On Certain Rings of Highest Weight Vectors

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Let \( R_{m,n} \) be the ring of highest weight vectors of the action of \( O_m \times GL_n \) on the polynomial algebra of \( m \times n \) matrices. We determine \( R_{m,2} \) and find generators for \( R_{m,n} \). In particular, the results about \( R_{m,2} \) give branching rules and information about the structure of holomorphic representations of \( Sp_{4} \).

1. Introduction

Let \( \mathbb{C}^{m,n} \) be the vector space of \( m \times n \) complex matrices and let \( \mathcal{A}(\mathbb{C}^{m,n}) \) be the algebra of complex-valued polynomials on \( \mathbb{C}^{m,n} \). Let \( GL_m \times GL_n \) act on \( \mathcal{A}(\mathbb{C}^{m,n}) \) by pre- and post-multiplication as

\[
(g_1, g_2) f(x) = f(g_1^{-1} x g_2),
\]

where \( x \in \mathbb{C}^{m,n}, (g_1, g_2) \in GL_m \times GL_n, f \in \mathcal{A}(\mathbb{C}^{m,n}) \). We choose a system of coordinates on \( \mathbb{C}^{m,n} \) as follows

\[
\begin{pmatrix}
x_{11} & x_{12} & \cdots & x_{1n} \\
x_{21} & x_{22} & \cdots & x_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
x_{m1} & x_{m2} & \cdots & x_{mn}
\end{pmatrix}
\]

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It is easy to describe the infinitesimal action of the Lie algebra $\mathfrak{gl}_m \oplus \mathfrak{gl}_n$ of $GL_m \times GL_n$ on $\mathcal{P}(\mathbb{C}^{m,n})$. Let

$$L_{jk} = \sum_{s=1}^{n} x_{js} \frac{\partial}{\partial x_{ks}}, \quad 1 \leq j, k \leq m,$$

$$R_{jk} = \sum_{s=1}^{m} x_{sj} \frac{\partial}{\partial x_{sk}}, \quad 1 \leq j, k \leq n. \quad (1.1)$$

Then

$$\mathfrak{gl}_m = \text{span}\{L_{jk}, 1 \leq j, k \leq m\},$$

$$\mathfrak{gl}_n = \text{span}\{R_{jk}, 1 \leq j, k \leq n\}.$$  

We have the Cartan decomposition

$$\mathfrak{gl}_m = \mathfrak{o}_m \oplus \mathfrak{p},$$

where $\mathfrak{o}_m$ is the Lie algebra of $O_m$ sitting in $GL_m$ and

$$\mathfrak{o}_m = \text{span}\{L_{jk} - L_{kj}, 1 \leq j < k \leq m\},$$

$$\mathfrak{p} = \text{span}\{L_{jk} + L_{kj}, 1 \leq j \leq k \leq m\}. \quad (1.2)$$

Let $\pi$ be a rational representation of a complex reductive algebraic group $G$ on $V$. Let $B = TU$ be a Borel subgroup of $G$, where $T$ is a maximal torus and $U$ is the unipotent radical of $B$. Let $\mathcal{P}(V)^U$ be the $U$-invariants in $\mathcal{P}(V)$, the polynomial algebra over $V$. Now $T$ acts semisimply on $\mathcal{P}(V)^U$, and we have a $T$-weight space decomposition

$$\mathcal{P}(V)^U = \sum_{\psi \in T^*} \mathcal{P}(V)^U_\psi,$$

where

$$\mathcal{P}(V)^U_\psi = \{f \in \mathcal{P}(V) | f(b^{-1}x) = \psi(b)f(x), b \in B\}$$

and $\psi$ is a character of $B$ that is trivial on $U$ and satisfies

$$\psi(tu) = \psi(t), \quad t \in T, u \in U.$$  

The vectors in $\mathcal{P}(V)^U_\psi$ are the highest weight vectors (with respect to the positive system determined by the choice of $B$) of the representations of $G$ appearing in $\mathcal{P}(V)$. By abuse of terminology, we shall call $\mathcal{P}(V)^U$ the ring of $G$ highest weight vectors in $\mathcal{P}(V)$. 

Our problem can be stated as follows:

**Problem.** Describe the ring $R_{m,n}$ of $O_m \times GL_n$ highest weight vectors in $\mathcal{P}(\mathbb{C}^{m,n})$.

Let

$$\Delta_{jk} = \sum_{s=1}^{m} \frac{\partial^2}{\partial x_{sj} \partial x_{sk}}, \quad 1 \leq j \leq k \leq n$$

and

$$r_{jk}^2 = \sum_{s=1}^{m} x_{sj} x_{sk}, \quad 1 \leq j \leq k \leq n. \tag{1.3}$$

Define the space of harmonics to be

$$\mathcal{H} = \{ f \in \mathcal{P}(\mathbb{C}^{m,n}) \mid \Delta_{jk} f = 0, 1 \leq j \leq k \leq n \}, \quad \tag{1.4}$$

and let

$$\mathcal{J} = \mathcal{P}(\mathbb{C}^{m,n})^{O_m}$$

be the ring of $O_m$ invariants. It is known ([8]) that

$$\mathcal{P}(\mathbb{C}^{m,n}) = \mathcal{J} \cdot \mathcal{H}. \tag{1.5}$$

The description of $R_{m,1}$ for $m \geq 3$ follows easily from the theory of spherical harmonics. It says that $\mathcal{H}$ is $O_m$ stable and the $O_m$ highest weight vectors in $\mathcal{H}$ (with respect to an appropriate choice of a positive system of roots, which we will elaborate on in Section 3) are $z_1^k$ for $k = 0, 1, \ldots$, where

$$z_1 = x_{11} - i x_{21}.$$ 

Furthermore, $\mathcal{J}$ is generated by $r_{11}^2$ as an algebra. The ring of $O_m$ highest weight vectors in $\mathcal{H}(\mathbb{C}^{m,1})$ is thus

$$\mathbb{C}[z_1, r_{11}].$$

For $n > 1$, the $GL_n$-structure of $\mathcal{J}$, and in particular its $O_m \times GL_n$ highest weight vectors, are well known from classical invariant theory (see Theorem 3.1 in Section 3). In [8], Kashiwara and Vergne generalised the theory of spherical harmonics to give the set of $O_m \times GL_n$ highest weight vectors in the space $\mathcal{H}$. These give partial information about $R_{m,n}$, and to get a complete picture, one has to understand how the highest weight vectors multiply as in (1.5) above. This will be the main theme of this
paper. It is also known that $R_{m,n}$ is Cohen–Macaulay ([4]), but besides these facts, very little is known about the explicit structure of the ring.

In this article, we determine the ring $R_{m,2}$. Several applications are given. In particular, we describe the $GL_2$ structure of all holomorphic representations of $Sp_4$ and compute the Poincaré series, Gelfand–Kirillov dimension ([17]) and Berenstein degree ([17]) for these representations. We also find a set of generators for $R_{m,3}$. Beyond $n = 3$, the situation gets too complicated. This is understandable because $R_{m,n}$ contains, in particular, information about the restriction of $GL_m$ representations (with depth at most $\min(m, n)$) to its subgroup $O_m$, which is known to be a difficult problem. Finally, we consider the subring of $R_{m,n}$ consisting of elements that are right-$SL_n$-invariant (see Section 6), and give a set of explicitly given generators. The latter is connected with the work of Sato ([12]).

We would like to remark that Howe ([7]) has investigated related rings of highest weight vectors from certain models of representations of $GL_n$ for $n = 2, 3$ and 4. In fact, our work was motivated by a talk given by him, where he stated the result of Theorem 4.2(d). We also want to thank him for helpful discussions.

2. ORBIT STRUCTURE AND THE KRULL DIMENSION

Let $N_R$ be the maximal unipotent subgroup of $GL_n$ consisting of upper triangular matrices with ones on the diagonal. We shall choose a maximal unipotent subgroup $N_\ell$ of $O_m$. First, we discuss the case when $m = 2\ell + 1$. Choose a Borel subalgebra for $o_m$ as follows

$$\begin{pmatrix}
A & B & \delta \\
0 & -A' & 0 \\
0 & -\delta' & 0
\end{pmatrix},$$

where $A$ is an upper triangular $\ell \times \ell$ matrix, $B$ is an $\ell \times \ell$ skew-symmetric matrix and $\delta$ is an $\ell \times 1$ matrix. The maximal unipotent subgroup $N_\ell$ will then be generated by

(i)$\begin{pmatrix}
A & 0 & 0 \\
0 & (A')^{-1} & 0 \\
0 & 0 & 1
\end{pmatrix}$

where $A$ sits in the unipotent subgroup of upper triangular matrices with
ones along the diagonal in $GL_l$,  

$$
\begin{bmatrix}
I_l & B & 0 \\
0 & I_l & 0 \\
0 & 0 & 1
\end{bmatrix}
$$

(ii)  

where $B$ is an $l \times l$ skew-symmetric matrix, and $I_l$ is the $l \times l$ identity matrix,

$$
\begin{bmatrix}
I_l & -\frac{\delta \delta'}{2} & \delta \\
0 & I_l & 0 \\
0 & -\delta' & 1
\end{bmatrix}
$$

(iii)  

where $\delta$ is an $l \times 1$ matrix.  

