Доклади на Българската академия на науките
Comptes rendus de l'Académie bulgare des Sciences

Tome 58, No 6, 2005

MATHEMATIQUES
Algèbre

DEFINING RELATIONS OF INVARIANTS OF TWO 3×3 MATRICES[1]

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(*Submitted on March 23, 2005*)

Abstract

Over a field of characteristic 0, Teranishi, 1986, found a minimal system of eleven generators of the algebra of invariants of two 3×3 matrices under simultaneous conjugation by GL_3. Nakamoto, 2002, obtained the explicit, but very complicated defining relation for a similar system of generators over $\mathbb{Z}$. In this paper we give another natural set of eleven generators of the algebra of invariants over a field of characteristic 0 and the defining relation with respect to this generating set. Our defining relation is much simpler than that of Nakamoto.
Key words: matrix invariants, defining relations, Hilbert series
2000 Mathematics Subject Classification: 16R30

Introduction. Let K be any field of characteristic 0 and let C_{nd} be the algebra of invariants of the general linear group $GL_n = GL_n(K)$ acting by simultaneous conjugation on d matrices of size $n \times n$. It is well known that C_{nd} is generated by the traces of a finite number of products $\mathrm{tr}(X_{i_1} \cdots X_{i_k})$, where $X_i = \left(x_{pq}^{(i)}\right)$, $p, q = 1, \ldots, n$, $i = 1, \ldots, d$, are d generic $n \times n$ matrices. The theory of PI-algebras gives an upper bound for the degree k of the generators of C_{nd}. By the Nagata–Higman theorem the polynomial identity $x^n = 0$ implies the identity $x_1 \cdots x_m = 0$. Then $k \leq m$ and for d sufficiently large this bound is sharp.

[1]The second author was partially supported by Grant MM1106/2001 of the Bulgarian National Science Fund.

For a background on the algebra of matrix invariants see, e.g. [7,5]. Explicit minimal sets of generators of C_{nd} are known for $n = 2$ and any d, see SIBIRSKII [10], FORMANEK [6], and DRENSKY [4], and for $n = 3, 4$ and $d = 2$, see TERANISHI [11]. ABEASIS and PITTALUGA [1] found a system of generators of C_{3d}, for any d, in terms of representation theory of the symmetric and general linear groups.

In the sequel we shall consider the case $n = 3$ and $d = 2$ only. We denote by X and Y the two generic 3×3 matrices. The system of generators of C_{32} found by Teranishi [11] is

$$\begin{gathered} \operatorname{tr}(X), \operatorname{tr}(Y), \operatorname{tr}(X^2), \operatorname{tr}(XY), \operatorname{tr}(Y^2), \\ \operatorname{tr}(X^3), \operatorname{tr}(X^2Y), \operatorname{tr}(XY^2), \operatorname{tr}(Y^3), \operatorname{tr}(X^2Y^2), \operatorname{tr}(X^2Y^2XY). \end{gathered} \tag{1}$$

The first ten of these generators form a homogeneous system of parameters of C_{32} and C_{32} is a free module with generators 1 and $\operatorname{tr}(X^2Y^2XY)$ over the polynomial algebra on these ten elements.

The algebra C_{32} has a natural bigrading which counts the degrees of the traces with respect to each of the generic matrices X, Y. The Hilbert series of C_{32} is the formal power series

$$H(C_{32}, t_1, t_2) = \sum_{k_1, k_2 \geq 0} \dim(C_{32}^{(k_1,k_2)}) t_1^{k_1} t_2^{k_2}$$

with coefficients equal to the dimensions of the homogeneous components $C_{32}^{(k_1,k_2)}$ of degree (k_1, k_2). It was also evaluated in [11] as a consequence of the description of C_{32}

$$H(C_{32}, t_1, t_2) = \frac{1 + t_1^3 t_2^3}{(1 - t_1)(1 - t_2) q_2(t_1, t_2) q_3(t_1, t_2)(1 - t_1^2 t_2^2)}, \tag{2}$$

where

$$q_2(t_1, t_2) = (1 - t_1^2)(1 - t_1 t_2)(1 - t_2^2),$$

$$q_3(t_1, t_2) = (1 - t_1^3)(1 - t_1^2 t_2)(1 - t_1 t_2^2)(1 - t_2^3).$$

The generator $\operatorname{tr}(X^2Y^2XY)$ satisfies a quadratic equation with coefficients depending on the other ten generators. The explicit (but very complicated) form of the equation was found by NAKAMOTO [8], over $\mathbb{Z}$, with respect to a slightly different system of generators. REVOY [9] studied the field of rational invariants of two 3×3 matrices and also found eleven generators with an explicitly given relation between them.

