The Analemma Dilemma

Solving visualization issues in astronomy using 3D graphics

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Abstract

This project was undertaken under the supervision of Professor Helmer Aslaksen and was completed as partial fulfilment of an independent study module. The report focuses on visualisation problems involved in modelling astronomical phenomena, with particular reference to how the analemma can be explained and depicted using graphical figures.

Contents

1 Introduction

The motion of the sun relative to the earth is a result of the combined effects of the rotation of the earth and the revolution of the earth around the sun. The rotation axis of the earth is tilted at approximately 23.45 degrees to the plane of the earth's orbit around the sun, and the revolution of the earth around the sun follows an elliptical path with eccentricity 0.0167. These irregularities in revolution and rotation allow for variations in the length of daylight and variations in sunrise and sunset times. The path of the sun at a specific time and location over a period of one year is referred to as the analemma. This phenomenon can be easily visualized and detected from the earth, where the only difficulty is that it requires a whole year to do so! Figure 1 provides one such photographic record of the analemma, where a photograph of the sun has been taken at 8:30am each morning.

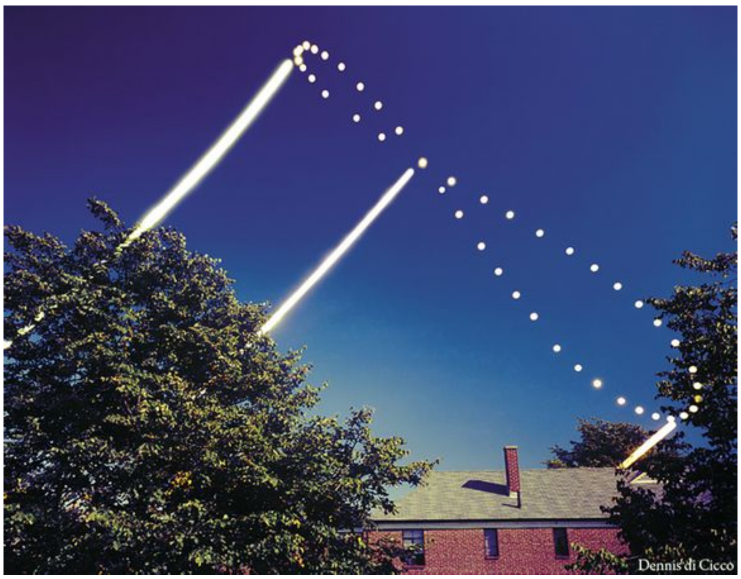


Figure 1: 8:30am analemma at 46°32'N[4]

Modelling the analemma requires the ability to determine the position of the sun in the sky when given a time, date and location. Graphical representations of the analemma are important, as it is not a phenomenon which can be seen at one single moment in time. The use of such graphics is an means of explaining related and more familiar phenomena, such as why the sun rises and sets at different times, even on the equator, and how sundials can tell the date as well as the time.

A difficulty arises in accurately depicting the analemma in graphical figures, such that the viewer is easily able to understand the phenomenon. Problems which arise in the depiction of the analemma are also common to models of other astronomical phenomena. Good astronomical graphics strive towards realism and scientific accuracy, although strategic modifications to scale and viewing position are often unavoidable. An external view of the universe is useful for depicting large scale solar motions, but is limited in its applications in usefully depicting the analemma. Corrections in viewpoint allow for the creation of graphical models which resemble real photographs of the analemma taken from the earth. It is possible to create both realistic and accurate models of the analemma, allowing for better visualisation and understanding of this phenomenon. It is useful to model the analemma using programmable figures, as this will allow for easy adaptation of models for the analemma as seen at different latitudes and times of day.

2 Background for Astronomical Modelling

All computer generated models in this report are created using Mathematica 7.0. The calendrical and astronomical algorithms required for these calcuations come from the package Calendrica, written in Lisp by Edward M. Reingold and Nachum Dershowitz and converted for use in Mathematica by Robert C. McNally.[3] The Wolfram Research package AstronomicalData served as a comparison between the analemma as generated by astronomical algorithms and the analemma as generated by real data points.

2.1 Visualising the Celestial Sphere

Generally, celestial bodies are either projected onto a sphere, such as the celestial sphere, or onto a plane by using one of numerous spherical projections. The earth lies at the centre of the celestial sphere and stars, planets and other bodies are projected onto the sphere itself. This is useful in explaining many phenomena but is also an unrealistic depiction of space. Typical diagrams show the celestial sphere as depicted in figure 2.

