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Generic 2 × 2 Matrices With Involution

Helmer Aslaksen and Eng-Chye Tan

Procesi has given a linear basis for the ring of m generic 2×2 matrices. We do the same for the ring of m generic 2×2 matrices with transpose involution.

1 Introduction

Let k be a field of characteristic 0 and let $M_2 = M_2(k)$ be the set of 2×2 matrices over k. Consider an involution $*$ on M_2 of the first kind. It follows from [14] that it is sufficient to study the symplectic involution and the transpose involution. The case of symplectic involution is essentially treated in Section 7 of [12], so we will only consider the transpose involution. We will use the following notation: M_2^+ is the set of symmetric matrices, M_2^- the skew-symmetric matrices, M_2^0 the matrices of trace 0, M_2^{0+} the symmetric traceless matrices and k the scalar matrices. Hence

$$M_2 = M_2^+ \oplus M_2^- = k \oplus M_2^0 = k \oplus M_2^{0+} \oplus M_2^-.$$

For $X_i \in M_2$ we define $s_i = X_i + X_i{}^t$, $S_i = s_i - \frac{1}{2} \operatorname{tr} X_i$ and $A_i = X_i - X_i{}^t$.

Let $F = k\{x_1, \dots, x_m, y_1, \dots, y_m\}$ be the free algebra in $2m$ variables. The ring R of m generic 2×2 matrices with transpose involution is the ring of polynomial functions on M_2^m induced from

F by letting $f(x_1, \dots, x_m, y_1, \dots, y_m) \in F$ correspond to the function $\overline{f}: M_2^m = (M_2^+)^m \oplus (M_2^-)^m \to M_2$ defined by

$$\overline{f}(s_1, \dots, s_m, A_1, \dots, A_m) = f(s_1, \dots, s_m, A_1, \dots, A_m),$$

where $s_i \in M_2^+, A_i \in M_2^-$.

We can think of R as the set of noncommutative polynomial in the m generic matrices X_i and their transposes or as noncommutative polynomials in the s_i and A_i. We see that elements in R are $O(2)$ equivariant, i.e.,

$$\bar{f}(gs_1g^{-1}, \dots, gs_mg^{-1}, gA_1g^{-1}, \dots, gA_mg^{-1}) = g\bar{f}(s_1, \dots, s_m, A_1, \dots, A_m)g^{-1},$$

for $g \in O(2)$.

Define S to be the $O(2)$-equivariant polynomial functions from M_2^m to M_2. These are the matrix concomitants. The scalar-valued equivariant functions are the invariants, and we denote them by T. It is well-known [10] that T is the center of S provided $m \geq 2$. If we set $t_i = \operatorname{tr} X_i$, then it follows from [11] that

$$S = R[t_1, \dots, t_m].$$

We will also consider $Z = R \cap T$, the ring of central noncommutative polynomials.

We can simplify our problem by factoring out the trace, so we define $\tilde{R}$ to be the set of $O(2)$-equivariant maps from $(M_2^0)^m$ to M_2. It is clear that $S = \tilde{R}[t_1, \dots, t_m]$. We also let $\tilde{R}^0$, $\tilde{R}^{0+}$ and $\tilde{R}^-$ be the $O(2)$-equivariant maps from $(M_2^0)^m$ to M_2^0, M_2^{0+} and M_2^- respectively. Then

$$\tilde{R} = \tilde{R}^0 \oplus T^0 = \tilde{R}^{0+} \oplus \tilde{R}^- \oplus T^0.$$

There is a natural $GL(m) \times GL(m)$ action on S, R and T. By classical invariant theory we can describe the $GL(m) \times GL(m)$ structure of T^0. Then we consider the $GL(m) \times GL(m)$-equivariant maps

$$\phi_1(f(S_1, \dots, S_m, A_1, \dots, A_m)) = \operatorname{tr}(f(S_1, \dots, S_m, A_1, \dots, A_m)S_{m+1}),$$
$$\phi_2(f(S_1, \dots, S_m, A_1, \dots, A_m)) = \operatorname{tr}(f(S_1, \dots, S_m, A_1, \dots, A_m)A_{m+1}).$$

Both maps take concomitants in m variables to invariants in $m+1$ variables, and are injective on $\tilde{R}^{0+}$ and $\tilde{R}^-$ respectively. By the theory of standard Young tableaux, we then get a description of the $GL(m) \times GL(m)$ structure of $\tilde{R}$ (see Theorem 1). This result has also been obtained by Drensky and Giambruno, using the method of cocharacters [6].

To study R, we investigate the $GL(m) \times GL(m)$-invariant ideals in $\tilde{R}$. In particular, we show that the ideal $[[\tilde{R}, \tilde{R}], \tilde{R}]$ sits in $[R, R]$ in such a way that it is a free $R/[R, R]$ module, thus giving a description of R.

Finally we give an outline of the paper. We will treat T^0 in Section 2 and use the results to give a description of of $\tilde{R}$ in Section 3. Section 4 gives the multiplicative structure of $\tilde{R}$ and the subring of $GL(m) \times GL(m)$-highest weight vectors in $\tilde{R}$. Section 5 describes the $GL(m) \times GL(m)$-invariant ideals in $\tilde{R}$ in a way similar to DeConcini, Eisenbud and Procesi [4]. In Section 6 we give a basis for R and in Section 7 we determine the central noncommutative polynomials Z. We follow Procesi [12] quite closely, but we do not discuss Poincaré series, since this has already been carried over to the case of matrices with involution by Berele [3] and Drensky and Giambruno [6].

