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RESTRICTED HOMOGENEOUS COORDINATES FOR THE CAYLEY PROJECTIVE PLANE

ABSTRACT. I. Porteous has shown that the Cayley projective plane can be coordinatized in a way resembling homogeneous coordinates. We will show how to construct line coordinates in a similar way. As an illustration, we give an explicit example to show that the Cayley projective plane is not Desarguean.

1. INTRODUCTION

It is well known that since the Cayley numbers are not associative, we cannot coordinatize the Cayley projective plane by homogeneous coordinates. Instead we can either use inhomogeneous coordinates as described in, for instance, [Ha] or represent the points as 3×3 Cayley valued Hermitian matrices as introduced by Jordan [Jo] and Freudenthal [Fr]. Porteous [Po] has shown, however, that we can use the fact that the Cayley numbers form an alternative division algebra to introduce a type of coordinates that we will call *restricted homogeneous coordinates*. Porteous only discusses point coordinates, and it is not immediately clear how to define line coordinates in a way that does not require associativity. The purpose of this brief article is to show how restricted homogeneous coordinates can be used to describe the lines in the Cayley projective plane. As an example of how to compute with these coordinates we will exhibit two triangles that are in perspective from a point but not from a line, thus illustrating the well-known fact that the Cayley projective plane is not Desarguean.

We will denote the Cayley numbers by $\mathbb{C}a$, express them in the form $a_1 + a_2 i + a_j j + a_4 k + a_5 e + a_6 ie + a_7 je + a_8 ke$ and consider $\mathbb{C}a^n$ as a left vector space over $\mathbb{C}a$. The formula for the Cayley–Dickson process is

$$(a+be)(c+de) = ac - \bar{d}b + (da + b\bar{c})e \quad \text{for } a, b, c, d \in \mathbb{H}.$$

In particular, we have

$$a(be) = (ba)e, (ae)b = (a\bar{b})e, (ae)(be) = -\bar{b}a \quad \text{for } a, b, c, d \in \mathbb{H}.$$

A theorem of Artin says that the subalgebra generated by two elements is associative. Hence

$$(1) \qquad (xy)y^{-1} = x \quad \text{and} \quad (xy)^{-1} = y^{-1}x^{-1}.$$

This will be crucial for our argument.

Geometriae Dedicata **40:** 245–250, 1991.

2. Restricted homogeneous coordinates

Porteous defines $\mathbb{C}aP^2$ as the set

(2) $$\{[1, y_1, z_1]\} \cup \{[x_2, 1, z_2]\} \cup \{[x_3, y_3, 1]\} \subset \mathbb{C}a^3$$

with the following identifications

(3) $$\begin{aligned}
[1, y_1, z_1] = [x_2, 1, z_2] \Leftrightarrow \quad & y_1 \neq 0 \text{ and } x_2 = y_1^{-1},\ z_2 = z_1 y_1^{-1} \\
& \text{or } x_2 \neq 0 \text{ and } y_1 = x_2^{-1},\ z_1 = z_2 x_2^{-1}, \\
[1, y_1, z_1] = [x_3, y_3, 1] \Leftrightarrow \quad & z_1 \neq 0 \text{ and } x_3 = z_1^{-1},\ y_3 = y_1 z_1^{-1} \\
& \text{or } x_3 \neq 0 \text{ and } z_1 = x_3^{-1},\ y_1 = y_3 x_3^{-1}, \\
[x_2, 1, z_2] = [x_3, y_3, 1] \Leftrightarrow \quad & z_2 \neq 0 \text{ and } y_3 = z_2^{-1},\ x_3 = x_2 z_2^{-1} \\
& \text{or } y_3 \neq 0 \text{ and } z_2 = y_3^{-1},\ x_2 = x_3 y_3^{-1}.
\end{aligned}$$

We need to show that these identifications are compatible, i.e. that

(4) $$\begin{gathered}
[1, y_1, z_1] = [x_2, 1, z_2] \quad \text{and} \quad [x_2, 1, z_2] = [x_3, y_3, 1] \\
\Downarrow \\
[1, y_1, z_1] \quad = \quad [x_3, y_3, 1],
\end{gathered}$$

or that

$$\begin{gathered}
x_2 = y_1^{-1}, \quad z_2 = z_1 y_1^{-1} \quad \text{and} \quad y_3 = z_2^{-1}, \quad x_3 = x_2 z_2^{-1} \\
\Downarrow \\
x_3 = z_1^{-1}, \quad y_3 = y_1 z_1^{-1}.
\end{gathered}$$

But this follows from (1) since we have

$$x_3 = x_2 z_2^{-1} = y_1^{-1}(z_1 y_1^{-1})^{-1} = y_1^{-1}(y_1 z_1^{-1}) = z_1^{-1}$$

and

$$y_3 = z_2^{-1} = (z_1 y_1^{-1})^{-1} = y_1 z_1^{-1}.$$

This method of coordinatizing $\mathbb{C}aP^2$ has several advantages. The coordinates are homogeneous in the sense that we do not treat points at infinity separately, and the construction is also quite similar to the construction of the real, complex and quaternionic projective planes. The use of Artin's theorem also shows exactly how much (or little) associativity we need. The fact that we cannot construct a Cayley projective space may seem more plausible after seeing this construction.