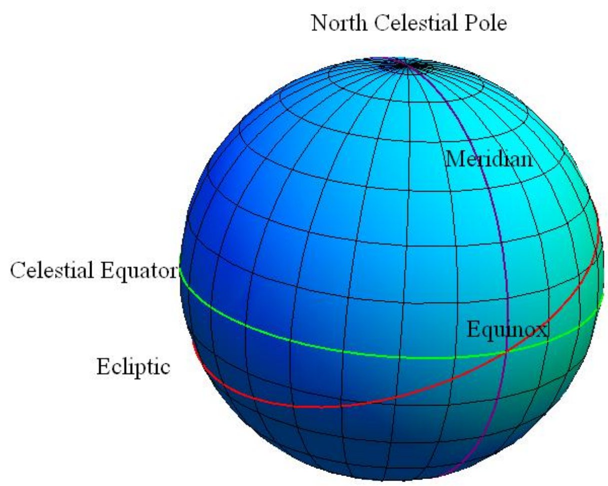


Figure 2: Celestial Sphere Model

The celestial equator is a projection of the earth's equator onto the celestial sphere. Similarly, latitudinal and longitudinal lines on the celestial sphere correspond to the projection of the earth's latitudinal and longitudinal lines. The ecliptic traces the apparent path of the sun throughout the year and lies at an angle of approximately 23.45 degrees to the celestial equator due to the tilt of the earth's axis. The two equinoxes, where the sun crosses the celestial equator, correspond to the vernal and spring equinox. On these dates at a location on the equator, the sun can be seen directly overhead.

Although this image of the imaginary celestial sphere is practical for explaining the apparent path of the sun, the first fundamental problem with this diagram is that nobody has even witnessed the sun's motion from this angle. The motion of the sun across the sky over one year as seen from the earth is dependent on latitude and appears very differently to this depiction from an external perspective. Another significant problem with this image is that it can be easily confused with the apparent motion of the sun throughout a day. This model of the celestial sphere also does not display a key factor contributing to the analemma's shape the eccentricity of the earth's orbit around the sun.

Whilst it is very easy to place the analemma on a celestial sphere in this manner, it is also not very useful in explaining the phenomenon. The shape and angle of the analemma are dependent on the latitude of the viewer and the time of day. The proceeding diagram shows the appearance of the analemma for a viewer at 9am at 20 degrees north, projected onto a sphere. This projection is misleading, because it implies that the analemma has a fixed angle and location in the sky for all viewers. Furthermore, it may be another confusion to note that in such diagrams the sphere is not on the same axes as the celestial sphere as shown in figure 2. The celestial equator is no longer marked, rather the horizon for a viewer at 20 degrees latitude at a given time of day.

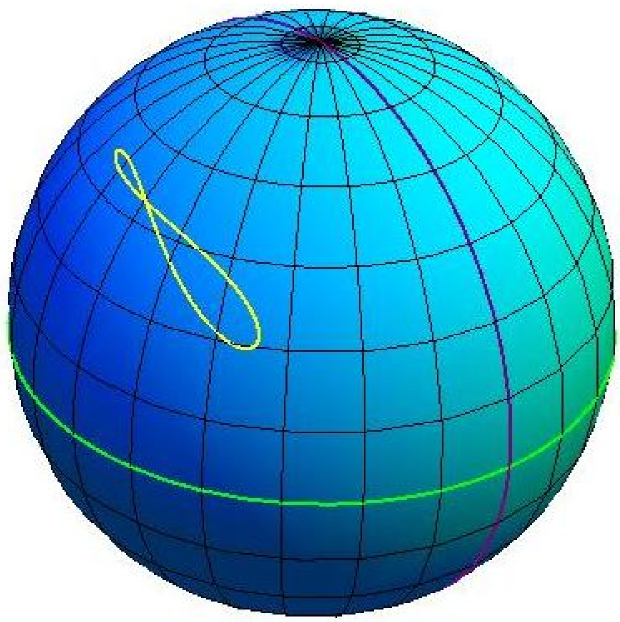


Figure 3: 3pm at 20°N analemma on a sphere[7]

Explanations of the analemma using an external view of the celestial sphere are, as a result of these issues, quite limited. In order to produce useful images of the analemma, it is essential to attempt to depict the analemma realistically. A more effective means of depicting the analemma is to 'open up' the celestial sphere and to view the analemma and the celestial sphere from the inside, as it would be seen on earth.