If $m$ is even, $N_L$ is generated by elements of type (i) and (ii) with the last row and column removed. It is possible to determine the $N_L \times N_R$ orbit structure using matrix reduction by elementary matrices. We will omit this. Instead, we will describe a Zariski open set, and then deduce the following result.  

**Theorem 2.1.** The Krull dimension of the ring of $O_m \times GL_n$ highest weight vectors in $\mathcal{A}(C^{m \times n})$ is given

If $m = 1$,  

$$1$$

if $m = 2$ and $n \geq 2$,  

$$\frac{n(n+3)}{2}$$

if $\left\lceil \frac{m}{2} \right\rceil \geq n$,  

$$\frac{\lfloor m/2 \rfloor (\lfloor m/2 \rfloor + 3)}{2} + \left( \left\lceil \frac{m}{2} \right\rceil + 1 \right) \left( n - \left\lceil \frac{m}{2} \right\rceil \right)$$

if $n \geq \left\lceil \frac{m}{2} \right\rceil \geq 1$ and $m$ is odd,  

$$\frac{\lfloor m/2 \rfloor (\lfloor m/2 \rfloor + 3)}{2} + \left\lceil \frac{m}{2} \right\rceil \left( n - \left\lfloor \frac{m}{2} \right\rfloor \right)$$

if $n \geq \left\lceil \frac{m}{2} \right\rceil \geq 2$ and $m$ is even.

**Proof.** If $m = 2l + 1$ and $1 \leq l \leq n$, consider the $N_L \times N_R$ orbit $\Theta_E$ through

$$E = \begin{bmatrix}
U & M_1 \\
D & 0 \\
0 & M_2
\end{bmatrix},$$

(2.1)

where $U$ is an $l \times l$ upper triangular matrix, $D$ is an $l \times l$ diagonal matrix and $M_1$ and $M_2$ are arbitrary $l \times (n - l)$ and $1 \times (n - l)$ matrices. The
set of $E$’s forms an affine variety $\mathcal{E}$. The $\Theta_{E}$’s are distinct except for a
Zariski closed subset of $\mathcal{E}$, and the collection of $\Theta_{E}$’s is Zariski open in
$\mathbb{C}^{m,n}$. Therefore, the Krull dimension of $\mathcal{P}(\mathbb{C}^{m,n})^{N_{1} \times N_{2}}$ is equal to the
dimension of $\mathcal{E}$ as an affine variety. This dimension is equal to

$$\frac{l(l + 3)}{2} + (l + 1)(n - l).$$

If $m = 2l$ and $2 \leq l \leq n$, we simply remove the entries in the last row in
(2.1) to get a Zariski open set with parameter space of dimension

$$\frac{l(l + 3)}{2} + l(n - l).$$

If $\lceil m/2 \rceil > n$, the collection of orbits through

$$\begin{bmatrix}
U \\
0 \\
D \\
0
\end{bmatrix},$$

where $U$ is $n \times n$ upper triangular and $D$ is $n \times n$ diagonal with nonzero
diagonal entries, will be Zariski open. This parameter space is of dimension

$$\frac{n(n + 3)}{2}.$$ 

The remaining cases are treated similarly.

We will first present an elementary result. Recall that we can parameterize polynomial representations of $GL_{n}$ by Young diagrams
$(a_{1}, a_{2}, \ldots, a_{n})$ with depth at most $n$.

**Proposition 2.2.** (a) $R_{1,n}$ is the polynomial ring in $x_{11}$.
(b) $R_{2,1}$ is the polynomial ring in $x_{11}$ and $x_{21}$.
(c) If $n \geq 2$, $R_{2,n}$ is freely generated by $x_{11}$, $x_{21}$ and

$$\begin{bmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{bmatrix}.$$

**Proof.** Parts (a) and (b) are easy. We will prove (c). In fact,

$$R_{2,n} = \mathcal{P}(\mathbb{C}^{2,n})^{N_{1} \times N_{2}}.$$
since $N_L$ is trivial. From the $GL_2 \times GL_n$ duality ([6]),

$$\mathcal{A}(\mathbb{C}^{2,n})|_{GL_2 \times GL_n} = \sum_{\sigma} V_\sigma \otimes \tilde{V}_\sigma,$$

where $\sigma = (a, b)$ is a Young diagram with depth at most 2 and $V_\sigma$ (respectively $\tilde{V}_\sigma$) is the $GL_2$ (respectively $GL_n$) representation determined by the Young diagram $\sigma$. If $v \in \mathcal{A}(\mathbb{C}^{2,n})$ is a $GL_n$ highest weight vector of type determined by the Young diagram $\sigma$, it must generate a $GL_2 \times GL_n$ module sitting in $V_\sigma \otimes \tilde{V}_\sigma$. Since the latter is irreducible, it is the whole space. If the Young diagram is $\sigma = (a, b)$, the $GL_2 \times GL_n$ highest weight vector in $V_\sigma \otimes \tilde{V}_\sigma$ is, up to a constant,

$$w = x_{11}^a x_{22}^b \begin{vmatrix} x_{11}^{12} & x_{12}^b \\ x_{21} & x_{22} \end{vmatrix}.$$

Thus,

$$(GL_2 \times GL_n) \cdot v = (GL_2 \times GL_n) \cdot w.$$  \hspace{1cm} (2.2)

Now, all the $GL_n$ highest weight vectors in $(GL_2 \times GL_n) \cdot w$ lie in the space $GL_2 \cdot w$, and so $v$ must lie in it, too. Thus any $GL_n$ highest weight vector lies in the $GL_2$ module generated by some polynomial (2.2). It is then not difficult to see that $R_{2,n}$ is freely generated by the 3 polynomials as stated above. □

Actually, it is quite interesting to describe the ring $\mathcal{A}(\mathbb{C}^{m,n})^{N\mathbb{H}}$, and we hope to take it up elsewhere.

3. Harmonic Polynomials

We recall some results from Kashiwara and Vergne ([8]). We choose $O_m$ so that its Lie algebra $\mathfrak{o}_m$ is described by (1.2). Now choose a Cartan subalgebra (different from the one in Section 2) for $\mathfrak{o}_m$ as follows

$$\mathfrak{h}_L = \left\{ H = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \end{bmatrix} , \quad A_j = \begin{bmatrix} 0 & i\hbar_j \\ -i\hbar_j & 0 \end{bmatrix}, \quad h_j \in \mathbb{C}, 1 \leq j \leq \left[ \frac{m}{2} \right] \right\}$$

\hspace{1cm} (3.1)
and also a Cartan subalgebra for $\mathfrak{gl}_n$

$$\mathfrak{h}_R = \left\{ \begin{bmatrix} d_1 & & \\ 0 & \ddots & 0 \\ & & d_n \end{bmatrix} \mid d_j \in \mathbb{C} \right\}.$$  

(3.2)

Define the following linear functionals on $\mathfrak{h}_L$

$$e_j(\text{above } H) = h_j, \quad 1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor.$$  

(3.3)

The roots of $\vartheta_m$ are

$$\Delta = \begin{cases} \{ \pm e_j \pm e_k, j \neq k \} \cup \{ \pm e_k \} & \text{if } m \text{ is odd}, \\ \{ \pm e_j \pm e_k, j \neq k \} & \text{if } m \text{ is even}. \end{cases}$$  

(3.4)

We choose a positive system as follows

$$\Delta^+ = \begin{cases} \{ e_j \pm e_k, j \neq k, j < k \} \cup \{ e_k \} & \text{if } m \text{ is odd}, \\ \{ e_j \pm e_k, j \neq k, j < k \} & \text{if } m \text{ is even}. \end{cases}$$  

(3.5)

For $GL_n$, we choose the Borel subalgebra of upper triangular matrices to fix a positive root system for $\mathfrak{gl}_n$.

We will parameterize the irreducible finite-dimensional representations of $SO_m$ by tuples of integers $(\alpha_1, \alpha_2, \ldots, \alpha_{\lfloor m/2 \rfloor})$ satisfying

$$\begin{cases} \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_{\lfloor m/2 \rfloor} \geq 0 & \text{if } m \text{ is odd}, \\ \alpha_1 \geq \alpha_2 \geq \cdots \geq |\alpha_{\lfloor m/2 \rfloor}| & \text{if } m \text{ is even}. \end{cases}$$

and parameterize the irreducible finite-dimensional representation of $GL_n$ by an $n$-tuple of integers $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$.

If $m = 2l + 1$, then $O_{2l+1}$ is the direct product $SO_{2l+1} \times \mathbb{Z}_2$ where $\mathbb{Z}_2 = \{ \pm 1 \}$. A representation of $O_{2l+1}$ can be parameterized by

$$(\alpha_1, \ldots, \alpha_l, \varepsilon) = (\alpha_1, \ldots, \alpha_l) \otimes \varepsilon,$$

where $(\alpha_1, \ldots, \alpha_l) \in SO_{2l+1}$ and $\varepsilon$ is a one-dimensional representation of $\mathbb{Z}_2$, trivial or nontrivial according to $\varepsilon = 1$ or $-1$. 