In this paper we have found another natural set of eleven generators of the algebra C_{32} and have given the defining relation with respect to this set. It has turned out that our relation is much simpler than that in [8].

The first change in the set of generators is well known. In the products of degree ≥ 2 we replace the generic matrices X, Y with generic traceless matrices x, y, so our generators become

$$\begin{gathered} \operatorname{tr}(X), \operatorname{tr}(Y), \operatorname{tr}(x^2), \operatorname{tr}(xy), \operatorname{tr}(y^2), \\ \operatorname{tr}(x^3), \operatorname{tr}(x^2y), \operatorname{tr}(xy^2), \operatorname{tr}(y^3), \operatorname{tr}(x^2y^2), \operatorname{tr}(x^2y^2xy). \end{gathered}$$

Then we use the result of Abeasis and Pittaluga [1] and replace $\mathrm{tr}(x^2y^2)$, $\mathrm{tr}(x^2y^2xy)$ with elements which generate one-dimensional GL_2-modules under the extension of the natural action of GL_2 on $K\cdot X+K\cdot Y$ to C_{32}. It becomes clear that the only defining relation between our generators spans a one-dimensional GL_2-submodule of the subalgebra of C_{32} generated by all traces of products of x and y. Applying representation theory of GL_2, we have determined the hypothetic candidate for the relation. Using standard procedures of Maple, we have obtained that it is really the relation.

The complete proofs of the paper will be published elsewhere. They are posted as the preprint [2] at the preprint server of Cornell University. The ideas and the results of this paper have been used and further developed by Benanti and Drensky [3] where they have found the defining relations of the noncommutative (or mixed) trace algebra of two 3×3 generic matrices.

Main results. It is a standard trick in the study of matrix invariants to replace the generic matrices in the traces of products with generic traceless matrices. We express X and Y in the form

$$X=\frac{1}{3}\,\mathrm{tr}(X)e+x\quad\text{and}\quad Y=\frac{1}{3}\,\mathrm{tr}(Y)e+y, \tag{3}$$

where e is the identity matrix and x, y are generic traceless matrices. By well known arguments, as for "ordinary" generic matrices, without loss of generality we may assume that x is a diagonal matrix. Till the end of the paper we fix the notation

$$x=\begin{pmatrix}x_1&0&0\\0&x_2&0\\0&0&-(x_1+x_2)\end{pmatrix},\qquad y=\begin{pmatrix}y_{11}&y_{12}&y_{13}\\y_{21}&y_{22}&y_{23}\\y_{31}&y_{32}&-(y_{11}+y_{22})\end{pmatrix}. \tag{4}$$

It follows from (1) that the algebra C_{32} can be generated by the system

$$\begin{gathered}\mathrm{tr}(X),\mathrm{tr}(Y),\mathrm{tr}(x^2),\mathrm{tr}(xy),\mathrm{tr}(y^2),\\ \mathrm{tr}(x^3),\mathrm{tr}(x^2y),\mathrm{tr}(xy^2),\mathrm{tr}(y^3),\mathrm{tr}(x^2y^2),\mathrm{tr}(x^2y^2xy).\end{gathered} \tag{5}$$

Further, we replace the traces $\mathrm{tr}(x^2y^2)$, $\mathrm{tr}(x^2y^2xy)$, respectively with the elements

$$v=\mathrm{tr}(x^2y^2)-\mathrm{tr}(xyxy), \tag{6}$$

$$w=\mathrm{tr}(x^2y^2xy)-\mathrm{tr}(y^2x^2yx). \tag{7}$$

The following equations, checked by Maple, express $\mathrm{tr}(x^2y^2)$ and $\mathrm{tr}(x^2y^2xy)$ in terms of v, w and the other generators

$$\mathrm{tr}(x^2y^2)=\frac{1}{3}v+\frac{1}{6}\,\mathrm{tr}(x^2)\,\mathrm{tr}(y^2)+\frac{1}{3}\,\mathrm{tr}^2(xy), \tag{8}$$

$$\begin{gathered}\mathrm{tr}(x^2y^2xy)=\tfrac{1}{2}w+\tfrac{1}{6}\,\mathrm{tr}(xy)v+\tfrac{1}{12}\,\mathrm{tr}(x^2)\,\mathrm{tr}(xy)\,\mathrm{tr}(y^2)\\ +\tfrac{1}{6}\,\mathrm{tr}^3(xy)-\tfrac{1}{6}\,\mathrm{tr}(x^3)\,\mathrm{tr}(y^3)+\tfrac{1}{2}\,\mathrm{tr}(x^2y)\,\mathrm{tr}(xy^2).\end{gathered} \tag{9}$$