2 Invariants

Let T^0 be the ring of $O(2)$-invariants on $(M_2^0)^m$. Procesi [12] considered the conjugation action of $GL(2)$ on M_2^0 and observed that it is equivalent to the natural action of $SO(3)$ on k^3. We will instead consider the restriction of this action to $O(2)$. Write

$$O(2) = SO(2) \cup \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} SO(2)$$

and for $w \in \mathbb{C}$ define

$$g(w) = \begin{pmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{pmatrix} \in SO(2).$$

Then

$$g(w) \begin{pmatrix} x & y \\ y & -x \end{pmatrix} g(w)^{-1} = g(2w) \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}x & y\\ y & -x\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}x & -y\\ -y & -x\end{pmatrix}.$$

It follows that conjugation by $O(2)$ on M_2^{0+} is

$$\phi_+(g)\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}1 & \\ & \det g\end{pmatrix}\lambda(g)\begin{pmatrix}x\\ y\end{pmatrix}, \qquad (1)$$

where λ is the natural representation of $SO(2)$. Since

$$g(w)\begin{pmatrix}0 & z\\ -z & 0\end{pmatrix}g(w)^{-1} = \begin{pmatrix}0 & z\\ -z & 0\end{pmatrix}$$

and

$$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0 & z\\ -z & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = \begin{pmatrix}0 & -z\\ z & 0\end{pmatrix},$$

it follows that conjugation by $O(2)$ on M_2^- is simply multiplication by

$$\phi_-(g) = \det g. \qquad (2)$$

Let $O(2)$ act on a vector space V and let $\mathcal{P}(V)$ be the polynomial functions on V. Define

$$\mathcal{P}(V)_+^{O(2)} = \{f \in \mathcal{P}(V) \mid g\cdot f = f,\ g \in O(2)\},$$
$$\mathcal{P}(V)_-^{O(2)} = \{f \in \mathcal{P}(V) \mid g\cdot f = (\det g)f,\ g \in O(2)\}.$$

Then

$$T^0 = \mathcal{P}\Big((M_2^0)^m\Big)^{O(2)} = \Big(\mathcal{P}\Big((M_2^{0+})^m\Big)_+^{O(2)} \otimes \mathcal{P}\Big((M_2^-)^m\Big)_+^{O(2)}\Big)$$
$$\oplus\Big(\mathcal{P}\Big((M_2^{0+})^m\Big)_-^{O(2)} \otimes \mathcal{P}\Big((M_2^-)^m\Big)_-^{O(2)}\Big).$$

We will now use classical invariant theory to describe T^0. Given a polynomial mapping $f(S_1,\dots,S_m,A_1,\dots,A_m)$ where $S_i \in M^{0+}$ and $A_i \in M^-$, we can perform linear substitutions of the variables S_i and A_i separately. This gives us an action of $GL(m)\times GL(m)$ on R, S and T that commutes with the conjugation action.

First let us recall some standard notations for representations of $GL(m)$. Every irreducible representation L_σ of $GL(m)$ is indexed by a Young diagram $\sigma = m^{\alpha_m}(m-1)^{\alpha_{m-1}}\cdots 2^{\alpha_2}1^{\alpha_1}$ with α_p rows of length p for $1 \le p \le m$. This diagram corresponds to the highest weight $\alpha_1\omega_1 + \alpha_2\omega_2 + \cdots + \alpha_m\omega_m$ where ω_i is the fundamental weight of the ith exterior power of the fundamental m-dimensional representation of $GL(m)$.

Let us first consider $\mathcal{P}\Big((M_2^{0+})^m\Big)^{SO(2)}$. Since this action clearly has the same invariants as the natural action, it follows from [5] that the standard tableaux of the form 2^a1^{2b} is a basis. The correspondence between tableaux and invariants is as follows. First we associate invariants to certain tableaux. Write

$$S_i = \begin{pmatrix}x_i & y_i\\ y_i & -x_i\end{pmatrix}$$

and set

$$(u_i, u_j) = \begin{pmatrix}x_i & x_j\\ y_i & y_j\end{pmatrix}.$$

We observe that

$$[S_i, S_j] = 2(x_iy_j - x_jy_i)J = \operatorname{pf}[S_i, S_j]J, \qquad (3)$$

where

$$J = \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}.$$

We then define

$$\begin{array}{|c|c|}\hline i & j\\ \hline\end{array} = \det(u_i, u_j) = x_iy_j - x_jy_i = \tfrac{1}{2}\operatorname{pf}[S_i, S_j],$$
$$\begin{array}{|c|}\hline i\\ \hline j\\ \hline\end{array} = \operatorname{tr}(S_iS_j) = 2(x_ix_j + y_iy_j), \qquad (4)$$
$$\begin{array}{|c|c|}\hline i & j\\ \hline h & k\\ \hline\end{array} = \det\begin{pmatrix}\begin{array}{|c|}\hline i\\ \hline h\\ \hline\end{array} & \begin{array}{|c|}\hline i\\ \hline k\\ \hline\end{array}\\ \begin{array}{|c|}\hline j\\ \hline h\\ \hline\end{array} & \begin{array}{|c|}\hline j\\ \hline k\\ \hline\end{array}\end{pmatrix}.$$

We associate to a tableau of the form $2^{2a}1^{2b}$ the product of a 2^2-tableaux and b 1^2-tableaux as defined above. We associate to a tableau of the form $2^{2a+1}1^{2b}$ the product of the 2-tableau corresponding to its top row and the $2^{2a}1^{2b}$-tableau corresponding to its bottom part.

It follows that $\mathcal{P}\left((M_2^{0+})^m\right)_+^{O(2)}$ is generated by the standard tableaux of the form $2^{2a}1^{2b}$, while $\mathcal{P}\left((M_2^{0+})^m\right)_-^{O(2)}$ is generated by the standard tableaux of the form $2^{2a+1}1^{2b}$.

Similarly, we write

$$A_i = \begin{pmatrix} 0 & z_i \\ -z_i & 0 \end{pmatrix},$$

and set

$$(v_i) = (z_i).$$

We define

$$\begin{aligned} \boxed{i} &= \det(v_i) = \operatorname{pf} A_i, \\ \begin{array}{|c|}\hline i \\ \hline j \\ \hline \end{array} &= \operatorname{tr}(A_i A_j) = -2 z_i z_j. \end{aligned} \tag{5}$$

Again, we associate to a tableau of the form 1^{2b} the product of b 1^2 tableaux, and to a tableau of the form 1^{2b+1} the product of the 1 tableau at the top and the 1^{2b} tableau corresponding to the bottom. It then follows that $\mathcal{P}\left((M_2^{-})^m\right)_+^{O(2)}$ is spanned by the standard tableaux of the form 1^{2b}, while $\mathcal{P}\left((M_2^{-})^m\right)_-^{O(2)}$ is spanned by the standard tableaux of the form 1^{2b+1}.

To sum up, we get the following decomposition of T^0 as a $GL(m) \times GL(m)$ representation

$$T^0 = \bigoplus_{a,b,c} (L_{2^{2a}1^{2b}} \otimes L_{1^{2c}}) \oplus \bigoplus_{a,b,c} (L_{2^{2a+1}1^{2b}} \otimes L_{1^{2c+1}}).$$

We notice that this argument gives a smaller set of generators for T^0 than the one in [15], and also gives an alternative proof of some of the results in [1].