2.2 Depictions of the Sun's Declination

The declination of the sun is the angle it makes with the plane of the celestial equator. At the equinoxes, the sun's declination is 0 degrees and at the summer and winter solstices, maximum and minimum declinations are reached at 23.45 degrees. The declination of the sun can be visualized by comparing the sun's path along the ecliptic with a path along the celestial equator. The mean sun refers to a fictional sun which travels across the celestial equator. This is equivalent to the sun's motion if the earth were not tilted and if its revolution around the sun were circular. The true sun follows the path of the ecliptic, tilted at 23.45 degrees to the celestial equator.

The graph of the declination angle δ of the sun over one year is a sine function involving the sun's maximum and minimum declination and the true sun's longitudinal position λ on the celestial sphere. The sun's longitudinal position may be extracted from a table or can be reproduced from astronomical algorithms.[5]

$$\sin\delta = 23.45\sin\lambda$$
$$\delta = \sin^{-1}(23.45\sin\lambda)$$

Plotting the declination angle δ against the true sun's longitudinal position λ produces the following graph, where the horizontal axes is the path of the mean sun ($\delta = 0$).

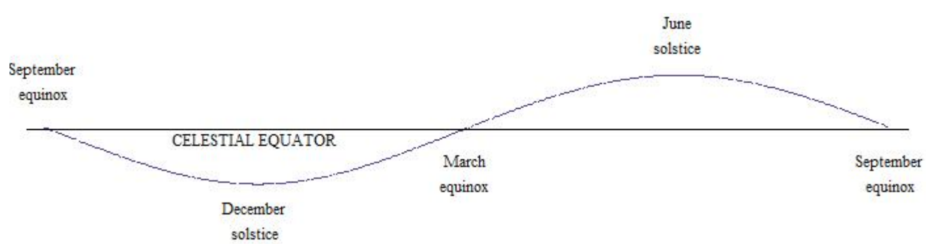


Figure 4: Path of the true sun[7]

Figure 4 is a projection of the ecliptic and the celestial equator onto a two dimensional plane. At the times of the summer and winter solstices, the sun reaches its northernmost and southernmost extremes respectively, as is clearly depicted by the two dimensional graphic.

This diagram is a clear and simple representation of the path of the true sun along the ecliptic as compared with the path of the mean sun along the celestial equator, however, lengths are incorrectly represented. By projecting this three dimensional phenomenon onto a two dimensional plane, the relative lengths of the celestial equator and the ecliptic become incorrect. The path of the true sun appears to be longer than the path of the mean sun and it seems that the true sun must travel further and faster than a sun following the celestial equator, although from the three dimensional figure 2, it is clear that this is not the case. The two suns travel at the same constant angular velocity around the earth over the same distance, but the relative velocity across lines of longitude differs. Initially, the mean sun travelling along the celestial equator would possess a greater velocity across lines of longitude. At the time of the June solstice, the two suns would be aligned and have equal longitude, but then the sun on the ecliptic would travel at a greater angular velocity across lines of longitude.

This key difference in velocity across longitudinal lines between the mean sun and the true sun is vital in order to correctly explain the analemma's figure-of-eight shape. The discrepancy between the mean sun and the true sun allows for changes in the length of daylight and changes in sunrise and sunset times. If the sun did not have differing velocities across the lines of longitude over the course of a year, then the position of the sun at the same time of each day would possess the same longitude on the celestial sphere. Similarly, if the sun did not have differing velocities across the lines of latitude, then the latitude of the sun on the celestial sphere would be the same at a specific time of each day. Images of the analemma, such as in figure 1, show that this cannot be the case.

A three dimensional diagram using an open celestial sphere may look like figure 5 below. This graphic correctly represents the important knowledge that the path of the mean sun and the path of the true sun are of the same length. Although the diagram is not as simple as the two dimensional version, it is a more accurate depiction of the true sun's path as compared to that of the mean sun.