If \( m = 2l \), then \( O_{2l} \) is the semi-direct product \( SO_{2l} \rtimes \mathbb{Z}_2 \), where \( \mathbb{Z}_2 = \langle J_{2l}, I_{2l} \rangle \) with

\[
J_{2l} = \begin{bmatrix}
I_{2(l-1)} & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}.
\]

If \( (\alpha_1, \ldots, \alpha_l) \in SO_{2l} \) and \( \alpha_l \neq 0 \), then \( \text{Ind}_{SO_{2l}}^{O_{2l}}(\alpha_1, \ldots, \alpha_l) \) is irreducible and we denote this by \( (\alpha_1, \ldots, \alpha_l; 1) \). If \( \alpha_l = 0 \), then there are two ways to extend \( (\alpha_1, \ldots, \alpha_l) \) to a representation of \( O_{2l} \), namely

\[
(\alpha_1, \ldots, \alpha_l; \epsilon) = (\alpha_1, \ldots, \alpha_l) \otimes \epsilon,
\]

where \( \epsilon = \pm 1 \). A representation of \( O_{2l} \) can then be parameterized by

\[
(\alpha_1, \ldots, \alpha_l; 1) \quad \text{if} \quad \alpha_l \neq 0,
\]

or

\[
(\alpha_1, \ldots, \alpha_l; \pm 1) \quad \text{if} \quad \alpha_l = 0.
\]

We will denote the two representations of \( O(1) \) by \( (\pm 1) \).

Now set

\[
z_{jk} = x_{2j-1,k} - ix_{2j,k} \tag{3.6}
\]

for \( 1 \leq j \leq \lfloor m/2 \rfloor \), \( 1 \leq k \leq n \), and define the following \( t \times t \) determinants for \( 1 \leq t \leq \min(m/2, n) \)

\[
\alpha_t = \begin{vmatrix}
z_{11} & z_{12} & \cdots & z_{1t} \\
\vdots & \vdots & \ddots & \vdots \\
z_{t1} & z_{t2} & \cdots & z_{tt}
\end{vmatrix}, \tag{3.7}
\]

\[
\gamma_t = \begin{vmatrix}
r_{11}^2 & r_{12}^2 & \cdots & r_{1t}^2 \\
\vdots & \vdots & \ddots & \vdots \\
r_{t1}^2 & r_{t2}^2 & \cdots & r_{tt}^2
\end{vmatrix}. \tag{3.8}
\]

It is not difficult to check that with respect to the positive systems for \( \mathfrak{o}_m \) and \( \mathfrak{gl}_n \), each \( \alpha_t \) is an \( O_m \times GL_n \) highest weight vector of weight

\[
\left(1, \ldots, 1, 0, \ldots, 0; (-1)^t\right) \otimes \left(1, \ldots, 1, 0, \ldots, 0\right)
\]

\( t \) copies \hspace{1cm} \( t \) copies
when \( m \) is odd, and an \( O_m \times GL_n \) highest weight vector of weight
\[
\left( \underbrace{1, \ldots, 1}_{t \text{ copies}}, 0, \ldots, 0, 1 \right) \otimes \left( \underbrace{1, \ldots, 1}_{t \text{ copies}}, 0, \ldots, 0 \right)
\]
when \( m \) is even. Similarly, \( \gamma_1 \) is an \( O_m \times GL_n \) highest weight vector of weight
\[
\left( 0, \ldots, 0; 1 \right) \otimes \left( \underbrace{2, \ldots, 2}_{t \text{ copies}}, 0, \ldots, 0 \right).
\]
For \( m = 3 \) and \( n = 2 \), we will also need
\[
\delta = \begin{bmatrix}
  z_{11} & z_{12} \\
  x_{31} & x_{32}
\end{bmatrix},
\]
which is an \( O_3 \times GL_2 \) highest weight vector of weight
\[
(1; 1) \otimes (1, 1).
\]
For \( m = 2 \) and \( n = 2 \), we will also need
\[
\zeta = \begin{bmatrix}
  z_{11} & z_{12} \\
  x_{21} & x_{22}
\end{bmatrix} = 2i \begin{bmatrix}
  x_{11} & x_{12} \\
  x_{21} & x_{22}
\end{bmatrix},
\]
which is an \( O_2 \times GL_2 \) highest weight vector of weight
\[
(0; -1) \otimes (1, 1).
\]
For \( m = 1 \) and \( n = 2 \), we will also need
\[
\omega = x_{11}
\]
which is an \( O_1 \times GL_2 \) highest weight vector of weight
\[
(-1) \otimes (1, 0).
\]
The following results are well known, and can be found in [8, 15, 2, 18].

**Theorem 3.1.** (a) The space of \( O_m \)-invariant polynomials in \( \mathcal{P}(C^{m,n}) \) is generated by \( r^i_j \), \( 1 \leq j \leq k \leq n \). The subring of \( O_m \times GL_n \) highest weight vectors in \( \mathcal{P} \) is generated freely by
\[
\gamma_1, \ldots, \gamma_{\min(m/2,n)}.
\]
(b) When \( m \geq 2n \), the ring of \( O_m \times GL_n \) highest weight vectors in \( \mathcal{P} \) is generated freely by
\[
\alpha_1, \ldots, \alpha_n.
\]
(c) When \( m = 3 \) and \( n = 2 \), the ring of \( \mathfrak{O}_m \times GL_n \) highest weight vectors in \( \mathcal{H} \) is spanned by \( \alpha_i^p \) and \( \alpha_i^p \delta \) for \( p \geq 0 \).

(d) \( \mathcal{H}(C^{m,n}) = \mathcal{H} \cdot \mathcal{I} \), and in particular, for \( m \geq 2n \), we have \( \mathcal{H}(C^{m,n}) = \mathcal{H} \oplus \mathcal{I} \).

Remarks. The results of [8] give generators for \( \mathfrak{O}_m \times GL_n \) highest weights in \( \mathcal{H} \) for all \( m \) and \( n \). When \( m < 2n \), we get additional generators (as in the case \( m = 3 \) and \( n = 2 \)).

4. Structure of \( R_{m,2} \) and Applications

4.1. Structure of \( R_{m,2} \)

We have already described \( R_{1,2} \) and \( R_{2,2} \). We will now treat the case when \( n = 2 \) and \( m \geq 4 \). First, we have a simple result about tensor products of \( GL_2 \) representations. Let

\[ gI_2 = \text{span}\{ R_{11}, R_{22}, R_{12}, R_{21} \}, \]

with the usual commutation relations

\[ [R_{ij}, R_{st}] = \delta_{is} R_{ti} - \delta_{it} R_{sj}. \]

We will select \( R_{12} \) to be the positive root vector in \( gI_2 \). As usual, if \( V \) is a representation of \( GL_2 \), we say that \( v \in V \) is a \( GL_2 \) highest weight vector if \( R_{12}v = 0 \) and \( v \) is an eigenvector for \( R_{11} \) and \( R_{22} \).

Lemma 4.1. Let \( V_{(\lambda_1, \lambda_2)} \) be an irreducible finite-dimensional representation of \( GL_2 \) with highest weight \((\lambda_1, \lambda_2)\). Then

\[ V_{(\lambda_1, \lambda_2)} \otimes V_{(1,0)} = \begin{cases} V_{(\lambda_1 + 1, \lambda_2)} \oplus V_{(\lambda_1, \lambda_2 + 1)} & \text{if } \lambda_1 \neq \lambda_2, \\ V_{(\lambda_1 + 1, \lambda_2)} & \text{if } \lambda_1 = \lambda_2, \end{cases} \]

and if \( \lambda_1 \neq \lambda_2 \), the highest weight vector of \( V_{(\lambda_1, \lambda_2 + 1)} \) is given (up to a scalar) by

\[ v = R_{21}v_{(\lambda_1, \lambda_2)} \otimes v_{(1,0)} - (\lambda_1 - \lambda_2)v_{(\lambda_1, \lambda_2)} \otimes R_{21}v_{(1,0)}, \]

where \( v_{(\lambda_1, \lambda_2)} \) and \( v_{(1,0)} \) are the highest weight vectors in \( V_{(\lambda_1, \lambda_2)} \) and \( V_{(1,0)} \), respectively.