3 Structure of $\widetilde{R}$

One of the crucial steps in [12] is the map (originally due to Kostant [8])

$$\phi(f(X_1, \dots, X_m)) = \operatorname{tr}(f(X_1, \dots, X_m) X_{m+1}),$$

that takes a concomitant in m variables to to an invariant in $m+1$ variables. In our case, we must instead consider the two $GL(m) \times GL(m)$-equivariant maps

$$\begin{aligned} \phi_1\left(f(S_1, \dots, S_m, A_1, \dots, A_m)\right) &= \operatorname{tr}\left(f(S_1, \dots, S_m, A_1, \dots, A_m) S_{m+1}\right), \\ \phi_2\left(f(S_1, \dots, S_m, A_1, \dots, A_m)\right) &= \operatorname{tr}\left(f(S_1, \dots, S_m, A_1, \dots, A_m) A_{m+1}\right), \end{aligned}$$

that take $f \in \widetilde{R}_m^0$ to T_{m+1}^0 (we will temporarily use subscripts to indicate the number of generic matrices).

It follows from the nondegeneracy of the trace and the fact that X_{m+1} is a generic matrix that ϕ is injective, but it is easy to see that ϕ_1 and ϕ_2 are not. Suppose that $f \in \widetilde{R}^-$. Then it follows from (8) that $\phi_2(f) = 0$, so $\phi\big|_{\widetilde{R}^{0+}} = \phi_1\big|_{\widetilde{R}^{0+}}$ and similarly, $\phi\big|_{\widetilde{R}^-} = \phi_2\big|_{\widetilde{R}^-}$. Since ϕ is injective, we see that $\ker \phi_1 = \widetilde{R}^-$ and $\ker \phi_2 = \widetilde{R}^{0+}$. It follows that

$$\phi(\widetilde{R}^0) = \phi_1(\widetilde{R}^{0+}) \oplus \phi_2(\widetilde{R}^-).$$

It remains to deduce the structure of $\widetilde{R}_m^{0+}$ and $\widetilde{R}_m^-$ from T_{m+1}^0. To understand the image of $\widetilde{R}_m^0$ in T_{m+1}^0, it is enough to determine the elements of each $L_\sigma \otimes L_\rho$ in T_{m+1}^0 that are linear in the last variable S_{m+1} or A_{m+1}. Recall that L_σ has a basis indexed by the standard tableaux of shape σ and filled with numbers $1, 2, \dots, m+1$ with possible repetitions in the columns. In the correspondence between tableaux and functions, the multiplicity of each index i in the tableaux σ (respectively ρ) is the degree in the variable S_i (respectively A_i). Thus we look for tableaux in T_{m+1}^0 that are linear in S_{m+1} (respectively A_{m+1}). This means that the $GL(m) \times GL(m)$ module structure of $\widetilde{R}^0$ can be obtained from T_{m+1}^0 by removing a corner from either of the diagrams σ or ρ in $L_\sigma \otimes L_\rho \in T_{m+1}^0$.

Using the fact that $S_i A_j = -A_j S_i$, we get

$$\begin{aligned} \boxed{i\,|\,j} \otimes \boxed{k} &= \frac{1}{2} \operatorname{pf}[S_i, S_j] \operatorname{pf} A_k = -\frac{1}{4} \operatorname{tr}([S_i, S_j] A_k) \\ &= -\frac{1}{4} \operatorname{tr}(S_j A_k S_i - S_j S_i A_k) = \frac{1}{2} \operatorname{tr}\left((S_i A_k) S_j\right). \end{aligned}$$

Ignoring the constants, we define

$$\begin{aligned} \boxed{i\,|\,j} \otimes 1 &= [S_i, S_j], \\ \boxed{i} \otimes \boxed{j} &= S_i A_j. \end{aligned} \tag{6}$$

Based on

$$\begin{array}{|c|}\hline i\\\hline j\\\hline\end{array} = \operatorname{tr}(S_iS_j),$$

we define

$$\boxed{i} = S_i,$$

and similarly for A_i.

From this we deduce that

$$\begin{aligned}\tilde{R}^{0+} &= (L_{2^{2a}1^{2b+1}} \otimes L_{1^{2c}}) \oplus (L_{2^{2a+1}1^{2b+1}} \otimes L_{1^{2c}})\\ &\quad \oplus (L_{2^{2a}1^{2b+1}} \otimes L_{1^{2c+1}}) \oplus (L_{2^{2a+1}1^{2b+1}} \otimes L_{1^{2c+1}}),\\ \tilde{R}^{-} &= (L_{2^{2a}1^{2b}} \otimes L_{1^{2c+1}}) \oplus (L_{2^{2a+1}1^{2b}} \otimes L_{1^{2c}}).\end{aligned}$$

Since

$$T^0 = \bigoplus_{a,b,c}(L_{2^{2a}1^{2b}} \otimes L_{1^{2c}}) \oplus \bigoplus_{a,b,c}(L_{2^{2a+1}1^{2b}} \otimes L_{1^{2c+1}})$$

and

$$\tilde{R} = \tilde{R}^{0+} \oplus \tilde{R}^{-} \oplus T^0,$$

we have proved the following theorem.

Theorem 1 *$\tilde{R}$ is generated by the S_i and the A_i. It has a basis indexed by the tensor products of the standard tableaux of the form $L_{2^a1^b}$ and L_{1^c} filled with the indices $1, 2, \ldots, m$. As a $GL(m) \times GL(m)$ module we have*

$$\tilde{R} = \bigoplus_{a,b,c}(L_{2^a1^b} \otimes L_{1^c}).$$

4 Multiplicative structure of $\tilde{R}$

We will now study the multiplicative structure of $\tilde{R}$. From the previous section we know that a basis for $L_{2^a1^b} \otimes L_{1^c} \in \tilde{R}$ is given by the set of tensor products of standard tableaux of shape 2^a1^b and 1^c. These tableaux correspond to noncommutative polynomials that are products of invariants in T^0 and matrices corresponding to tableaux of shape $1 \otimes 0$, $0 \otimes 1$, $1 \otimes 1$ or $2 \otimes 0$. (We let 0 denote the trivial representation or the null diagram.) Since the invariants are central, we know the multiplicative structure of $\tilde{R}$ once we know the multiplication table for the standard tableaux of the form $\boxed{i} \otimes 1$, $1 \otimes \boxed{i}$, $\boxed{i} \otimes \boxed{j}$ and $\begin{array}{|c|c|}\hline i & j\\\hline\end{array} \otimes 1$.