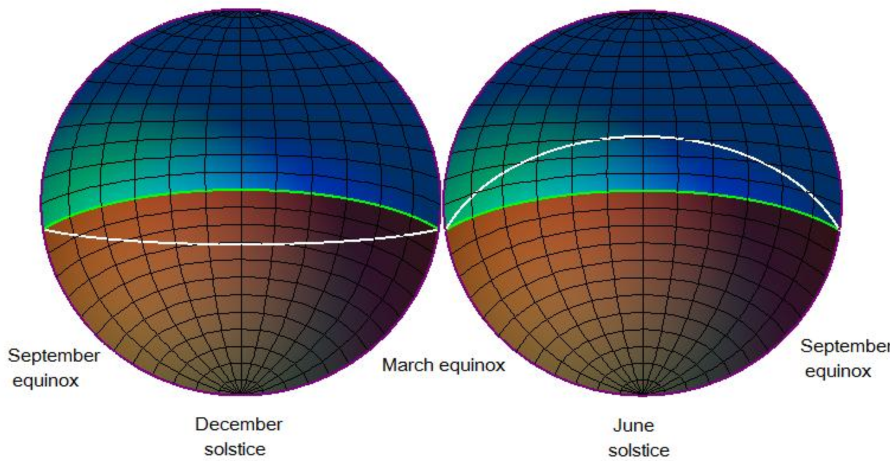


Figure 5: Declination in three dimensions

By knowing how far 'behind' or 'in front' of the mean sun the true sun is, a conversion can be made to determine the position of the true sun at a specified moment given the position of the mean sun. The longitudinal velocity of the mean sun is 360 degrees per day or 15 degrees per hour. Using this fact, a relationship between longitudinal difference and time difference between the mean and true suns can be established. The relationship alleviates the need for a three dimensional graphic, because time, as opposed to longitude, lives on two dimensional axes.

2.2.1 Equation of Time

Until now, the models of the sun's motion have ignored the fact that the earth's revolution around the sun is elliptical. The physical discrepancy between the true and mean sun has already been graphically represented in figure 4, and the two dimensional graph of the difference in solar time of a tilted earth and mean solar time naturally has a similar sine curve shape. In the previous section, it was stated that the angular velocity of the true sun is constant; however, as a result of the earth's eccentric orbit, the earth must travel at varying speeds in order to maintain a constant angular velocity. The earth's orbit has eccentricity 0.0167, which means it is almost circular. At the perihelion, the earth is closest to the sun and travels fastest, whilst at the aphelion, the earth is furthest from the sun and travels slowest.

Approximating the location of the sun assuming a circular orbit does not suffice for producing accurate images of the analemma. As with declination, the elliptical orbit can be compared with a simplified circular motion of the mean sun, and the physical discrepancy between these two suns can be expressed in units of time. The time difference between an sun travelling on a circular orbit and an sun travelling on an elliptical orbit also follows a sine curve, where there is no time difference at the perihelion and aphelion.

The total time difference between a mean sun travelling along the celestial equator and the true sun travelling along the ecliptic in an eccentric orbit is given by the equation of time. This is simply the summation of the two sine curves derived from the time difference between a sun on the celestial equator and a sun on the ecliptic and the time difference between a circular and elliptical orbit. The equation of time inputs the day of year and outputs time difference between the mean and true sun in fraction of a day. It enables a conversion between position of the mean sun and position of the true sun.

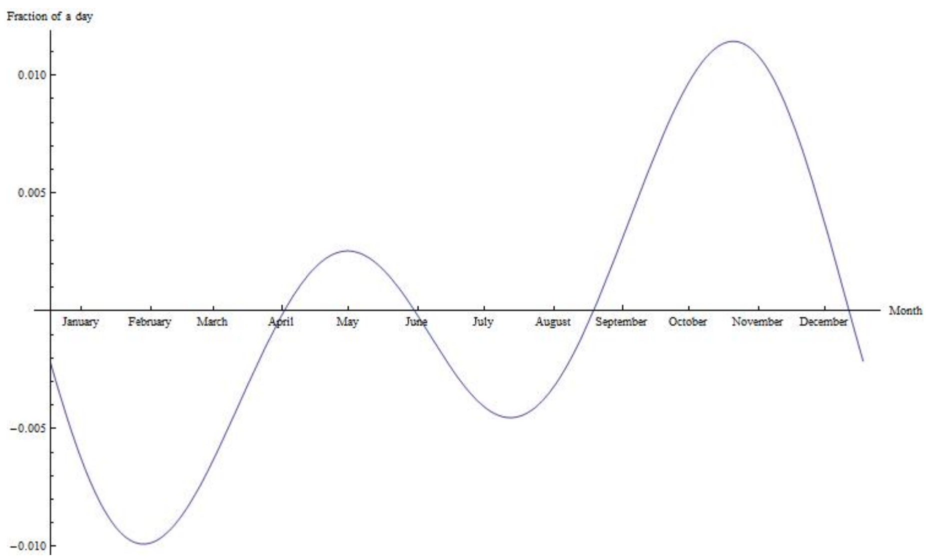
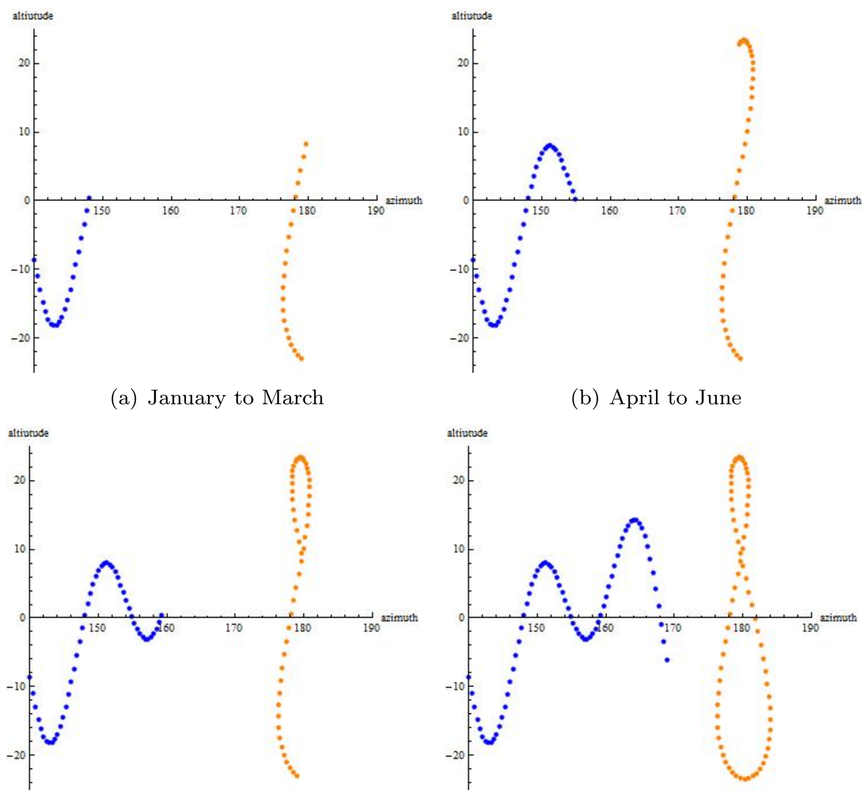