Proof. That there are at most 2 components in the tensor product follows from the Clebsch–Gordan Formula. To find the highest weight
vector for $V_{(\lambda_1, \lambda_2 + 1)}$, set
\[ v = R_{21} v_{(\lambda_1, \lambda_2)} \otimes v_{(1, 0)} + a v_{(\lambda_1, \lambda_2)} \otimes R_{21} v_{(1, 0)}. \]

We only have to set $R_{12} v = 0$, which gives $a = -(\lambda_1 - \lambda_2)$. \[ \]

From now on, we restrict our discussion to vectors in $\mathcal{P}(\mathbb{C}^{m,n})$. Consider the map
\[ \Phi: \mathcal{F} \otimes \mathcal{S} \rightarrow \mathcal{P}(\mathbb{C}^{m,n}) \]
by polynomial multiplication. Theorem 3.1(d) says that the map is onto. To find the $O_n \times GL_n$ representations appearing in $\mathcal{P}(\mathbb{C}^{m,n})$, consider $1 \otimes \sigma_1$ appearing in $\mathcal{F}$ and $\rho \otimes \sigma_2$ appearing in $\mathcal{S}$. Since
\[(1 \otimes \sigma_1) \otimes (\rho \otimes \sigma_2) = \rho \otimes (\sigma_1 \otimes \sigma_2) \]
as $O_n \times GL_n$ representations, it suffices to understand the projection of $\rho \otimes \sigma$ under $\Phi$, where $\sigma$ is a $GL_n$ subrepresentation of $\sigma_1 \otimes \sigma_2$. More precisely, to find the $O_n \times GL_n$ highest weights in $\mathcal{P}(\mathbb{C}^{m,n})$, it suffices to find the $GL_n$ highest weights in $\rho \otimes (\sigma_1 \otimes \sigma_2)$.

Let us work in $\mathcal{P}(\mathbb{C}^{m,n})$. Let $\langle f \rangle$ denote the $GL_2$ module in $\mathcal{P}(\mathbb{C}^{m,n})$ generated by the polynomial $f$. From the discussion above, it follows that we must find the $GL_2$ highest weight vectors in the tensor product
\[ \langle \alpha_1^x \alpha_2^y \rangle \otimes \langle \gamma_1^x \gamma_2^y \rangle. \]
Since $\gamma_2$ and $\alpha_2$ each generate a one-dimensional representation of $GL_2$, it suffices to consider
\[ \langle \alpha_1^x \rangle \otimes \langle \gamma_1^x \rangle. \]

Observe from Lemma 4.1, that if $v_1$ and $v_2$ are two $GL_2$ highest weight vectors of weights $(\lambda_1, \lambda_2)$ and $(\lambda_1', \lambda_2')$, then the highest weight vectors of
\[ \langle \alpha_1 \rangle \otimes \langle v_1 v_2 \rangle \]
are $\alpha_1 v_1 v_2$ and
\[ (R_{21}(v_1 v_2)) \alpha_1 - (\lambda_1 + \lambda_1' - \lambda_2 - \lambda_2') v_1 v_2 (R_{21} \alpha_1) \]
\[ = \left[ (R_{21} v_1) \alpha_1 - (\lambda_1 - \lambda_2) v_1 (R_{21} \alpha_1) \right] v_2 \]
\[ + \left[ (R_{21} v_2) \alpha_1 - (\lambda_1' - \lambda_2') v_2 (R_{21} \alpha_1) \right] v_1. \]
(4.1)
By applying Lemma 4.1 to \( \langle \alpha_1 \rangle \otimes \langle \gamma_1 \rangle \), we get

\[
\beta_1 = \begin{vmatrix}
z_{11} & z_{12} \\
r^2_{11} & r^2_{12}
\end{vmatrix}
\]

which has \( SO_m \times GL_2 \) weight \((1,0,\ldots,0) \otimes (2,1)\). Another application of Lemma 4.1 to \( \langle \alpha_1 \rangle \otimes \langle \beta_1 \rangle \) gives

\[
\beta_2 = \begin{vmatrix}
0 & z_{11} \\
z_{11} & r^2_{11} \\
z_{12} & r^2_{21}
\end{vmatrix},
\]

which has \( SO_m \times GL_2 \) weight \((2,0,\ldots,0) \otimes (2,2)\). Since \( \beta_2 \) transforms under \( GL_2 \) by the determinant representation, we have

\[
\langle \alpha_1 \rangle \otimes \langle \beta_2 \rangle = \langle \alpha_1 \beta_2 \rangle.
\]

The relation

\[
\alpha_1^2 \gamma_2 + \beta_1^2 + \gamma_1 \beta_2 = 0
\]

follows from a direct computation.

Recall the definition of \( \delta, \xi, \) and \( \omega \) in Section 3 (see (3.9), (3.10), and (3.11)).

**Theorem 4.2.** (a) \( R_{1,2} \) is the polynomial ring in \( \omega = x_{11} \).

(b) \( R_{2,2} \) is freely generated by \( \alpha_1, \overline{\alpha_1}, \) and \( \xi \).

(c) Let \( S \) be the ring generated by \( \alpha_1, \beta_1, \beta_2, \gamma_1 \) and \( \gamma_2 \) with the relation

\[
\alpha_1^2 \gamma_2 + \beta_1^2 + \gamma_1 \beta_2 = 0. \quad (4.2)
\]

Then \( R_{3,2} = S \oplus S \delta \).

(d) If \( m \geq 4 \), \( R_{m,2} \) is generated by \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \) and \( \gamma_2 \) with the relation (4.2).

**Proof:** We have shown (a) and (b) in Section 2. We will first prove (d). Because

\[
\langle \alpha_1^{n} \rangle \otimes \langle \gamma_1^{n} \rangle \hookrightarrow \langle \alpha_1 \rangle \otimes \langle \alpha_1^{-1} \rangle \otimes \langle \gamma_1 \rangle,
\]

we can use Eq. (4.1) and a straightforward induction to conclude that \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \) and \( \gamma_2 \) generate \( R_{m,2} \). By Theorem 2.1, the Krull dimension of \( R_{m,2} \) is 5. (It is simple to see that \( \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \) and \( \gamma_2 \) are algebraically independent.) The ideal of relations among \( \alpha_1, \alpha_2, \beta_1, \beta_2, \)
γ₁, and γ₂ will therefore have to be a principal ideal. Since
\[ α₁^2γ₂ + β₁^2 + γ₁β₂ \]
is an irreducible polynomial in the variables α₁, α₂, β₁, β₂, γ₁, γ₂, it generates this ideal. The proof of (c) is similar.

**Corollary 4.3.** The ring of \( O_m × \text{GL}_2 \) highest weight vectors in \( ℙ(ℂ^{m,2}) \) is Cohen–Macaulay and its Poincaré series, \( P(t) \), is as follows

\[
P(t) = \begin{cases} 
1 + t^3 & \text{for } m \geq 4, \\
\frac{(1 - t)^2(1 - t^2)^2}{(1 - t^3)^2} & \text{for } m = 3, \\
\frac{1}{(1 - t)^3(1 - t^2)} & \text{for } m = 2, \\
\frac{1}{1 - t} & \text{for } m = 1.
\end{cases}
\]

**Remark.** That the ring \( R_{m,n} \) is Cohen–Macaulay is a special case of a well-known result of Hochschild and Mostow (14).

**Proof.** Immediate from Theorem 4.2.

4.2. A Branching Rule from \( \text{GL}_m \) to \( O_m \)

A consequence of Theorem 4.2 is a branching law for \( O_m \) representations appearing in a \( GL_m \) representation of depth at most two. Since the result is probably well known for \( m = 2 \) or 3, we will only state it for \( m \geq 4 \).

**Corollary 4.4.** We have the following formulae for the restriction of a representation \( (a, b, 0, \ldots, 0) \) of \( GL_m \) to \( O_m \) for \( m \geq 4 \).

\[
(a, b, 0, \ldots, 0)|_{O_m} = \sum_{s \in S} (x_s, y_s, 0, \ldots, 0; (-1)^s),
\]

where

\[
S = \{(a_1, a_2, b_1, b_2, c_1, c_2) ∈ ℤ^6_+ \mid a_1 + a_2 + 2b_1 + 2b_2 + 2c_1 + 2c_2 = a, \]
\[
a_2 + b_1 + 2b_2 + 2c_2 = b, \quad b_1 = 0, 1\}.
\]
and for \( s \in S \), we have

\[
x_s = a_1 + a_2 + b_1 + 2b_2, \quad y_s = a_2, \quad \varepsilon_s = \begin{cases} a_1 + b_1 & \text{if } m \text{ is odd}, \\ 0 & \text{otherwise}. \end{cases}
\]

**Proof.** This follows from the \( GL_m \times GL_2 \) duality ([16])

\[
\mathcal{P}(C^{m,2})|_{GL_m \times GL_2} = \sum V_\sigma \otimes \tilde{V}_\sigma,
\]

where \( \sigma \) is a Young diagram with at most 2 rows and \( V_\sigma \) (respectively \( \tilde{V}_\sigma \)) is a \( GL_m \) (respectively \( GL_2 \)) representation indexed by the Young diagram \( \sigma \). The \( O_m \) highest weights in \( V_\sigma \) are those that transform under \( GL_2 \) by a representation of type \( \tilde{V}_\sigma \). The result now follows from the description of the ring of \( O_m \times GL_2 \) highest weights in Theorem 4.2.