Let us list some formulas and properties of matrices that we will need for these computations. For $X, Y, Z \in M_2^0$ we have

$$\begin{aligned} XY = \tfrac{1}{2}\operatorname{tr} XY + \tfrac{1}{2}[X,Y], \quad X^2 = \tfrac{1}{2}\operatorname{tr} X^2,\\ 2XYZ = \operatorname{tr}(YZ)X - \operatorname{tr}(XZ)Y + \operatorname{tr}(XY)Z - \operatorname{tr}(XYZ).\end{aligned} \tag{7}$$

If $S_p \in M_2^{0+}$ and $A_q \in M_2^-$, then

$$\begin{aligned} A_iA_j = A_jA_i, \quad A_iS_j = -S_jA_i, \quad \operatorname{tr}(S_iA_j) = 0,\\ [S_i, S_j]S_k = -S_k[S_i, S_j], \quad A_i[S_j, S_k] = [S_j, S_k]A_i\end{aligned} \tag{8}$$

The last two relations in (8) follow from $[M_2^{0+}, M_2^{0+}] \subset M_2^-$ ((3)).

As an example, we will compute the following.

$$\begin{aligned} &(\boxed{i} \otimes \boxed{j}) \cdot (\begin{array}{|c|c|}\hline k & l\\\hline\end{array} \otimes 1)\\ &= S_iA_j[S_k, S_l]\\ &= S_i[S_k, S_l]A_j\\ &= S_iS_kS_lA_j - S_iS_lS_kA_j\\ &= (\tfrac{1}{2}\operatorname{tr}(S_iS_k) + \tfrac{1}{2}[S_i, S_k])S_lA_j - (\tfrac{1}{2}\operatorname{tr}(S_iS_l) + \tfrac{1}{2}[S_i, S_l])S_kA_j\\ &= \tfrac{1}{2}\operatorname{tr}(S_iS_k)S_lA_j - \tfrac{1}{2}\operatorname{tr}(S_iS_l)S_kA_j + \tfrac{1}{2}([S_i, S_k]S_l - [S_i, S_l]S_k)A_j.\end{aligned}$$

But

$$\begin{aligned} &[S_i, S_k], S_l - [S_i, S_l]S_k\\ &= \tfrac{1}{2}([[S_i, S_k], S_l] - [[S_i, S_l], S_k])\\ &= \tfrac{1}{2}(-[[S_k, S_l], S_i] - [[S_l, S_i], S_k] - [[S_i, S_l], S_k])\\ &= -\tfrac{1}{2}[[S_k, S_l], S_i] = S_i[S_k, S_l],\end{aligned}$$

so

$$\begin{aligned} &(\boxed{i} \otimes \boxed{j}) \cdot (\begin{array}{|c|c|}\hline k & l\\\hline\end{array} \otimes 1)\\ &= \operatorname{tr}(S_iS_k)S_lA_j - \operatorname{tr}(S_iS_l)S_kA_j\\ &= \left(\begin{array}{|c|}\hline i\\\hline k\\\hline\end{array} \otimes 1\right) \boxed{l} \otimes \boxed{j} - \left(\begin{array}{|c|}\hline i\\\hline l\\\hline\end{array} \otimes 1\right) \boxed{k} \otimes \boxed{j}.\end{aligned}$$

We can similarly compute the other entries in the multiplication table in Table 1.

We will also describe the subring B of $GL(m) \times GL(m)$ highest weight vectors in $\widetilde{R}$. We have $B = \widetilde{R}^{U^+(m)\times U^+(m)}$, where $U^+(m) \subset GL(m)$ is the subgroup of upper triangular matrices. The canonical tableau K_σ of shape σ is the unique tableau with indices i in the ith column, and is the highest weight vector in L_σ. Hence B is generated by the tensor products of the canonical tableaux. We will write $K_{\sigma,\rho} = K_\sigma \otimes K_\rho$. Let F be the center of B. It is easy to see that F is a polynomial ring in $K_{1^2,0}$, $K_{0,1^2}$ and $K_{2,1}$. (We can express $K_{2^2,0}$ in terms of $K_{1^2,0}$.) We also know that B is generated as an F algebra by $K_{0,1}$, $K_{1,0}$ and $K_{2,0}$. In order to generate B as an F module, we must also include the products $K_{1,1} = K_{1,0}K_{0,1}$, $K_{21,0} = K_{2,0}K_{1,0}$ and $K_{21,1} = K_{21,0}K_{0,1} = K_{2,0}K_{1,1}$. The rules from Table 1 give us the multiplication table in Table 2.

A generalized quaternion algebra over a ring D is the four-dimensional D space with basis $1, i, j, k$, where $i^2 = a$, $j^2 = b$ and $ij = -ji = k$. We see that $\{1, K_{1,0}, K_{2,0}, K_{21,0}\}$ generate a generalized quaternion algebra over the center F. Because of the relations

$$\begin{aligned} K_{2,1}K_{1,0} + K_{21,1} &= 0, \\ K_{2,1}K_{2,0} + K_{2^2,0}K_{0,1} &= 0, \\ K_{2,1}K_{21,0} + K_{2^2,0}K_{1,1} &= 0, \end{aligned}$$

we see that $\{1, K_{21,1}, K_{0,1}, K_{1,1}\}$ "almost" generate a generalized quaternion algebra over F. (Note that $K_{21,1}K_{0,1} = -K_{2,1}K_{1,1}$ instead of $K_{1,1}$.) But we get generators for quaternion algebras by considering the sets $\{1, K_{1,0}, K_{0,1}, K_{1,1}\}$, $\{1, K_{2,0}, K_{1,1}, K_{21,1}\}$ and $\{1, K_{21,0}, K_{0,1}, K_{21,1}\}$. If we let F' be the field of fractions of F, then $B' = F'B$, the quotient ring of B, is a generalized quaternion algebra over F'.