Now we define the following elements of C_{32}:

$$u = \begin{vmatrix} \operatorname{tr}(x^2) & \operatorname{tr}(xy) \\ \operatorname{tr}(xy) & \operatorname{tr}(y^2) \end{vmatrix}, \tag{10}$$

$$w_1 = u^3, \quad w_2 = u^2 v, \quad w_4 = uv^2, \quad w_7 = v^3, \tag{11}$$

$$w_5 = v \begin{vmatrix} \operatorname{tr}(x^2) & \operatorname{tr}(xy) & \operatorname{tr}(y^2) \\ \operatorname{tr}(x^3) & \operatorname{tr}(x^2y) & \operatorname{tr}(xy^2) \\ \operatorname{tr}(x^2y) & \operatorname{tr}(xy^2) & \operatorname{tr}(y^3) \end{vmatrix}, \tag{12}$$

$$w_6 = \begin{vmatrix} \operatorname{tr}(x^3) & \operatorname{tr}(xy^2) \\ \operatorname{tr}(x^2y) & \operatorname{tr}(y^3) \end{vmatrix}^2 - 4 \begin{vmatrix} \operatorname{tr}(y^3) & \operatorname{tr}(xy^2) \\ \operatorname{tr}(xy^2) & \operatorname{tr}(x^2y) \end{vmatrix} \begin{vmatrix} \operatorname{tr}(x^3) & \operatorname{tr}(x^2y) \\ \operatorname{tr}(x^2y) & \operatorname{tr}(xy^2) \end{vmatrix}, \tag{13}$$

$$w_3' = u \begin{vmatrix} \operatorname{tr}(x^2) & \operatorname{tr}(xy) & \operatorname{tr}(y^2) \\ \operatorname{tr}(x^3) & \operatorname{tr}(x^2y) & \operatorname{tr}(xy^2) \\ \operatorname{tr}(x^2y) & \operatorname{tr}(xy^2) & \operatorname{tr}(y^3) \end{vmatrix}, \tag{14}$$

$$\begin{aligned} w_3'' &= 5[\operatorname{tr}^3(y^2)\operatorname{tr}^2(x^3) + \operatorname{tr}^3(x^2)\operatorname{tr}^2(y^3)] \\ &-30[\operatorname{tr}^2(y^2)\operatorname{tr}(xy)\operatorname{tr}(x^2y)\operatorname{tr}(x^3) + \operatorname{tr}^2(x^2)\operatorname{tr}(xy)\operatorname{tr}(y^3)\operatorname{tr}(xy^2)] \\ &+3\{[4\operatorname{tr}(y^2)\operatorname{tr}^2(xy) + \operatorname{tr}^2(y^2)\operatorname{tr}(x^2)][3\operatorname{tr}^2(x^2y) + 2\operatorname{tr}(xy^2)\operatorname{tr}(x^3)] \\ &+[4\operatorname{tr}^2(xy)\operatorname{tr}(x^2) + \operatorname{tr}^2(x^2)\operatorname{tr}(y^2)][3\operatorname{tr}^2(xy^2) + 2\operatorname{tr}(x^2y)\operatorname{tr}(y^3)]\} \\ &-2[2\operatorname{tr}^3(xy) + 3\operatorname{tr}(x^2)\operatorname{tr}(xy)\operatorname{tr}(y^2)][9\operatorname{tr}(xy^2)\operatorname{tr}(x^2y) + \operatorname{tr}(x^3)\operatorname{tr}(y^3)], \end{aligned} \tag{15}$$

where v is defined in (6). The element w_3'' can be expressed in the following simple way. Recall that a linear mapping δ of an algebra R is a derivation if $\delta(rs) = \delta(r)s + r\delta(s)$ for all $r, s \in R$. Defining $\delta(X) = \delta(x) = 0$, and $\delta(Y) = X$, $\delta(y) = x$, we obtain a derivation δ of C_{32} which sends x to 0 and y to x. Then

$$w_3'' = \frac{1}{144} \sum_{i=0}^{6} (-1)^i \delta^i(\operatorname{tr}^3(y^2)) \delta^{6-i}(\operatorname{tr}^2(y^3)). \tag{16}$$

The following theorem is the main result of our paper.