4.3. Poincaré Series for Highest Weight Modules of \( Sp_4 \)

Since \( \mathcal{P}(C^{m,2}) \) is the Fock space for the dual pair \( (O_m, Sp_4) \) ([3]), we have

\[
\mathcal{P}(C^{m,2}) = \sum_{\tau \in O_m} \tau \otimes W_\tau,
\]

where the \( W_\tau \) are certain highest weight modules (or holomorphic representations) of the metaplectic cover of \( Sp_4 \), denoted by \( \widetilde{Sp}_4 \). (When \( m \) is even, this factors through to give a representation of \( Sp_4 \). Since this technicality will not affect our later discussion of Poincaré series and highest weight vectors, we will ignore this distinction and simply write \( Sp_4 \).) The space \( W_\tau \) admits a grading induced from the usual grading on \( \mathcal{P}(C^{m,2}) \), so it makes sense to define the Poincaré series of \( W_\tau \) as

\[
P_{W_\tau}(t) = \sum \dim V_{(\lambda_1, \lambda_2)} t^{\lambda_1 + \lambda_2},
\]

where the sum is over all \( GL_2 \)-types \( V_{(\lambda_1, \lambda_2)} \) in \( W_\tau \). It is well known ([3]) that all holomorphic representations of \( Sp_4 \) arise through the duality correspondence with some representation of \( O_m \). In the following, we will describe the \( GL_2 \) structures of these representations and compute their Poincaré series.
Proposition 4.5. Let \( \tau \in \mathcal{O}_m \) be defined as
\[
\tau = \begin{cases} 
(x, y, 0, \ldots, 0; (-1)^r) & \text{if } m \geq 4, \\
(x; (-1)^r) & \text{if } m = 3 \text{ or } 2, \\
((-1)^r) & \text{if } m = 1.
\end{cases}
\] \hspace{1cm} (4.4)

The representation \( W_\tau \) of \( \text{Sp}_4 \) occurs in the duality correspondence
\[
\mathcal{P}(\mathbb{C}^{m,2}) = \sum_{\tau \in \mathcal{O}_m} \tau \otimes W_\tau
\]
if and only if
\[
\varepsilon = \begin{cases} 
(x - y \text{ mod } 2) & \text{if } m \geq 5 \text{ is odd}, \\
0 \text{ mod } 2 & \text{if } m \geq 4 \text{ is even}, \\
0 \text{ or } 1 \text{ mod } 2 & \text{if } m = 3 \text{ and } x \neq 0, \\
0 \text{ mod } 2 & \text{if } m = 3 \text{ and } x = 0, \\
0 \text{ mod } 2 & \text{if } m = 2 \text{ and } x \neq 0, \\
0 \text{ or } 1 \text{ mod } 2 & \text{if } m = 2 \text{ and } x = 0, \\
0 \text{ or } 1 \text{ mod } 2 & \text{if } m = 1.
\end{cases}
\] \hspace{1cm} (4.5)

Proof. An important tool here is Howe’s reciprocity formula ([5])
\[
m(V_{(a,b), W_\tau}) = m(\tau, (a, b, 0, \ldots, 0)),
\]
where \( m(V_{(a,b), W_\tau}) \) is the multiplicity of \( V_{(a,b)} \in \text{GL}_2 \) in \( W_\tau \) and \( m(\tau, (a, b, 0, \ldots, 0)) \) is the multiplicity of \( \tau \in \mathcal{O}_m \) in \( (a, b, 0, \ldots, 0) \in \text{GL}_m \).
Assume that \( m \geq 4 \) and let \( \tau = (x, y, 0, \ldots, 0; (-1)^r) \in \mathcal{O}_m \). The vector space of \( \mathcal{O}_m \times \text{GL}_2 \) highest weights of fixed \( \mathcal{O}_m \) weight \( \tau \) in \( \mathcal{P}(\mathbb{C}^{m,2}) \) is a \( \mathbb{C}[\gamma_1, \gamma_2] \) module. It is spanned by vectors
\[
\alpha_1^{a_1} \alpha_2^{a_2} \beta_1^{b_1} \beta_2^{b_2} \gamma_1^{c_1} \gamma_2^{c_2},
\]
where \( a_1, a_2, b_1, b_2, c_1, c_2 \) are positive integers satisfying
\[
x = a_1 + a_2 + b_1 + 2b_2, \\
y = a_2,
\]
\[
\varepsilon = \begin{cases} 
a_1 + b_1 \text{ mod } 2 & \text{if } m \text{ is odd}, \\
0 \text{ mod } 2 & \text{if } m \text{ is even},
\end{cases}
\] \hspace{1cm} (4.6)
\[
b_1 = 0 \text{ or } 1.
\]
The $GL_2$-types of $W_z$ are parameterized by $(a, b) \in \overline{GL}_2$ given by
\[
\begin{align*}
a &= a_1 + a_2 + 2b_1 + 2b_2 + 2c_1 + 2c_2, \\
b &= a_2 + b_1 + 2b_2 + 2c_2,
\end{align*}
\]
(4.7)
where $(a_1, a_2, b_1, b_2, c_1, c_2)$ satisfies (4.6). It is easy to see that solutions
$(a_1, a_2, b_1, b_2, c_1, c_2) \in \mathbb{Z}_+^6$ of equations (4.6) and (4.7) exist if and only if
\[
e = a_1 + b_1 = x - y + 2b_2 \equiv x - y \mod 2
\]
if $m$ is odd and $e \equiv 0 \mod 2$ if $m$ is even. The proofs for $m = 3, 2$ or 1 are similar. \[\Box\]

**Proposition 4.6.** Suppose that $\tau \in \overline{O}_m$ satisfies (4.4) and (4.5) in Proposition 4.5. The Poincaré series of $W_z$ is then
\[
P_{W_z}(t) = \begin{cases} 
\frac{(1 + x - y)t^{x+y}}{(1 - t^2)^3} & \text{if } m \geq 4, \\
\frac{(1 + x)t^x}{(1 - t^2)^3} & \text{if } m = 3, x \equiv 0 \mod 2 \text{ and } e \equiv 0 \mod 2, \\
\frac{xt^{x+1}}{(1 - t^2)^3} & \text{if } m = 3, x \equiv 1 \mod 2 \text{ and } e \equiv 0 \mod 2, \\
\frac{(1 + x)t^x - t^{2x+1}}{(1 - t^2)^3} & \text{if } m = 3, x \equiv 1 \mod 2 \text{ and } e \equiv 1 \mod 2, \\
\frac{(1 + |x| + (1 - |x|)t^2)t^{1+|x|}}{(1 - t^2)^3} & \text{if } m = 2 \text{ and } x \neq 0, \\
\frac{1}{(1 - t^2)^3} & \text{if } m = 2, x = 0 \text{ and } e \equiv 0 \mod 2, \\
\frac{t^2}{(1 - t^2)^3} & \text{if } m = 2, x = 0 \text{ and } e \equiv 1 \mod 2, \\
\frac{1 + t^2}{(1 - t^2)^2} & \text{if } m = 1 \text{ and } e \equiv 0 \mod 2, \\
\frac{2t}{(1 - t^2)^3} & \text{if } m = 1 \text{ and } e \equiv 1 \mod 2.
\end{cases}
\]
Remark. It is known (for \( m \geq 4 \)) that these representations of \( Sp_4 \) are the holomorphic discrete series representations or their limits (\( \mathbb{L} \)), and their \( GL_2 \) structure is simply the tensor product of \( \mathbb{F} = W_0 \) with the lowest \( GL_2 \)-type in \( W_\tau \). Using this fact, the Poincaré series could be derived from knowing the lowest \( GL_2 \)-type. Here we compute the series without appealing to this fact.

Proof. We will give the proof for \( m \geq 4 \). The proof for the remaining cases are similar, so we will leave out the details. If \( m \geq 4 \) and \( \tau = (x, y, 0, \ldots, 0; (-1)^n) \) with \( \epsilon \) as above, then the \( GL_2 \) highest weight vectors of \( W_\tau \) are of the form

\[
\alpha_1^{x-y-2b} \alpha_2^{2b} \beta_2^{y \gamma_1 \gamma_2^c} \quad \text{or} \quad \alpha_1^{x-y-2b-1} \alpha_2^{2b} \beta_1 \beta_2^{y \gamma_1 \gamma_2^c}.
\]

Note that if \( \tau = 0 \) is the trivial representation, then the \( GL_2 \) highest weight vectors of \( W_0 = \mathcal{H}(\mathbb{C}^{m+2})^{O_m} \) are \( \gamma_1 \gamma_2^c \), which is well known. Thus, the Poincaré series for \( W_\tau \) is given by

\[
\sum_{c_1, c_2 \geq 0, x-y \geq 2b, b \geq 0} \dim(V_{(x+2c_1+2c_2, y+2b+2c_2)}) t^{x+y+2b+2c_1+4c_2} \\
+ \sum_{c_1, c_2 \geq 0, x-y \geq 2b+1, b \geq 0} \dim(V_{(x+2c_1+2c_2+1, y+2b+2c_2+1)}) t^{x+y+2b+2c_1+4c_2+2}
\]

\[
= \sum_{c_1, c_2 \geq 0, x-y \geq 2b, b \geq 0} (x-y+2c_1-2b+1)t^{x+y+2b+2c_1+4c_2}
\]

\[
+ \sum_{c_1, c_2 \geq 0, x-y \geq 2b+1, b \geq 0} (x-y+2c_1-2b+1)t^{x+y+2b+2c_1+4c_2+2}
\]

\[
= \sum_{x-y \geq 2b, b \geq 0} \left( \frac{(x-y-2b+1)t^{x+y+2b}}{(1-t^2)(1-t^4)} + \frac{2t^{x+y+2b+2}}{(1-t^2)^2(1-t^4)} \right)
\]

\[
+ \sum_{x-y \geq 2b+1, b \geq 0} \left( \frac{(x-y-2b+1)t^{x+y+2b+2}}{(1-t^2)(1-t^4)} + \frac{2t^{x+y+2b+4}}{(1-t^2)^2(1-t^4)} \right)
\]

\[
= \frac{(x-y+1)t^{x+y}}{(1-t^2)^3}.
\]

\[\square\]