5 Invariant ideals in $\widetilde{R}$

In this section we will determine the $GL(m) \times GL(m)$-invariant ideals in $\widetilde{R}$. We write $L_{(\sigma,\rho)} = L_\sigma \otimes L_\rho$ and define

$$\begin{aligned} M_1 &= L_{(1,0)} \cdot L_{(\sigma,\rho)}, & M_2 &= L_{(\sigma,\rho)} \cdot L_{(1,0)}, & M_3 &= [L_{(1,0)}, L_{(\sigma,\rho)}], \\ M_4 &= L_{(0,1)} \cdot L_{(\sigma,\rho)}, & M_5 &= L_{(\sigma,\rho)} \cdot L_{(0,1)}, & M_6 &= [L_{(0,1)}, L_{(\sigma,\rho)}]. \end{aligned}$$

Let

$$\begin{aligned} j_i : L_{(1,0)} \otimes L_{(\sigma,\rho)} &\longrightarrow M_i, \quad i = 1,2,3, \\ j_i : L_{(0,1)} \otimes L_{(\sigma,\rho)} &\longrightarrow M_i, \quad i = 4,5,6. \end{aligned}$$

be the projections defined by

$$j_i(x \otimes y) = \begin{cases} x \cdot y & \text{if } i = 1,4, \\ y \cdot x & \text{if } i = 2,5, \\ [x,y] & \text{if } i = 3,6, \end{cases}$$

where $x \in L_{(1,0)}$ or $L_{(0,1)}$ and $y \in L_{(\sigma,\rho)}$. If σ is 2^a1^b, then

$$L_1 \otimes L_\sigma = L_1 \otimes L_{2^a1^b} = \begin{cases} L_{2^a1^{b+1}} \oplus L_{2^{a+1}1^{b-1}} & \text{if } b > 0, \\ L_{2^a1^{b+1}} & \text{if } b = 0, \end{cases}$$

where $L_{2^a1^{b+1}}$ is generated by

$$K_{\sigma^+} = S_1 \otimes K_\sigma$$

and $L_{2^{a+1}1^{b-1}}$ is generated by

$$K_{\sigma^-} = S_1 \otimes \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \vdots & \vdots \\ \hline 1 & 2 \\ \hline 1 & \\ \cline{1-1} \vdots & \\ \cline{1-1} 1 & \\ \cline{1-1} 2 & \\ \cline{1-1} \end{array} - S_2 \otimes \begin{array}{|c|c|} \hline 1 & 2 \\ \hline \vdots & \vdots \\ \hline 1 & 2 \\ \hline 1 & \\ \cline{1-1} \vdots & \\ \cline{1-1} 1 & \\ \cline{1-1} 1 & \\ \cline{1-1} \end{array} .$$

To study the images of the projections j_i, we check if $K_{\sigma\pm,\rho}$ or $K_{\sigma,\rho\pm}$ survive the map j_i. Since the j_i are $GL(m)\times GL(m)$ equivariant, we conclude that if $j_i(K_{\sigma,\rho\pm}) \neq 0$ (respectively $j_i(K_{\sigma\pm,\rho}) \neq 0$), then there is a copy of $L_{(\sigma,\rho\pm)}$ (respectively, $L_{(\sigma\pm,\rho)}$) in M_i.

Write

$$K_{\sigma,\rho,1} = \begin{cases} K_{\sigma+,\rho} & \text{when considering } L_{(1,0)}, \\ K_{\sigma,\rho+} & \text{when considering } L_{(0,1)}, \end{cases}$$

and

$$K_{\sigma,\rho,2} = \begin{cases} K_{\sigma-,\rho} & \text{when considering } L_{(1,0)}, \\ K_{\sigma,\rho-} & \text{when considering } L_{(0,1)}. \end{cases}$$

We want to compute $j_i(K_{\sigma,\rho,1})$ and $j_i(K_{\sigma,\rho,2})$. The results are listed in Tables 3 and 4.

From Tables 3 and 4 we can extract the following Theorem.

Theorem 2 *We have*

$$L_{(1,0)} \cdot L_{(2^a1^b,1^c)} = L_{(2^a1^b,1^c)} \cdot L_{(1,0)} = L_{(2^a1^{b+1},1^c)} \oplus L_{(2^{a+1}1^{b-1},1^c)},$$

and

$$L_{(0,1)} \cdot L_{(2^a1^b,1^c)} = L_{(2^a1^b,1^c)} \cdot L_{(0,1)} = L_{(2^a1^b,1^{c+1})}.$$

The brackets $[L_{(1,0)}, L_{(2^a1^b,1^c)}]$ and $[L_{(0,1)}, L_{(2^a1^b,1^c)}]$ are given by Table 5.

We can order Young diagrams by containment, and we say that $\mathcal{K}$ is an ideal of Young diagrams if $\omega \in \mathcal{K}$ and $\tau \supset \omega$, then $\omega \in \mathcal{K}$. We will also extend this ordering to pairs of tableaux. So $(\sigma_1, \rho_1) \subset (\sigma_2, \rho_2)$ if $\sigma_1 \subset \sigma_2$ and $\rho_1 \subset \rho_2$.

We can deduce the following Corollary.

Corollary 3 *A $GL(m) \times GL(m)$-invariant left (or right) ideal of $\tilde{R}$ is necessarily a two-sided ideal of the form*

$$\bigoplus_{\sigma,\rho\in\mathcal{I}} L_\sigma \otimes L_\rho,$$

where $\mathcal{I}$ is an ideal of pairs of Young diagrams.

6 Structure of R

In this section we will describe the structure of R. We will break the proof into a series of lemmas.

Lemma 4 *Let $x_2, y_1, y_2 \in R$ and consider*

$$h = \big[y_1, [x_2, y_2]\big] + \big[y_2, [x_2, y_1]\big] \in R.$$

For any $z \in R$ we have

$$\operatorname{tr}(z)h \in R.$$

Proof. We start with the identity

$$[x,y]^2 = -\det[x,y].$$

Polarizing, we get the central polynomial

$$\begin{aligned} f(x_1,x_2,y_1,y_2) &= [x_1,y_1][x_2,y_2] + [x_2,y_2][x_1,y_1] \\ &\quad + [x_1,y_2][x_2,y_1] + [x_2,y_1][x_1,y_2]. \end{aligned}$$

Since every term in f contains an x_1 term, we can use the cyclic property of trace to write

$$\operatorname{tr}(f) = \operatorname{tr}(x_1 h'), \tag{9}$$

where $h' \in R$. It follows from the nondegeneracy of the trace and the fact that $x_1 \in R$ is a generic matrix that h' is uniquely given by $h' = h$.