Theorem. The algebra of invariants C_{32} of two 3×3 matrices has the following presentation. It is generated by

$$\begin{gathered}\mathrm{tr}(X),\mathrm{tr}(Y),\mathrm{tr}(x^2),\mathrm{tr}(xy),\mathrm{tr}(y^2),\\ \mathrm{tr}(x^3),\mathrm{tr}(x^2y),\mathrm{tr}(xy^2),\mathrm{tr}(y^3),v,w\end{gathered}\tag{17}$$

subject to the defining relation

$$f=w^2-\left(\frac{1}{27}w_1-\frac{2}{9}w_2+\frac{4}{15}w_3'+\frac{1}{90}w_3''+\frac{1}{3}w_4-\frac{2}{3}w_5-\frac{1}{3}w_6-\frac{4}{27}w_7\right)=0,\tag{18}$$

where the elements v, w, w_1, w_2, w_3', w_3'', w_4, w_5, w_6, w_7 are given in (6), (7), (11), (12), (13), (14) and (15).

SKETCH OF THE PROOF. Replacing the system of generators (1) with the system (5), the equations (8) and (9) insure that the elements (17) are generators of the algebra C_{32}. The relation (18) is checked with the help of Maple, where x and y are as in (4).

Let us consider the polynomial algebra P_{11} in the 11 variables (17) written in the form $P_{11}=(S[\mathrm{tr}(X),\mathrm{tr}(Y)])[w]$, where

$$S=K[\mathrm{tr}(x^2),\mathrm{tr}(xy),\mathrm{tr}(y^2),\mathrm{tr}(x^3),\mathrm{tr}(x^2y),\mathrm{tr}(xy^2),\mathrm{tr}(y^3),v],\tag{19}$$

and the principal ideal $I=(f)$ of P_{11} generated by the element f from (18). The factor algebra P_{11}/I is a graded free $S[\mathrm{tr}(X),\mathrm{tr}(Y)]$-module, generated by 1 and w. Its Hilbert series is

$$\frac{1+t_1^3t_2^3}{(1-t_1)(1-t_2)(1-t_1^2)(1-t_1t_2)(1-t_2^2)(1-t_1^3)(1-t_1^2t_2)(1-t_1t_2^2)(1-t_2^3)(1-t_1^2t_2^2)},$$

and is equal to the Hilbert series (2) of the algebra C_{32}. The algebra C_{32} is a homomorphic image of P_{11}/I and the coefficients of its Hilbert series are bounded from above by the coefficients of the Hilbert series of P_{11}/I. Since the two Hilbert series coincide, the algebras C_{32} and P_{11}/I are isomorphic and this completes the proof.

Acknowledgements. This project was started when the second author visited the Department of Mathematics of the National University of Singapore. He is very grateful for the hospitality and the creative atmosphere during his stay in Singapore.

REFERENCES

[1] ABEASIS S., M. PITTALUGA. Commun. Algebra, **17**, 1989, 487–499. [2] ASLAKSEN H., V. DRENSKY, L. SADIKOVA. http://xxx.lanl.gov/abs/math.RA/0405389. [3] BENANTI F., V. DRENSKY. http://xxx.lanl.gov/abs/math.RA/0501219. [4] DRENSKY V. Algebras and Representation Theory, **6**, 2003, 193–214. [5] DRENSKY V., E. FORMANEK. Polynomial Identity Rings, Advanced Courses in Mathematics, CRM Barcelona, Birkhäuser, Basel–Boston, 2004. [6] FORMANEK E. J. Algebra, **89**, 1984, 178–223. [7] Id. The Polynomial Identities and Invariants of $n\times n$ Matrices, CBMS Regional Conf. Series in Math., **78**, Published for the

Confer. Board of the Math. Sci. Washington DC, AMS, Providence RI, 1991. [8] Nakamoto K. J. Pure Appl. Algebra, **166**, 2002, No 1–2, 125–148. [9] Revoy P. Compt. rend. Acad. Sci. Paris, Sér. I, **323**, 1996, No 1, 1–6. [10] Sibirskii K. S. Sib. Mat. Zh., **9**, 1968, No 1, 152–164. Translation: Siber. Math. J., **9**, 1968, 115–124. [11] Teranishi Y. Nagoya Math. J., **104**, 1986, 149–161.

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