Corollary 4.7. Let \( \tau \in \overline{O}_m \) be defined as in (4.4) and (4.5). Then

the Gelfand–Kirillov dimension of \( W_\tau \) = \[
\begin{cases} 
3 & \text{if } m \geq 2, \\
2 & \text{if } m = 1.
\end{cases}
\]
and

the Berenstein degree of \( W \),

\[
\begin{cases} 
  x - y + 1 & \text{if } m \geq 4, \\
  x + 1 & \text{if } m = 3, x = 0 \mod 2 \text{ and } \epsilon = 0 \mod 2, \\
  x & \text{for the remaining } m = 3 \text{ cases}, \\
  2 & \text{if } m = 2 \text{ and } x \neq 0, \\
  1 & \text{if } m = 2 \text{ and } x = 0, \\
  2 & \text{if } m = 1.
\end{cases}
\]

Proof. The Gelfand–Kirillov dimension ([17]) can be read from the denominator of the Poincaré series while the Berenstein degree ([17]) is the value of the numerator at \( t = 1 \).

In principle, one could even write down the character of \( W \), and exhibit its global character ([16], [10]).

5. Generators for \( R_{m,3} \)

We have already described \( R_{1,3} \) and \( R_{2,3} \). We proceed to study the case \( n = 3 \) and \( m \geq 6 \). As in the previous section, we use Theorem 3.1 to reduce our study to \( GL_3 \) tensor products of the form

\[
\langle \alpha_1^a \alpha_2^b \alpha_3^c \rangle \otimes \langle \gamma_1^d \gamma_2^e \gamma_3^f \rangle.
\]

Since \( \langle \alpha_3 \rangle \) and \( \langle \gamma_3 \rangle \) each generate a one-dimensional representation of \( GL_3 \), it suffices to study

\[
\langle \alpha_1^a \alpha_2^b \rangle \otimes \langle \gamma_1^d \gamma_2^e \rangle.
\]

Because

\[
\langle \alpha_1^a \alpha_2^b \rangle \otimes \langle \gamma_1^d \gamma_2^e \rangle \rightarrow \langle \alpha_1 \rangle \otimes (\langle \alpha_1^{a-1} \alpha_2^b \rangle \otimes \langle \gamma_1^d \gamma_2^e \rangle)
\]

if \( a \neq 0 \) and

\[
\langle \alpha_1^a \alpha_2^b \rangle \otimes \langle \gamma_1^d \gamma_2^e \rangle \rightarrow \langle \alpha_2 \rangle \otimes (\langle \alpha_1^a \alpha_2^{b-1} \rangle \otimes \langle \gamma_1^d \gamma_2^e \rangle)
\]

if \( b \neq 0 \), we shall first consider decompositions of the \( GL_3 \) modules

\[
\langle \alpha_1 \rangle \otimes V \quad \text{and} \quad \langle \alpha_2 \rangle \otimes V
\]

for an arbitrary irreducible \( GL_3 \) module \( V \).
Let us introduce some notations. Recall our selection of the set of diagonal matrices

$$h_R = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix}, \quad d_j \in \mathbb{C}$$

as the Cartan subalgebra in $\mathfrak{gl}_3$. Define the functionals

$$\epsilon_j \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} = d_j, \quad j = 1, 2, 3.$$  

The nonzero roots of $\mathfrak{l}_3$ are

$$\Delta = \{ \pm (\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq 3 \},$$

where the root vectors of $\epsilon_i - \epsilon_j$ and $-(\epsilon_i - \epsilon_j)$ are the matrix units $E_{ij}$ and $E_{ji}$, respectively. These root vectors act on $\mathcal{B}(C^{m,3})$ as $R_{ij}$ described by (1.1). Select a positive system as follows

$$\Theta = \{ \epsilon_1 = \epsilon_3 - \epsilon_2, \epsilon_2 = \epsilon_2 - \epsilon_3 \}.$$  

If $h$ is a $GL_3$ highest weight vector in $\mathcal{B}(C^{m,3})$, we shall denote by $\langle h \rangle$ the $GL_3$ module generated by $h$. If $V_1$ and $V_2$ are two $GL_3$ modules in $\mathcal{B}(C^{m,3})$, we have the multiplication map

$$\Phi: V_1 \otimes V_2 \to \mathcal{B}(C^{m,3}),$$

$$\Phi(v_1 \otimes v_2) = v_1 v_2, \quad v_1 \in V_1, v_2 \in V_2.$$  

If $V_{\lambda_1}$ and $V_{\lambda_2}$ are $GL_3$ modules of highest weight $\lambda_1$ and $\lambda_2$, respectively, standard representation theory says that

$$V_{\lambda_1} \otimes V_{\lambda_2} = \sum_{\lambda \in P} m_{\lambda} V_{\lambda_1 + \lambda_2 - \lambda} = \sum_{\lambda \in P} W_{\lambda_1 + \lambda_2 - \lambda},$$

where $P$ is the set of sums of positive roots (including the zero sum), $m_\lambda$ is the multiplicity of the representation $V_{\lambda_1 + \lambda_2 - \lambda}$ in the tensor product and $W_{\lambda_1 + \lambda_2 - \lambda}$ is the $V_{\lambda_1 + \lambda_2 - \lambda}$-isotypic component. For $\lambda \in P$, let

$$\pi_\lambda: V_{\lambda_1} \otimes V_{\lambda_2} \to W_{\lambda_1 + \lambda_2 - \lambda}$$

be the projection onto the $V_{\lambda_1 + \lambda_2 - \lambda}$-isotypic component, and define another map

$$F_\lambda: V_{\lambda_1} \otimes V_{\lambda_2} \to \Phi(W_{\lambda_1 + \lambda_2 - \lambda}),$$

$$F_\lambda = \Phi \circ \pi_\lambda.$$
The following lemma is straightforward.

**Lemma 5.1.** Let \( V_1 = \langle v_1 \rangle \) and \( V_2 = \langle v_2 \rangle \) be irreducible \( GL_3 \) modules generated by highest weight vectors \( v_1 \) and \( v_2 \) of weight \((\lambda_1, \lambda_2, \lambda_3)\) and \((\mu_1, \mu_2, \mu_3)\), respectively. We have the following formulae.

(a) \( F_{e_1}^v(v_1 \otimes v_2) = (\lambda_1 - \lambda_2) v_1 (R_{21} v_2) - (\mu_1 - \mu_2) (R_{21} v_1) v_2, \)

(b) \( F_{e_2}^v(v_1 \otimes v_2) = (\lambda_2 - \lambda_3) v_1 (R_{32} v_2) - (\mu_2 - \mu_3) (R_{32} v_1) v_2, \)

(c) \( F_{e_1 + e_2}^v(\alpha_1 \otimes v_1) = \alpha_1 (R_{21} R_{32} v_1) - (\lambda_2 - \lambda_3) \alpha_2 (R_{32} v_1) \)
\[ + (\lambda_1 - \lambda_3 + 1) (R_{21} \alpha_1) (R_{32} v_1) + (\lambda_2 - \lambda_3) \lambda_1 - \lambda_3 + 1) \]
\[ (R_{32} \alpha_1) v_1, \]

(d) \( F_{e_1 + e_2}^v(\alpha_2 \otimes v_1) = \alpha_2 (R_{21} R_{32} v_1) - (\lambda_1 - \lambda_2 + 1) \alpha_2 (R_{32} v_1) \)
\[ + (\lambda_1 - \lambda_3 + 1) (R_{21} \alpha_2) (R_{32} v_1) + (\lambda_1 - \lambda_2) \lambda_1 - \lambda_3 + 1) \]
\[ (R_{32} \alpha_2) v_1. \]

**Proposition 5.2.** Let \( V_1 = \langle v_1 \rangle \) and \( V_2 = \langle v_2 \rangle \) be irreducible \( GL_3 \) modules generated by highest weight vectors \( v_1 \) and \( v_2 \) of weight \((\lambda_1, \lambda_2, \lambda_3)\) and \((\mu_1, \mu_2, \mu_3)\), respectively. We have the following formulae.