Porteous only discusses point coordinates, and it is not immediately clear how to define line coordinates in a way that does not require associativity.

We will define the lines in $\mathbb{C}aP^2$ to be the following subsets of $\mathbb{C}aP^2$:

$$\begin{aligned}(5)\qquad &\langle 1, b_1, c_1\rangle = \{[x_2, 1, z_2] \mid x_2 + b_1 + c_1 z_2 = 0\} \cup \{[-c_1, 0, 1]\},\\ &\langle a_2, 1, c_2\rangle = \{[x_3, y_3, 1] \mid a_2 x_3 + y_3 + c_2 = 0\} \cup \{[1, -a_2, 0]\},\\ &\langle a_3, b_3, 1\rangle = \{[1, y_1, z_1] \mid a_3 + b_3 y_1 + z_1 = 0\} \cup \{[0, 1, -b_3]\}.\end{aligned}$$

We want to make identifications similar to (3), but notice that we take the inverses on the left.

$$\begin{aligned}(6)\qquad \langle 1,b_1,c_1\rangle = \langle a_2,1,c_2\rangle \Leftrightarrow\quad & b_1 \neq 0 \text{ and } a_2 = b_1^{-1},\ c_2 = b_1^{-1}c_1\\ &\text{or } a_2 \neq 0 \text{ and } b_1 = a_2^{-1},\ c_1 = a_2^{-1}c_2,\\ \langle 1,b_1,c_1\rangle = \langle a_3,b_3,1\rangle \Leftrightarrow\quad & c_1 \neq 0 \text{ and } a_3 = c_1^{-1},\ b_3 = c_1^{-1}b_1\\ &\text{or } a_3 \neq 0 \text{ and } c_1 = a_3^{-1},\ b_1 = a_3^{-1}b_3,\\ \langle a_2,1,c_2\rangle = \langle a_3,b_3,1\rangle \Leftrightarrow\quad & c_2 \neq 0 \text{ and } b_3 = c_2^{-1},\ a_3 = c_2^{-1}a_2\\ &\text{or } b_3 \neq 0 \text{ and } c_2 = b_3^{-1},\ a_2 = b_3^{-1}a_3.\end{aligned}$$

We must check that the relations in (6) are compatible with the definitions in (5). We have

$$\begin{aligned}(7)\qquad \langle b_1^{-1}, 1, b_1^{-1}c_1\rangle &= \{[x_3, y_3, 1] \mid b_1^{-1}x_3 + y_3 + b_1^{-1}c_1 = 0\}\\ &\quad \cup \{[1, -b_1^{-1}, 0]\}\\ &= \{[x_3 y_3^{-1}, 1, y_3^{-1}] \mid b_1^{-1}x_3 + y_3 + b_1^{-1}c_1 = 0, y_3 \neq 0\}\\ &\quad \cup \{[x_3, 0, 1] \mid b_1^{-1}x_3 + b_1^{-1}c_1 = 0\} \cup \{[1, -b_1^{-1}, 0]\}\\ &= \{[x_3 y_3^{-1}, 1, y_3^{-1}] \mid x_3 + b_1 y_3 + c_1 = 0,\ y_3 \neq 0\}\\ &\quad \cup \{[-c_1, 0, 1]\} \cup \{[1, -b_1^{-1}, 0]\}\\ &= \{[x_2, 1, z_2] \mid x_2 + b_1 + c_1 z_2 = 0,\ z_2 \neq 0\}\\ &\quad \cup \{[1, -b_1^{-1}, 0]\} \cup \{[-c_1, 0, 1]\}\\ &= \{[x_2, 1, z_2] \mid x_2 + b_1 + c_1 z_1 = 0\} \cup \{[-c_1, 0, 1]\}\\ &= \langle 1, b_1, c_1\rangle,\end{aligned}$$

and similarly for the other relations.

Notice that we expressed the points in (5) in such a way that the equations only contained one summand that was a product. If we had set

$$\langle 1, b_1, c_1\rangle = \{[1, y_1, z_1] \mid 1 + b_1 y_1 + c_1 z_1 = 0\} \cup \{[0, -c_1, b_1]\},$$

we would not have been able to derive (7).

