuming that  $[C]$  is indecomposable, and hence  $R'$  is also indecomposable. The next lemma is proved by inspection of the extended Dynkin diagrams.

**Lemma 8.** *The only indecomposable roots systems  $[C] \subset R'$  where  $C$  can be obtained by a sequence of elementary transformations of a basis of  $R'$  are listed below.*

$$A_3 \subset B_3, \quad D_n \subset B_n \text{ for } n \geq 4, \quad A_7 \subset E_7, \\ A_8 \subset E_8, \quad D_8 \subset E_8, \quad B_4 \subset F_4, \quad D_4 \subset F_4, \quad A_2 \subset G_2.$$

$D_4 \subset F_4$  is obtained by using two elementary transformations, while all the other only require one.

Since our pairs must sit inside a root system,  $R$ , of higher rank, and the diagram of  $C$  must be a subdiagram of the diagram of  $R$ , there are only two cases that satisfies our conditions,  $A_3 \subset B_3 \subset B_n$  and  $A_7 \subset E_7 \subset E_8$ .

The only way  $C$  can fail to extend is if  $C$  is obtained from an  $R'$  diagram by an elementary transformation so that  $C$  includes the lowest root of  $R'$ . We will see in the next theorem that in that case we can use the fact that the lowest root is a linear combination of all the roots in the basis to show that  $[C] \neq R'$ .

**Theorem 9.** *An indecomposable  $\pi$ -system,  $C$ , in  $R$  can be extended to a basis  $B$  of  $R$ , unless  $[C] \subset R$  is on of the following two cases.*

1.  $A_3 \subset B_n$  for  $n \geq 4$  and  $C = \{r_1, r_2, r_3\}$  has Dynkin diagram

$$\begin{array}{c} r_1 \quad r_2 \quad r_3 \\ \circ \text{---} \circ \text{---} \circ \end{array}$$

and  $(r_1 + 2r_2 + r_3)/2$  is a root in  $B_n$ .

2.  $A_7 \subset E_8$  and  $C = \{r_1, \dots, r_7\}$  has Dynkin diagram

$$\begin{array}{c} r_1 \quad r_2 \quad r_3 \quad r_4 \quad r_5 \quad r_6 \quad r_7 \\ \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \text{---} \circ \end{array}$$

and

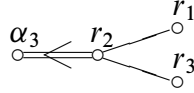
$$(r_1 + 2r_2 + 3r_3 + 4r_4 + 3r_5 + 2r_6 + r_7)/2$$

is a root in  $E_8$ .

*Proof.* We cannot use the pairs in Theorem 8 involving  $E_8$ ,  $F_4$  and  $G_2$ , because we cannot fit them into any bigger root systems. We also cannot fit a  $D_k$  diagram inside  $B_n$ , so we are left with  $A_3 \subset B_3 \subset B_n$  and  $A_7 \subset E_7 \subset E_8$ .

We will use the bases for the root systems listed in [Bou] and denote the lowest root by  $\alpha_0$ .

For  $A_3 \subset B_n$ , either  $A_3$  is the Y-branch at the end of the extended diagram of  $B_3$  or  $A_3$  is part of the diagram of  $B_n$ . In the first case,  $r_2 = \alpha_2$  and either  $r_1$  or  $r_3$  must be the lowest root  $-(\alpha_1 + 2\alpha_2 + 2\alpha_3)$  and the other must be  $\alpha_1$ . In either case, we can recover the deleted root from the lowest root, giving us  $(r_1 + 2r_2 + r_3)/2 = -\alpha_3 = -e_3$ , which is a root of  $B_n$ , but not in  $[C]$ . So  $V'$  is the span of  $\{e_1, e_2, e_3\}$ , and  $R' = B_3$  while  $[C] = A_3$ . Hence  $C$  cannot be extended to a basis of  $B_n$  by Theorem 7.



However, if  $C = \{\alpha_1, \alpha_2, \alpha_3\}$  corresponds to an  $A_3$  that is a subdiagram of the diagram of  $B_n$ , then  $(r_1 + 2r_2 + r_3)/2$  is not a root of  $B_n$ . In this case  $V'$  does not contain any short roots, so  $R' = [C]$  and  $C$  can be extended to a basis of  $B_n$ .

For  $A_7 \subset E_8$ , the  $A_7$  is either part of the diagram of  $E_8$  or is part of the extended diagram of  $E_7$ . In the  $E_8$  case we have

$$C = \{\alpha_1, \dots, \alpha_8\} - \{\alpha_2\},$$

and in the extended  $E_7$  case we have

$$C = \{\alpha_0, \alpha_1, \dots, \alpha_7\} - \{\alpha_2\},$$

where  $\alpha_0 = -2\alpha_1 - 2\alpha_2 - 3\alpha_3 - 4\alpha_4 - 3\alpha_5 - 2\alpha_6 - \alpha_7$  is the lowest root of  $E_7$ .

In the extended  $E_7$  case we don't know whether  $r_1$  or  $r_7$  is the extended root, but again we can recover the deleted root from the lowest root, giving us

$$(r_1 + 2r_2 + 3r_3 + 4r_4 + 3r_5 + 2r_6 + r_7)/2 = -\alpha_2.$$

In the  $E_8$  case this expression will not be a root. It follows that in the extended  $E_7$  case  $C$  can be extended to a basis of  $E_8$ , while in the  $E_8$  case  $C$  can only be extended to a basis of  $A_8$ .  $\square$

Notice that  $C = \{\alpha_1, \alpha_2, \alpha_0\} = \{e_1 - e_2, e_2 - e_3, -e_1 - e_2\}$ , where  $\alpha_0$  is the lowest short root in  $C_n$  is *not* a  $\pi$ -system in  $C_n$ , since  $e_1 - e_2 - (-e_1 - e_2) = 2e_1$  is a root in  $C_n$ . We initially considered linearly independent sets of roots with nonpositive inner products, i.e., linearly independent, admissible (nonpositive inner product) sets of roots instead of  $\pi$ -systems.  $\pi$ -systems are always admissible, and for simply laced root systems, indecomposable  $\pi$ -systems are admissible, since the only way the difference between two roots can have the same length as the two roots is if the angle between them is  $\pi/3$ . However,  $C \subset C_n$  shows that this is false for multiply laced root systems. In particular, Exercise 34 on

page 177 of [Oni-Vin] appears to be incorrect. (They do not require  $\pi$ -systems to be linearly independent, but that does not make any difference.)

Notice that  $D_k \subset B_k$  is listed in Lemma 8, while  $D_k \subset C_k$  is not. There are two standard ways of constructing equal rank inclusions of root systems. One is to use elementary transformations and is used by Borel and de Siebenthal ([Bor-Sib]). The other is to consider the set of short and long roots in multiply laced root systems.  $D_k$  forms the long roots in  $B_k$  and the short roots in  $C_k$ , but only the  $B_k$  inclusion can be obtained by an elementary transformation.

Notice also that we only talk about root systems, and not about Lie subalgebras. Unless we know something about the Cartan subalgebras, inclusions of root systems and inclusions of Lie algebras will not necessarily correspond. The fact that  $D_k \subset C_k$  does not imply that  $\mathfrak{so}(2n) \subset \mathfrak{sp}(2n)$ .

## References

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