(a) \( F_{e_1}^v(\alpha_1 \otimes (v_1 v_2)) = v_1 F_{e_1}^v(\alpha_1 \otimes v_2) + v_2 F_{e_1}^v(\alpha_1 \otimes v_1), \)

(b) \( F_{e_2}^v(\alpha_2 \otimes (v_1 v_2)) = v_1 F_{e_2}^v(\alpha_2 \otimes v_2) + v_2 F_{e_2}^v(\alpha_2 \otimes v_1), \)

(c) \( F_{e_1 + e_2}^v(\alpha_1 \otimes (v_1 v_2)) \)
\[ = \frac{\left[F_{e_2}(F_{e_1}(\alpha_1 \otimes v_2) \otimes v_1) + (\lambda_2 + \mu_2 - \lambda_3 - \mu_3 + 1) v_1 F_{e_1 + e_2}(\alpha_1 \otimes v_2)\right]}{\left(\mu_2 - \mu_3 + 1\right)} \]
\[ + \frac{\left[F_{e_2}(F_{e_1}(\alpha_1 \otimes v_1) \otimes v_2) + (\lambda_2 + \mu_2 - \lambda_3 - \mu_3 + 1) v_2 F_{e_1 + e_2}(\alpha_1 \otimes v_1)\right]}{\left(\lambda_2 - \lambda_3 + 1\right)}, \]

(d) \( F_{e_1 + e_2}^v(\alpha_2 \otimes (v_1 v_2)) \)
\[ = \frac{\left[F_{e_2}(F_{e_1}(\alpha_2 \otimes v_1) \otimes v_2) + (\lambda_1 + \mu_1 - \lambda_2 - \mu_2 + 1) v_2 F_{e_1 + e_2}(\alpha_2 \otimes v_1)\right]}{\left(\mu_1 - \mu_2 + 1\right)} \]
\[ + \frac{\left[F_{e_2}(F_{e_1}(\alpha_2 \otimes v_2) \otimes v_1) + (\lambda_1 + \mu_1 - \lambda_2 - \mu_2 + 1) v_1 F_{e_1 + e_2}(\alpha_2 \otimes v_2)\right]}{\left(\lambda_1 - \lambda_2 + 1\right)}. \]
Proof. The first two formulae follow directly from Lemma 5.1. We shall prove (c) and omit the proof of (d), which is similar.

\[ F_{r_1, r_2}(\alpha_1 @ (r'_{12})) \]

\[ - \alpha_1((R_{23}v_{12})(R_{32}v_{12}) + v_1(R_{21}R_{32}v_{12}) + (R_{23}R_{32}v_{12})v_2 + (R_{32}v_{12})(R_{23}v_{12})) \]

\[ - (\lambda_2 + \mu_2 - \lambda_3 - \mu_3)\alpha_1(v_1(R_{31}v_{12}) + v_2(R_{31}v_{12})) \]

\[ - (\lambda_1 + \mu_1 - \lambda_3 - \mu_3 + 1)(R_{21}v_{12})(v_1(R_{31}v_{12}) + v_2(R_{31}v_{12})) \]

\[ + (\lambda_2 + \mu_2 - \lambda_3 - \mu_3)(\lambda_1 + \mu_1 - \lambda_3 - \mu_3 + 1)(R_{31}v_{12})(v_1v_{12}) \]

\[ = r_1F_{r_1, r_2}(\alpha_1 @ v_{12}) + r_2F_{r_1, r_2}(\alpha_1 @ v_{12}) \]

\[ - (\lambda_2 - \lambda_3)v_1[\alpha_1(R_{31}v_{12}) + (R_{21}v_{12})(R_{31}v_{12}) - (\mu_1 - \mu_3)(R_{31}v_{12})v_{12}] \]

\[ - (\mu_1 - \mu_3)v_2[\alpha_1(R_{31}v_{12}) + (R_{21}v_{12})(R_{31}v_{12}) - (\lambda_1 - \lambda_3)(R_{31}v_{12})v_{12}] \]

\[ + F_{r_1}(\alpha_1 @ v_{12})(R_{31}v_{12}) + (R_{31}v_{12})F_{r_1, r_2}(\alpha_1 @ v_{12}) \]

\[ - v_1F_{r_1, r_2}(\alpha_1 @ v_{12}) + v_2F_{r_1, r_2}(\alpha_1 @ v_{12}) \]

\[ + \frac{F_{r_1}(\alpha_1 @ v_{12}) @ v_{12}}{(\lambda_2 - \lambda_3 + 1)} + \frac{F_{r_1}(\alpha_1 @ v_{12}) @ v_{12}}{(\mu_2 - \mu_3 + 1)} \]

\[ + \frac{(\lambda_2 - \lambda_3)v_1[\alpha_1(R_{31}R_{21}v_{12}) - (\mu_1 - \mu_3)(R_{31}v_{12})v_{12} - (\mu_1 - \mu_3)(R_{21}v_{12})(R_{31}v_{12})]}{(\mu_2 - \mu_3 + 1)} \]

\[ + \frac{(\mu_2 - \mu_3)v_2[\alpha_1(R_{31}R_{21}v_{12}) - (\lambda_1 - \lambda_3)(R_{31}v_{12})v_{12} - (\lambda_1 - \lambda_3)(R_{21}v_{12})(R_{31}v_{12})]}{(\lambda_2 - \lambda_3 + 1)} \]

\[ - (\lambda_2 - \lambda_3)v_1[\alpha_1(R_{31}v_{12}) + (R_{21}v_{12})(R_{31}v_{12}) - (\mu_1 - \mu_3)(R_{31}v_{12})v_{12}] \]

\[ - (\mu_1 - \mu_3)v_2[\alpha_1(R_{31}v_{12}) + (R_{21}v_{12})(R_{31}v_{12}) - (\lambda_1 - \lambda_3)(R_{31}v_{12})v_{12}] \]

\[ = \frac{F_{r_1}(\alpha_1 @ v_{12}) + (\lambda_2 + \mu_2 - \lambda_3 - \mu_3 + 1)v_1F_{r_1, r_2}(\alpha_1 @ v_{12})}{(\mu_2 - \mu_3 + 1)} \]

\[ + \frac{F_{r_1}(\alpha_1 @ v_{12}) + (\lambda_2 + \mu_2 - \lambda_3 - \mu_3 + 1)v_2F_{r_1, r_2}(\alpha_1 @ v_{12})}{(\lambda_2 - \lambda_3 + 1)}. \]
Define

\[
\beta_1 = \begin{bmatrix}
z_{11} & z_{12} \\
 r_{11}^2 & r_{12}^2 
\end{bmatrix},
\beta_2 = \begin{bmatrix}
z_{11} & z_{12} & z_{13} \\
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} 
\end{bmatrix},
\beta_3 = \begin{bmatrix}
z_{11} & z_{12} & z_{13} \\
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} 
\end{bmatrix},
\beta_4 = \begin{bmatrix}
z_{11} & z_{12} & z_{13} \\
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} 
\end{bmatrix}
\]

\[
\beta_5 = \begin{bmatrix}
z_{11} & z_{12} & z_{13} \\
z_{21} & z_{22} & z_{23} \\
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} 
\end{bmatrix},
\beta_6 = \begin{bmatrix}
z_{11} & z_{12} & z_{13} \\
z_{11} & r_{12} & r_{13} \\
z_{21} & r_{22} & r_{23} \\
r_{11} & r_{12} & r_{13} 
\end{bmatrix},
\beta_7 = \begin{bmatrix}
z_{11} & z_{12} & z_{13} \\
z_{21} & z_{22} & z_{23} \\
r_{11} & r_{12} & r_{13} \\
r_{21} & r_{22} & r_{23} 
\end{bmatrix},
\beta_8 = \begin{bmatrix}
z_{11} & z_{12} & z_{13} \\
z_{11} & r_{12} & r_{13} \\
z_{21} & r_{22} & r_{23} \\
r_{11} & r_{12} & r_{13} 
\end{bmatrix}
\]

The \( O_m \times GL_3 \) weights of the above polynomials are given in Table 1.

**Theorem 5.3.** If \( m \geq 6 \), the ring of \( O_m \times GL_3 \) highest weight vectors in \( \mathcal{P}(\mathbb{C}^m, 3) \) is generated by

\[\alpha_i, \ i = 1, 2, 3, \quad \beta_j, \ j = 1, \ldots, 9, \quad \gamma_k, \ k = 1, 2, 3.\]

**Remarks.** Since the Krull dimension of \( R_{m, 3} \) is 9, we expect to find a set of 6 relations generating the ideal of relations. We suspect that the following set of relations is enough.