Figure 6: Equation of Time

The Calendrica package for Mathematica performs many basic astronomical and calendrical calculations from algorithms, one of which is the equation of time. Unlike approximations for the equation of time, the Calendrica package also accounts for changes in the alignment of perihelion and aphelion with the solstices each year. For simplification and in need of an accurate representation, the Calendrica function EquationOfTime is used to generate values of this function in figure 6. The figure shows the equation of time for the year 2011.

2.2.2 The Analemma as a Depiction of the Equation of Time

The analemma is a direct consequence of the equation of time. The deviation in position of the true sun from the mean sun at a given time on each day of the year is provided in the equation of time graph in units of time. It however takes an intuitive leap to be able to mentally transform the equation of time into the analemma, as the relationship is quite complicated.



(a) January to March

(b) April to June

(c) July to September

(d) October to December

Figure 7: Animation of a noon analemma and the equation of time

Figure 7 is an animation showing how the analemma is a physical representation of the equation of time. The equation of time has been scaled to the same axes as the analemma, but the significant points of the equation of time, such as the peaks and zeros, occur at the correct timing. When the equation of time graph is negative, the true sun is moving longitudinally behind the mean sun and the left section of this particular analemma is formed. The true sun is seen to be east of the mean sun. When the equation of time is positive, the sun is longitudinally ahead of the mean sun and the right portion of this analemma is formed. The true sun is seen to be west of the mean sun. The height of the peaks and the depth of the troughs determine the 'width' of each section of the analemma, hence the largest peak and trough correspond to the bottom section of the analemma. This is where there is the greatest time difference between the true and the mean suns and so the true sun is longitudinally furthest from the mean sun.

3 2D Modelling

Two dimensional models of the analemma involve some form of projection of the analemma as shown on the celestial sphere to a plane. The projection involved in this section maps great circles on the celestial sphere to lines and neither area nor distance is conserved. Azimuth and altitude angles are simply given on a linear scale. Unlike latitude and longitude, azimuth and altitude angles are relative to the position of the observer. Altitude angle ranges from 0 degrees on the horizon to 90 degrees at the zenith and is considered negative if the object is below the horizon. Azimuth angle is defined to be 0 degrees at due north, 90 degrees at due east, 180 degrees at due south and 270 degrees at due west.

3.1 Astronomical functions

To produce figures of the analemma, one must plot altitude against azimuth angle for the position of the sun at each day over the course of a year. In Mathematica, it is possible to use a package called AstronomicalData, which takes realistic data points for the location of the sun at any given location and time. This is the simplest way of modelling the analemma. However, it is also possible to derive the position of the sun in the sky from the equation of time, the position of the observer and by using three dimensional geometry.