Let z be any element of R. Then

$$\operatorname{tr}(fz) = f\operatorname{tr}(z) = \tfrac{1}{2}\operatorname{tr}(f)\operatorname{tr}(z), \tag{10}$$

since f is a scalar. In the same way as above, we can find a unique $g(z) \in R$ such that

$$\operatorname{tr}(fz) = \operatorname{tr}(x_1 g(z)).$$

We can now rewrite (10) as

$$\operatorname{tr}(x_1 g(z)) = \tfrac{1}{2}\operatorname{tr}(x_1 h)\operatorname{tr}(z),$$

or

$$\operatorname{tr}\left(x_1(g(z) - \tfrac{1}{2}h \operatorname{tr} z)\right) = 0.$$

Using the nondegeneracy of the trace again, we get that

$$h \operatorname{tr} z = 2g(z). \quad \square$$

The point of this, is that the polynomial h absorbs traces. The left side $h \operatorname{tr} z$ is an element of S, but lies in fact in R.

Lemma 5 *Set* $\mathcal{L}_1 = L_{(1,0)} \oplus L_{(0,1)}$. *Then*

$$\mathcal{L}_2 = [\mathcal{L}_1, \mathcal{L}_1] = L_{(2,0)} \oplus L_{(1,1)}$$

and

$$\mathcal{L}_{21} = \left[\mathcal{L}_1, [\mathcal{L}_1, \mathcal{L}_1]\right] = L_{(21,0)} \oplus L_{(1^2,1)} \oplus L_{(1,1^2)}.$$

Proof. Follows from the results of Section 5. $\square$

Let A be the space obtained by evaluation of h on M_2^3. Since scalars will cancel the brackets, it is sufficient to evaluate on $(M_2^0)^3$, and we see that $A = \mathcal{L}_{21}$. We will now determine the ideal in S generated by A. (We let $[R,R]$ denote the ideal generated by the set $[R,R]$.)

Theorem 6 *We have*

$$AS = SA = I[t_1, \dots, t_m] = J \subset [R,R],$$

where

$$I = \bigoplus_{\substack{(21,0)\subset(\sigma,\rho) \text{ or} \\ (1^2,1)\subset(\sigma,\rho) \text{ or} \\ (1,1^2)\subset(\sigma,\rho)}} L_{(\sigma,\rho)}.$$

Proof. The polarized form of Cayley-Hamilton Theorem says that

$$xy + yx - \operatorname{tr}(x)y - \operatorname{tr}(y)x - \operatorname{tr}(xy) + \operatorname{tr}(x)\operatorname{tr}(y) = 0, \qquad (11)$$

and it follows that any product of of traces can be written in the form $\sum u_i \operatorname{tr}(v_i)$ with $u_i, v_i \in R$. But this is precisely the case considered in Lemma 4, where we showed that $A \operatorname{tr}(z) \subset R$. And since it is easy to see that every monomial in $g(z)$ will contain at least one bracket, we have in fact

$$A \operatorname{tr}(z) \subset [R,R].$$

Hence $AS = SA \subset [R,R]$. It follows from Corollary 3 that the $\tilde{R}$-ideal generated by A is I. But since $S = \tilde{R}[t_1, \dots, t_m]$, the theorem follows. $\square$

We will consider the filtration

$$R \supset [R,R] \supset J.$$

Since R is the ring of the noncommutative polynomials in s_i and A_i, we see that $R/[R,R]$ is the ring of commutative polynomials in in s_i and A_i. We will now consider $[R,R]/J$. We first observe that

$$[R,R] \subset [S,S] = [\tilde{R},\tilde{R}][t_1, \dots, t_m] = I_0[t_1, \dots, t_m],$$

where

$$I_0 = \bigoplus_{\substack{(2,0)\subset(\sigma,\rho) \text{ or} \\ (1,1)\subset(\sigma,\rho)}} L_{(\sigma,\rho)}.$$

It follows that

$$\begin{aligned} [S,S]/J &= I_0[t_1, \dots, t_m]/I[t_1, \dots, t_m] && (12) \\ &= (L_{(2,0)} \oplus L_{(2,1)} \oplus L_{(1,1)})[t_1, \dots, t_m]. && (13) \end{aligned}$$

Since

$$[xy, z] = x[y,z] + [x,z]y = x[y,z] + y[x,z] - \left[y, [x,z]\right] \in R\mathcal{L}_2 + J,$$

we have

$$[R,R]/J = R\mathcal{L}_2/J = R(L_{(2,0)} \oplus L_{(1,1)})/J. \qquad (14)$$

Lemma 7 *We have* $[R,R][R,R] \subset \left[R, [R,R]\right]$.

Proof. We have

$$\begin{aligned} [x,y][u,v] &= xy[u,v] - yx[u,v] \\ &= x\left[y,[u,v]\right] + x[u,v]y - y\left[x,[u,v]\right] - y[u,v]x. \end{aligned}$$

But

$$x[u,v]y - y[u,v]x = \Big[x,[u,v]\Big]y + \Big[[u,v]x,y\Big],$$

so we get

$$[x,y][u,v] = x\Big[y,[u,v]\Big] - y\Big[x,[u,v]\Big] + \Big[x,[u,v]\Big]y + \Big[[u,v]x,y\Big].$$

□

Lemma 8 *$[R,R]/J$ is an $R/[R,R]$ module generated by $L_{(2,0)} \oplus L_{(1,1)}$.*

Proof. It follows from Lemma 7 that the action of R on $\mathcal{L}_2$ factors through $[R,R]$. Then use (14). □

We claim that $[R,R]/J$ is a *free* $R/[R,R]$ module generated by $L_{(2,0)} \oplus L_{(1,1)}$. To see this, we consider

$$\Big([S,S]/J\Big)\Big/L_{(2,1)}[t_1,\dots,t_m] = \mathcal{L}_2[t_1,\dots,t_m].$$

Inside this module, we have

Lemma 9 *If $u \in \mathcal{L}_2$, then*

$$2x_iu \equiv t_iu \pmod{(J + L_{(2,1)}[t_1,\dots,t_m])}.$$

Proof. Set $u = [u_1,u_2]$. Using (11), we get

$$\begin{aligned} 0 &= (x_i - t_i)u + (u - \operatorname{tr} u)x_i + t_i \operatorname{tr} u - \operatorname{tr}(x_iu) \\ &= (2x_i - t_i)[u_1,u_2] + \Big[[u_1,u_2],x_i\Big] - \operatorname{tr}(x_i[u_1,u_2]). \end{aligned}$$

But the last two terms are in J and $L_{(2,1)}[t_1,\dots,t_m]$ respectively. □

Proposition 10 *$[R,R]/J$ is a free $R/[R,R]$ module generated by $L_{(2,0)} \oplus L_{(1,1)}$.*

Proof. Lemma 9 shows that in $[R,R]/J\Big/L_{(2,1)}[t_1,\dots,t_m]$, multiplication by elements of R correspond to scalar multiplication. It follows that $[R,R]/J$ is free. □

We can now state the main theorem.