1. \( \beta_1^2 + \beta_2 \gamma_1 + \alpha_1^2 \gamma_2 = 0, \)
2. \( \beta^2 + \beta_6 \gamma_2 - \beta_2 \gamma_3 = 0, \)
3. \( \beta_3^2 + \alpha_2^2 \gamma_1 \gamma_3 + \beta_3^2 \gamma_2 - \beta_4 \gamma_1 \gamma_2 = 0, \)
TABLE I

\begin{center}
\begin{tabular}{|c|c|}
\hline
$SO_{3}$ weight & $GL_{3}$ weight \\
\hline
$\sigma_{1}$ & (1, 0, \ldots, 0) \\
$\sigma_{2}$ & (1, 1, 0, \ldots, 0) \\
$\sigma_{3}$ & (1, 1, 1, 0, \ldots, 0) \\
$\gamma_{1}$ & (0, \ldots, 0) \\
$\gamma_{2}$ & (0, \ldots, 0) \\
$\gamma_{3}$ & (0, \ldots, 0) \\
$\beta_{1}$ & (1, 0, \ldots, 0) \\
$\beta_{2}$ & (1, 0, \ldots, 0) \\
$\beta_{3}$ & (1, 1, 0, \ldots, 0) \\
$\beta_{4}$ & (1, 1, 1, 0, \ldots, 0) \\
$\beta_{5}$ & (2, 0, \ldots, 0) \\
$\beta_{6}$ & (2, 0, \ldots, 0) \\
$\beta_{7}$ & (2, 1, 0, \ldots, 0) \\
$\beta_{8}$ & (2, 2, 0, \ldots, 0) \\
$\beta_{9}$ & (2, 2, 0, \ldots, 0) \\
\hline
\end{tabular}
\end{center}

(4) \hspace{1cm} \beta_{7}^{2} + \beta_{2} \beta_{9} - \sigma_{3} \beta_{6} = 0,

(5) \hspace{1cm} \beta_{5}^{2} + \sigma_{3} \beta_{6} \gamma_{3} + \beta_{8} \beta_{6} \gamma_{1} - \beta_{3}^{2} \beta_{6} = 0,

(6) \hspace{1cm} \beta_{1} \beta_{3} - \beta_{7} \gamma_{1} - \sigma_{1} \beta_{5} = 0.

These relations were found using the computer algebra programs [1] and [19] and the methods of [14]. In particular, the last relation is a consequence of following identity

\[ \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ y_{11} & y_{12} & y_{13} \end{vmatrix} = y_{11} \begin{vmatrix} 0 & x_{11} & x_{12} & x_{13} \\ 0 & x_{21} & x_{22} & x_{23} \\ x_{11} & y_{11} & y_{12} & y_{13} \\ x_{12} & y_{21} & y_{22} & y_{23} \end{vmatrix} + x_{11} \begin{vmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ 0 & y_{11} & y_{12} & y_{13} \\ 0 & y_{21} & y_{22} & y_{23} \end{vmatrix}. \] (5.2)

To prove (5.2), we start with the matrix

\[ M = \begin{vmatrix} x_{11} & 0 & 0 & 0 & y_{11} \\ 0 & x_{11} & x_{12} & x_{13} & 0 \\ 0 & x_{21} & x_{22} & x_{23} & 0 \\ 0 & y_{11} & y_{12} & y_{13} & 0 \\ x_{12} & y_{21} & y_{22} & y_{23} & y_{21} \end{vmatrix}. \]
It is easy to see that det $M$ equals the left hand side of (5.2), but if we add the first row to the fourth row before evaluating the determinant of $M$, we obtain the right hand side of (5.2). (These computations can be carried out using the usual expansions along rows or columns, or by applying the Lewis Carroll identity ([11]) to $M$.)

**Proof.** By Proposition 5.2, it suffices to find a set of highest weights $S$ that is closed under $F_{e_i}$ and $F_{e_j}$, i.e., $x, y \in S$ implies that $F_{e_i}(x \otimes y)$ is generated by elements in $S$. That the 15 elements listed above satisfy this criterion is shown in Table II. \[ \square \]

6. A Subring of the Ring of $O_m \times GL_n$ Highest Weight Vectors

As in previous sections, we assume that $m \geq 2n$. Let $S_{m,n}$ be the following subring of the ring $R_{m,n}$

$$S_{m,n} = \{ f \in R_{m,n} | f(xh) = f(x), \forall h \in SL_n \}$$

$$= \{ f \in \mathcal{P}(C^{m,n}) | f(u^{-1}xh) = f(x), \forall u \in N_L, h \in SL_n \}. \quad (6.1)$$

Clearly, an $O_m \times GL_n$ highest weight vector $f$ is in $S_{m,n}$ only if the corresponding $GL_n$ highest weight is of the form $(t, \ldots, t)$ for $t \in \mathbb{Z}$. Conversely, if $f$ is an $O_m \times GL_n$ highest weight vector with $GL_n$ highest weight $(t, \ldots, t)$, then $f$ generates an irreducible representation of $GL_n$, which must be a power of the determinant representation, and so $f$ is right $SL_n$ invariant, i.e., $f \in S_{m,n}$.

This ring was studied by Sato in [12]. He found among other things a set of generators of $S_{m,n}$. In this section we will give a different set of generators, which are more in line with the spirit of this article.

Let us write $z_i = (z_{i1}, z_{i2}, \ldots, z_{im})$ and define the following $(n + t) \times (n + t)$ determinant for $1 \leq t \leq (n - 1)$,

$$\delta_t = \begin{vmatrix} 0 & \cdots & 0 & z_{11} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & z_t \\ z_{1i} & \cdots & z_{ti} & \gamma \end{vmatrix} \quad (6.2)$$
TABLE II

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$\gamma_1$</th>
<th>$\gamma_2$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
<th>$\beta_3$</th>
<th>$\beta_4$</th>
<th>$\beta_5$</th>
<th>$\beta_6$</th>
<th>$\beta_7$</th>
<th>$\beta_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1$</td>
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<td>—</td>
<td>$\beta_1$</td>
<td>—</td>
<td>$\beta_2$</td>
<td>—</td>
<td>$\beta_7$</td>
<td>—</td>
<td>$\alpha_1\beta_4$</td>
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<td>$\alpha_1\beta_6$</td>
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<td>$\alpha_2$</td>
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<td>$\gamma_2$</td>
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<td>2$\alpha_1\gamma_2$</td>
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<td>2$\beta_5$</td>
<td>—</td>
<td>2$\beta_5\gamma_2$</td>
<td>—</td>
<td>2$\alpha_1\alpha_2\gamma_3 - 2\beta_3\beta_4$</td>
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</tr>
<tr>
<td>$\beta_1$</td>
<td>$\alpha_1\beta_3$</td>
<td>—</td>
<td>$\beta_4\gamma_3$</td>
<td>—</td>
<td>2$\alpha_4\beta_2$</td>
<td>—</td>
<td>2$\alpha_4\beta_4$</td>
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<td>2$\alpha_4\beta_5$</td>
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<td>2$\alpha_4\beta_7$</td>
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<tr>
<td>$\beta_2$</td>
<td>$\alpha_1\beta_7$</td>
<td>—</td>
<td>$-\beta_1\beta_4$</td>
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<td>$\beta_9$</td>
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<td>$\beta_5$</td>
<td>$\beta_5\gamma_3 - \beta_3^3$</td>
<td>—</td>
<td>$\alpha_1\gamma_3 + \beta_6\gamma_1$</td>
<td>$-\beta_1\beta_6$</td>
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<td>$\beta_7$</td>
<td>$\alpha_1\beta_9$</td>
<td>$\alpha_1\beta_8$</td>
<td>$-\alpha_1\beta_8$</td>
<td>$-\alpha_1\beta_8$</td>
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</tbody>
</table>

Note. Let $i < j$. The $(j, i)$ (respectively $(i, j)$) entry is the image of the map $F_{\gamma}$ (respectively $F_{\alpha_i}$) of the tensor product of the $i$th entry of the first column with the $j$th entry of the first row. Note that $F_i(v \otimes v) = 0$ for all highest weight vectors $v$ and $i = 1, 2$. $-$ or $*$ means that the entry is 0.
where
\[ \tilde{\gamma} = \begin{bmatrix} r_{11}^2 & r_{12}^2 & \cdots & r_{1n}^2 \\ r_{11} & r_{12} & \cdots & r_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{n1}^2 & r_{n2}^2 & \cdots & r_{nn}^2 \end{bmatrix}. \]

It is easy to see that \( \delta_i \) is an \( O_m \times GL_n \) highest weight vectors of weight
\[ \left( 2, \ldots, 2, 0, \ldots, 0; 1 \right) \otimes \left( 2, \ldots, 2 \right), \]
\( t \) copies \( \otimes \) \( n \) copies (6.3)

We observe that \( \gamma_n, \delta_1, \ldots, \delta_{n-1}, \alpha_n \) are all in \( S_{m,n} \) (see (3.7) and (3.8) for the definitions of \( \alpha_n \) and \( \gamma_n \)). Define
\[ P_0 = \gamma_n, \]
\[ P_t = \delta_t, \quad 1 \leq t \leq n - 2, \]
\[ P_{n-1} = \begin{cases} \delta_{n-1} & \text{if } m > 2n, \\ \delta_{n-1} / \alpha_n & \text{if } m = 2n, \end{cases} \] (6.4)
\[ P_n = \alpha_n. \]

Adapting the proof of the similar theorem in [12], we have the following.

**Theorem 6.1.** If \( m \geq 2n \), the ring \( S_{m,n} \) is freely generated by \( P_0, \ldots, P_n \).

**References**

[1] D. Bayer and M. Stillman, Macaulay: A system for computation in algebraic geometry and commutative algebra. Source and object code available for Unix and Macintosh computers. Contact the authors, or download from zariski.harvard.edu via anonymous ftp.