We will now compare this with the usual inhomogeneous coordinates as described by Hall [Ha, pp. 353–355]. (For an equivalent, but slightly different way of assigning inhomogeneous coordinates, see [FF].) If we think of the line of infinity as a one-dimensional projective space we can choose a point of infinity on it. Hall denotes this by (∞), and the other points at the line of infinity can be expressed in the form (c). The points not on the line of infinity are then assigned coordinates of the form (a, b). We can then introduce line coordinates by defining

$$\begin{aligned}\langle a, b\rangle &= \{(x, ax + b)\} \cup (a),\\ \langle c\rangle &= \{(c, y)\} \cup (\infty),\\ \langle\infty\rangle &= \{(c)\} \cup (\infty).\end{aligned}$$

It is clear that

$$(a, b) = [a, b, 1], \quad (c) = [1, c, 0], \quad (\infty) = [0, 1, 0],$$

and we also have

$$\begin{aligned}\langle a, b\rangle &= \{(x, ax+b)\} \cup (a) = \{[x, ax + b, 1]\} \cup \{[1, a, 0]\}\\ &= \langle -a, 1, -b\rangle,\\ \langle c\rangle &= \{(c, y)\} \cup (\infty) = \{[c, y, 1]\} \cup \{[0, 1, 0]\}\\ &= \{[cy^{-1}, 1, y^{-1}] \mid y \neq 0\} \cup \{[c, 0, 1]\} \cup \{[0, 1, 0]\}\\ &= \{[cz, 1, z]\} \cup \{[c, 0, 1]\}\\ &= \langle 1, 0, -c\rangle,\\ \langle\infty\rangle &= \{(c)\} \cup (\infty) = \{[1, c, 0]\} \cup \{[0, 1, 0]\}\\ &= \langle 0, 0, 1\rangle.\end{aligned}$$

This shows that the lines defined by the restricted homogeneous line coordinates coincide with the lines defined by the inhomogeneous line coordinates.

3. Examples

In the previous section we showed how to introduce coordinates in a way resembling the associative case. We feel that this is of interest in itself, but the definition might seem to be awkward to work with. That is not necessarily the case, and in this section we will illustrate this by giving an example that shows that the Cayley projective plane is not Desarguean.

A simple calculation gives the following corollary to the above definitions.

COROLLARY 1. *The coordinates of the line joining the two points* $[u, 1, v]$ *and* $[u', 1, v']$ *are given by*

$$\langle 1,\ -u-((u-u')(v'-v)^{-1})v,\ (u-u')(v'-v)^{-1}\rangle \quad \textit{for } v' \neq v,$$

$$\langle 0,\ 1,\ -v^{-1}\rangle \quad \textit{for } v' = v \neq 0,$$

$$\langle 0,\ 0,\ 1\rangle \quad \textit{for } v' = v = 0.$$

Consider now the line joining $[u, 1, v]$ and $[p, q, 1]$ or $[u, 1, v]$ and $[1, r, s]$. If $q \neq 0$ or $r \neq 0$ this can be reduced to the above corollary. The remaining cases are covered by the next corollary.

COROLLARY 2. *The coordinates of the line joining the two points* $[u, 1, v]$ *and* $[p, 0, 1]$ *are given by* $\langle 1, pv-u, -p\rangle$, *while the line joining* $[u, 1, v]$ *and* $[1, 0, s]$ *is given by* $\langle 1, s^{-1}v-u, -s^{-1}\rangle$ *for* $s \neq 0$ *and by* $\langle 0, -v, 1\rangle$ *for* $s = 0$.

This gives us the coordinates of all the lines through points of the form $[u, 1, v]$. By cyclic permutation we can find the coordinates of lines through points of the form $[1, u, v]$ and lines through points of the form $[u, v, 1]$.

We will now consider the two triangles $\triangle ABC$ and $\triangle A'B'C'$ where

$$A = [0, i/2, 1], \qquad B = [1, e/2, 0], \qquad C = [1, j, 1],$$
$$A' = [0, -i/2, 1], \qquad B' = [1, -e/2, 0], \qquad C' = [1, k, 1].$$

Using the above corollaries we find that the two triangles are in perspective from the point $[0, 1, 0]$ and that

$$P = AB \cap A'B' = [-ie, 0, 1],$$

$$Q = AC \cap A'C' = [(1-j-k)/3, (1+j/2+k/2)/3, 1],$$

$$R = BC \cap B'C' = [1-je+ke, (j+k)/2, 1].$$

We then find that the line PR is given by

$$\langle(-j-k-2ie-je+ke)/8,\ 1,\ (2-j-k+je-ke)/8\rangle,$$

and we see that Q does not lie on the line PR, illustrating the fact that $\mathbb{C}aP^2$ is not Desarguean.

REFERENCES

[FF] Faulkner, J. R. and Ferrar, J. C., 'Generalizing the Moufang plane', *Rings and Geometry* (Istanbul, 1984), *NATO Adv. Sci. Inst. Ser. C: Math. Phys. Sci.*, **160**, Kluwer, Dordrecht, 1985, pp. 235–288.

[Fr] Freudenthal, H., *Oktaven, Ausnahmegruppen und Oktavengeometrie*, Mathematisch Intituut der Rijksuniversiteit te Utrecht, 1951.

[Ha] Hall, M., Jr, *The Theory of Groups*, Macmillan, New York, 1959.
[Jo] Jordan, P., 'Über eine nicht-desarguessche ebene projektive Geometrie', *Abh. Math. Sem. Univ. Hamburg* **16** (1949), 74–76.
[Po] Porteous, I. R., *Topological Geometry*, Van Nostrand Reinhold, New York, 1969 (second edition, Cambridge Univ. Press, Cambridge, 1981).

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(Received, February 8, 1991; revised version, April 2, 1991)