Theorem 11 *The structure of R is described by the filtration*

$$R \supset [R,R] \supset J.$$

Proof. $R/[R,R]$ is is the ring of commutative polynomials in s_i and A_i. $[R,R]/J$ is described by Lemma 10, and J is given by Theorem 6. □

Corollary 12 *We have the following basis for R:*

(i) $S_{i_1}^{h_1}\cdots S_{i_s}^{h_s}A_{j_1}^{l_1}\cdots A_{j_t}^{l_t}$,

(ii) $S_{i_1}^{h_1}\cdots S_{i_s}^{h_s}[S_i,S_j]A_{j_1}^{l_1}\cdots A_{j_t}^{l_t}$, *and* $S_{i_1}^{h_1}\cdots S_{i_s}^{h_s}S_iA_jA_{j_1}^{l_1}\cdots A_{j_t}^{l_t}$, *for* $i<j$,

(iii) $P_Dt_1^{h_1}\cdots t_k^{h_k}$, *where P_D is a monomial obtained from the Young tableau D (see Section 3) of type $D=(\rho,\sigma)$, where (ρ,σ) contains $(21,0)$, $(1^2,1)$ or $(1,1^2)$.*

7 The center of R

Let $Z = R \cap T$ be the center of R, and let Z^+ be the subring of homogeneous, central polynomial of positive degree. Set

$$J_0 = J \cap T.$$

It is clear that $Z \supset J_0[t_1,\dots,t_m]$. In fact, we have the following theorem.

Theorem 13 *The center of R is given by*

$$Z = J_0[t_1,\dots,t_m] + k.$$

Proof. Let $f \in Z^+$. It follows from [7] that $f \in [R,R]$ (i.e., after making the variables in f commute, we get 0). We know that $[R,R]/J$ is a free module generated by $L_{(2,0)} \oplus L_{(1,1)}$, but none of this are projections of invariants. Hence $Z^+ \subset J$, and the lemma follows. □

<table>
<tr><td></td><td>$\boxed{k}\otimes 1$</td><td>$1\otimes\boxed{k}$</td><td>$\boxed{k}\otimes\boxed{l}$</td><td>$\begin{array}{|c|c|}\hline k&l\\\hline\end{array}\otimes 1$</td></tr>
<tr><td>$\boxed{i}\otimes 1$</td><td>$\frac{1}{2}\begin{array}{|c|}\hline i\\\hline k\\\hline\end{array}\otimes 1$
$+\frac{1}{2}\begin{array}{|c|c|}\hline i&k\\\hline\end{array}\otimes 1$</td><td>$\boxed{i}\otimes\boxed{k}$</td><td>$\frac{1}{2}\left(\begin{array}{|c|}\hline i\\\hline k\\\hline\end{array}\otimes 1\right)$
$1\otimes\boxed{l}$
$+\frac{1}{2}\begin{array}{|c|c|}\hline i&k\\\hline\end{array}\otimes\boxed{l}$</td><td>$-\left(\begin{array}{|c|}\hline i\\\hline l\\\hline\end{array}\otimes 1\right)$
$\boxed{k}\otimes 1$
$+\left(\begin{array}{|c|}\hline i\\\hline k\\\hline\end{array}\otimes 1\right)$
$\boxed{l}\otimes 1$</td></tr>
<tr><td>$1\otimes\boxed{i}$</td><td>$-\boxed{k}\otimes\boxed{i}$</td><td>$\frac{1}{2}\left(1\otimes\begin{array}{|c|}\hline i\\\hline k\\\hline\end{array}\right)$</td><td>$-\frac{1}{2}\left(1\otimes\begin{array}{|c|}\hline i\\\hline l\\\hline\end{array}\right)$
$\boxed{k}\otimes 1$</td><td>$\begin{array}{|c|c|}\hline k&l\\\hline\end{array}\otimes\boxed{i}$</td></tr>
<tr><td>$\boxed{i}\otimes\boxed{j}$</td><td>$-\frac{1}{2}\left(\begin{array}{|c|}\hline i\\\hline k\\\hline\end{array}\otimes 1\right)$
$1\otimes\boxed{j}$
$-\frac{1}{2}\begin{array}{|c|c|}\hline i&k\\\hline\end{array}\otimes\boxed{j}$</td><td>$\frac{1}{2}\left(1\otimes\begin{array}{|c|}\hline j\\\hline k\\\hline\end{array}\right)$
$\boxed{i}\otimes 1$</td><td>$-\frac{1}{4}\left(1\otimes\begin{array}{|c|}\hline j\\\hline l\\\hline\end{array}\right)$
$\begin{array}{|c|c|}\hline i&k\\\hline\end{array}\otimes 1$
$-\frac{1}{4}\left(\begin{array}{|c|}\hline i\\\hline k\\\hline\end{array}\otimes 1\right)$
$\left(1\otimes\begin{array}{|c|}\hline j\\\hline l\\\hline\end{array}\right)$</td><td>$-\left(\begin{array}{|c|}\hline i\\\hline l\\\hline\end{array}\otimes 1\right)$
$\boxed{k}\otimes\boxed{j}$
$+\left(\begin{array}{|c|}\hline i\\\hline k\\\hline\end{array}\otimes 1\right)$
$\boxed{l}\otimes\boxed{j}$</td></tr>
<tr><td>$\begin{array}{|c|c|}\hline i&j\\\hline\end{array}\otimes 1$</td><td>$\left(\begin{array}{|c|}\hline j\\\hline k\\\hline\end{array}\otimes 1\right)$
$\boxed{i}\otimes 1$
$-\left(\begin{array}{|c|}\hline i\\\hline k\\\hline\end{array}\otimes 1\right)$
$\boxed{j}\otimes 1$</td><td>$\begin{array}{|c|c|}\hline i&j\\\hline\end{array}\otimes\boxed{k}$</td><td>$\left(\begin{array}{|c|}\hline j\\\hline k\\\hline\end{array}\otimes 1\right)$
$\boxed{i}\otimes\boxed{l}$
$-\left(\begin{array}{|c|}\hline i\\\hline k\\\hline\end{array}\otimes 1\right)$
$\boxed{j}\otimes\boxed{l}$</td><td>$-\begin{array}{|c|c|}\hline i&j\\\hline k&l\\\hline\end{array}\otimes 1$</td></tr>
</table>