Given latitude of the observer ϕ, declination δ and hour angle H, azimuth angle A and altitude angle h are generated by the proceeding equations.[5][7]

$$\tan A = \frac{\sin H}{\cos H \sin \phi - \tan \delta \cos \phi}$$

$$A = \tan^{-1} \frac{\sin H}{\cos H \sin \phi - \tan \delta \cos \phi}$$

$$\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos H$$

$$h = \sin^{-1} (\sin \phi \sin \delta + \cos \phi \cos \delta \cos H)$$

The hour angle H is given by the following equation, where x is the number of hours after noon using mean solar time (12pm in Greenwich).

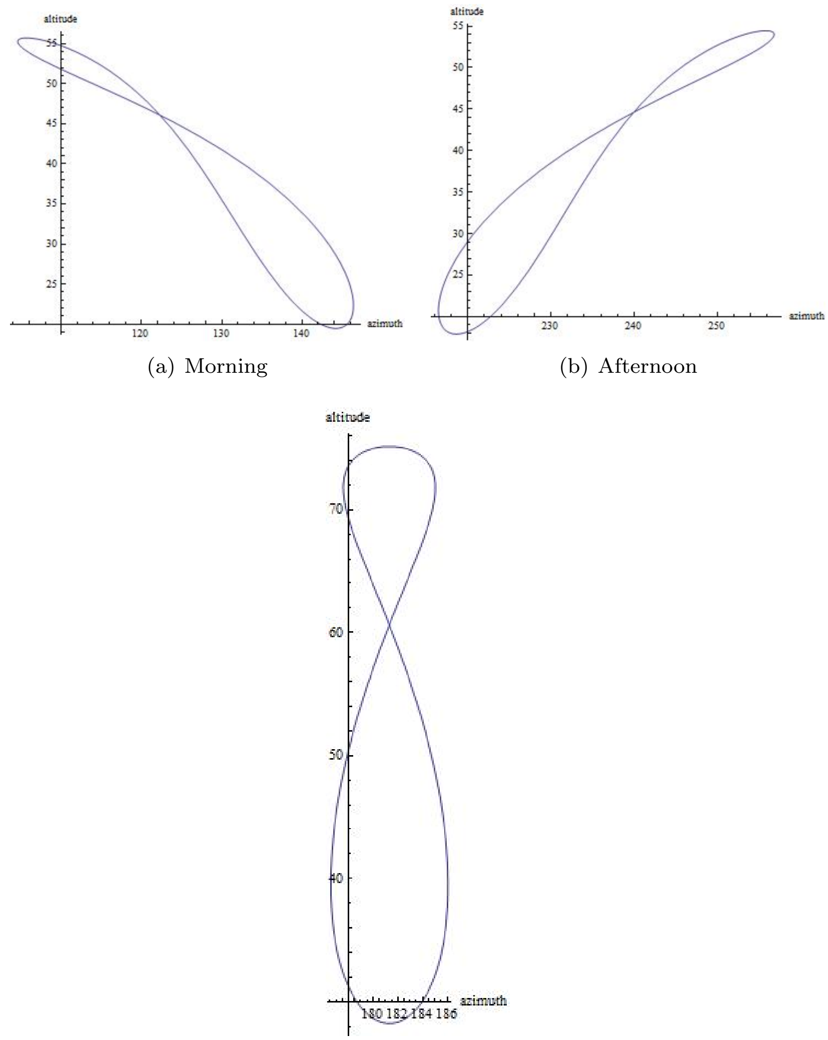
$$H = 15(x + \text{EquationOfTime}[date] \times 24)$$

The equation of time outputs time in units of fraction of a day and so this is multiplied by 24 to convert to hours. The summation of mean solar time and the equation of time gives true solar time. This is multiplied by 15 as the sun moves 15 degrees per hour. Here, the equation of time is being used to 'correct' the time as measured by a clock (mean solar time) to true solar time, and hence the position of the true sun.

An input variable in these equations is the latitude of the observer. It should be noted here that the latitude of the observer is taken to be exactly equal to the declination of the sun when it is seen directly overhead. This is not exactly true because the earth is not an infinitely small body inside the celestial sphere, but as the sun is significantly far away from this earth, this is a very reasonable approximation.

Once altitude and azimuth of the analemma have been calculated, they can be simply plotted on a two dimensional plane. However, a similar problem occurs here as when latitude and longitude of geographical locations are plotted on a map. Analemmas found in high altitude angles become extremely distorted.