Table 1

	$K_{1,0}$	$K_{0,1}$	$K_{1,1}$	$K_{2,0}$	$K_{21,0}$	$K_{21,1}$
$K_{1,0}$	$\frac{1}{2}K_{1^2,0}$	$K_{1,1}$	$\frac{1}{2}K_{1^2,0}K_{0,1}$	$K_{21,0}$	$\frac{1}{2}K_{1^2,0}K_{2,0}$	$-\frac{1}{2}K_{21^2,1}$
$K_{0,1}$	$-K_{1,1}$	$\frac{1}{2}K_{0,1^2}$	$-\frac{1}{2}K_{0,1^2}K_{1,0}$	$K_{2,1}$	$-K_{21,1}$	$K_{2,1}K_{1,1}$
$K_{1,1}$	$-\frac{1}{2}K_{1^2,0}K_{0,1}$	$\frac{1}{2}K_{0,1^2}K_{1,0}$	$-\frac{1}{4}K_{1^2,1^2}$	$-K_{21,1}$	$-\frac{1}{2}K_{21^2,1}$	$\frac{1}{4}K_{1^2,1^2}K_{2,0}$
$K_{2,0}$	$-K_{21,0}$	$K_{2,1}$	$K_{21,1}$	$-K_{2^2,0}$	$K_{2^2,0}K_{1,0}$	$-K_{2^2,0}K_{1,1}$
$K_{21,0}$	$-\frac{1}{2}K_{1^2,0}K_{2,0}$	$K_{21,1}$	$-\frac{1}{2}K_{21^2,1}$	$-K_{2^2,0}K_{1,0}$	$\frac{1}{2}K_{2^21^2,0}$	$-\frac{1}{2}K_{2^21^2,0}K_{0,1}$
$K_{21,1}$	$-\frac{1}{2}K_{21^2,1}$	$-K_{2,1}K_{1,1}$	$-\frac{1}{4}K_{1^2,1^2}K_{2,0}$	$K_{2^2,0}K_{1,1}$	$\frac{1}{2}K_{2^21^2,0}K_{0,1}$	$-\frac{1}{4}K_{2^21^2,1^2}$

Table 2

$L_{(\sigma,\rho)}$	$j_1(K_{\sigma,\rho,1})$	$j_2(K_{\sigma,\rho,1})$	$j_3(K_{\sigma,\rho,1})$	$j_4(K_{\sigma,\rho,1})$	$j_5(K_{\sigma,\rho,1})$	$j_6(K_{\sigma,\rho,1})$
$L_{(1,0)}$	$\frac{1}{2}K_{1^2,0}$	$\frac{1}{2}K_{1^2,0}$	0	$-K_{1,1}$	$K_{1,1}$	$2K_{1,1}$
$L_{(0,1)}$	$K_{1,1}$	$-K_{1,1}$	$2K_{1,1}$	$\frac{1}{2}K_{0,1^2}$	$\frac{1}{2}K_{0,1^2}$	0
$L_{(1,1)}$	$\frac{1}{2}K_{1^2,1}$	$-\frac{1}{2}K_{1^2,1}$	$K_{1^2,1}$	$-\frac{1}{2}K_{1^2,1}$	$\frac{1}{2}K_{1^2,1}$	$-K_{1^2,1}$
$L_{(2,0)}$	$\frac{1}{2}K_{21,0}$	$-\frac{1}{2}K_{21,0}$	$K_{21,0}$	$-K_{2,1}$	$-K_{2,1}$	0
$L_{(21,0)}$	$K_{21^2,0}$	$-K_{21^2,0}$	$2K_{21^2,0}$	$-K_{21,1}$	$K_{21,1}$	$-2K_{21,1}$
$L_{(21,1)}$	$K_{21^2,1}$	$K_{21^2,1}$	0	$-\frac{1}{2}K_{21,1^2}$	$\frac{1}{2}K_{21,1^2}$	$-K_{21,1^2}$

Table 3

$L_{(\sigma,\rho)}$	$j_1(K_{\sigma,\rho,2})$	$j_2(K_{\sigma,\rho,2})$	$j_3(K_{\sigma,\rho,2})$	$j_4(K_{\sigma,\rho,2})$	$j_5(K_{\sigma,\rho,2})$	$j_6(K_{\sigma,\rho,2})$
$L_{(1,0)}$	$\frac{1}{2}K_{2,0}$	$-\frac{1}{2}K_{2,0}$	$K_{2,0}$	0	0	0
$L_{(0,1)}$	0	0	0	0	0	0
$L_{(1,1)}$	$\frac{1}{2}K_{2,1}$	$\frac{1}{2}K_{1^2,1}$	0	0	0	0
$L_{(2,0)}$	0	0	0	0	0	0
$L_{(21,0)}$	$-K_{2^2,0}$	$-K_{2^2,0}$	0	0	0	0
$L_{(1^2,0)}$	$-K_{21,0}$	$-K_{21,0}$	0	0	0	0
$L_{(21^2,0)}$	$K_{2^21,0}$	$-K_{2^21,0}$	$2K_{2^21,0}$	0	0	0
$L_{(21,1)}$	$-K_{2^2,1}$	$K_{2^2,1}$	$-2K_{2^2,1}$	0	0	0

Table 4

a	b	c	$[L_{(1,0)}, L_{(2^a1^b,1^c)}]$	$[L_{(0,1)}, L_{(2^a1^b,1^c)}]$
+	+	+	0	0
+	+	−	$L_{(2^a1^{b+1},1^c)}$	0
+	−	+	$L_{(2^{a+1}1^{b-1},1^c)}$	$L_{(2^a1^b,1^{c+1})}$
+	−	−	$L_{(2^a1^{b+1},1^c)}$	$L_{(2^a1^b,1^{c+1})}$
−	+	+	$L_{(2^a1^{b+1},1^c)} \oplus L_{(2^{a+1}1^{b-1},1^c)}$	0
−	+	−	0	0
−	−	+	$L_{(2^a1^{b+1},1^c)}$	$L_{(2^a1^b,1^{c+1})}$
−	−	−	$L_{(2^{a+1}1^{b-1},1^c)}$	$L_{(2^a1^b,1^{c+1})}$

Table 5 : Here + denotes an even number and − denotes an odd number.

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Department of Mathematics
National University of Singapore
Singapore 0511
Republic of Singapore
e-mail: `mathelmr(mattanec)@nusunix.nus.sg`

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