3.1.1 Calendrica



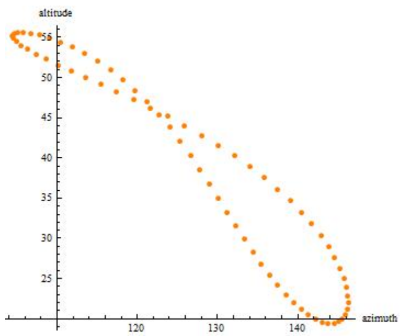
(a) Morning

(b) Afternoon

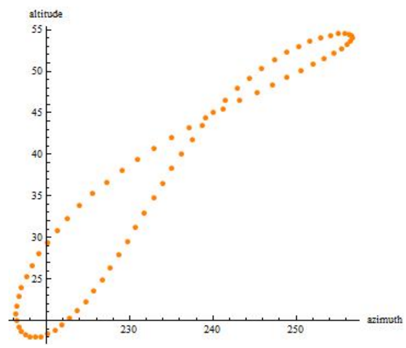
(c) Noon

Figure 8: Two-dimensional analemmas at 38°18'N

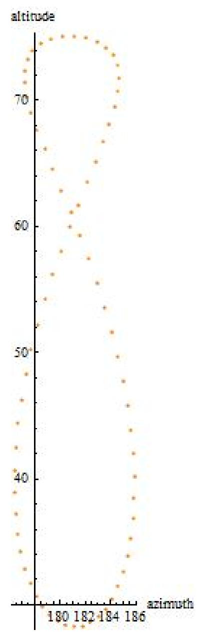
3.1.2 AstronomicalData



(a) Morning



(b) Afternoon



(c) Noon

Figure 9: Two-dimensional analemmas at 38°18'N using AstronomicalData

Both the models using Calendrica and using AstronomicalData do not resemble the straight and symmetrical photo of the analemma provided in the introduction in figure 1. One useful piece of information that can be gained from these graphics is the values for altitude and azimuth angle of the analemma. The absolute angular difference between the end of the 'top' loop and the end of the 'bottom' loop of the figure eight's shape is always $23.45° + 23.45° = 46.9°$. This is because the declination of the sun varies from $+23.45°$ to $-23.45°$ over one year.

Most importantly, it is apparent from these figures that the Calendrica package and the Astronomical package return exactly the same results. This implies accuracy in the equations and models involving Calendrica in determining the shape of the analemma.

4 3D Modelling

A parametrisation for a sphere can be given by:

$$x = r\cos v\cos u$$
$$y = -r\cos v\sin u$$
$$z = r\sin v$$

Where r is the radius of the sphere and $u, v \in [0, 2\pi)$.

Using this parametrisation, the two dimensional models in the previous section can be placed on a sphere of radius $r = 1$, where v is the altitude angle h and u is the azimuth angle A at a given time t, latitude ϕ and date D.

It then follows that a parametrisation for the analemma curve at a given time t and latitude ϕ would be:

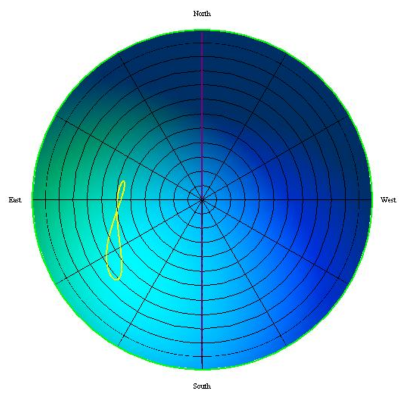
$$x = \cos(h(t,\phi,D))\cos(A(t,\phi,D))$$
$$y = -\cos(h(t,\phi,D))\sin(A(t,\phi,D))$$
$$z = \sin(h(t,\phi,D))$$

Where $D \in [D_0, D_0 + 365]$.

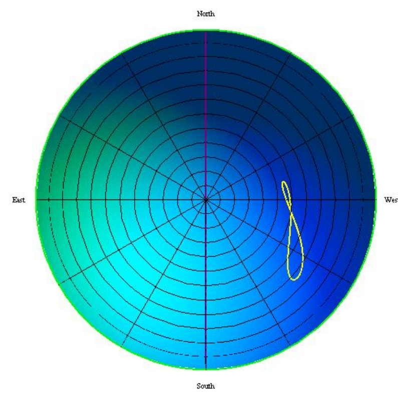
This parametrisation will be used to create three dimensional models by projecting the position of the sun as given in azimuth and altitude angles onto a sphere.

4.1 Analemma from below

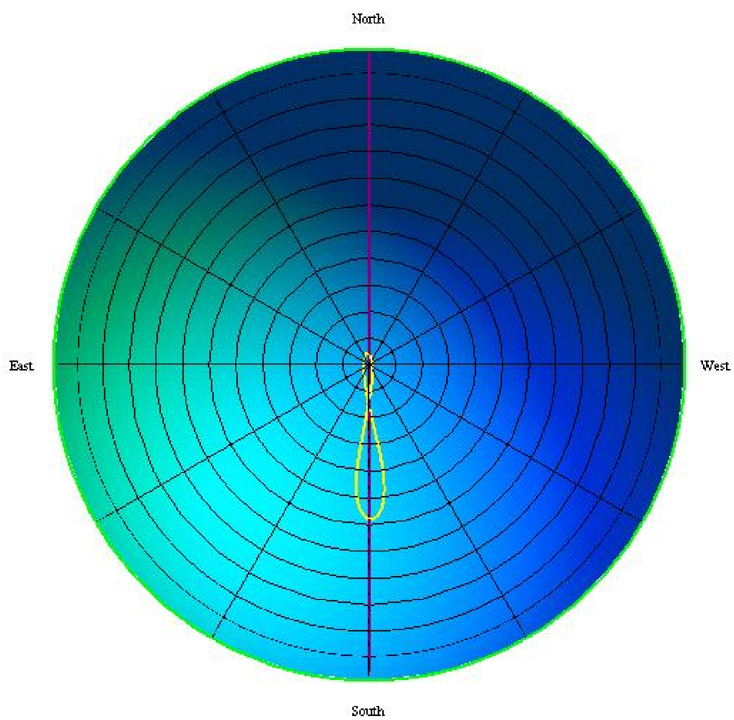
Astronomical maps and star charts generally present the northern or southern skies from a view point at the opposite pole on the celestial sphere. This is practical because an entire hemisphere can be seen in one image and the azimuth and altitude angles of celestial bodies can be easily read from the chart. The analemma can also be presented in this manner in order to represent its position in the sky for a viewer at a given latitude. The meshed lines in the following graphics imply the concavity of the hemisphere, so the inside of the hemisphere is being viewed.



(a) 9am



(b) 3pm



(c) 12pm

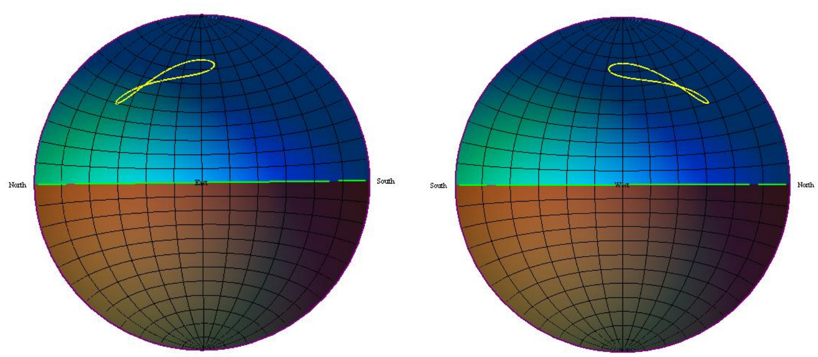
Figure 10: View from below at 20°N

Images such as these are beneficial as they give a clear indication of where the analemma is to be found in the sky at a certain time of day. It is important to notice that east is on the left hand side because the observer is looking upwards at the sky not downwards onto a map! In figure 10(a), the analemma is located in the east, and so it is a morning analemma (9am). The sun spends most of the year in the southern sky and so it is evident that the observer is in the northern hemisphere (20 degrees north).

A significant problem with such graphics is that the view point is from the opposite end of the celestial sphere. The human eye cannot see a complete 360 degree view whilst focusing on the zenith. Creating such an image in Mathematica is only possible, if one is to stand outside the interior region of the celestial hemisphere, in this case at the opposite pole of the celestial sphere. It is quite unnatural to look upwards at the sun during the daytime in this manner, as usually we notice the sun with respect to the horizon. In these graphics, points below the horizon are simply omitted and so there is little scope for depiction of events such as sunset and sunrise. When the sun nears the horizon, the skewing of images becomes most drastic and therefore such a view point is only practical when the sun is overhead.

4.2 Analemma from the horizon

By dissecting the celestial sphere through the zenith, the horizon becomes better depicted in the graphics and can be used as a reference point. This seems to be a more natural depiction of the analemma. Such images may be most useful for explaining the relationship between the analemma and sunrise and sunset. As this analemma is viewed from a latitude in the southern hemisphere, the figure eight shape has become inverted.



(a) 10am

(b) 2pm

Figure 11: View from the horizon at 20°S

4.3 Visualisation issues

In the three dimensional hemispherical models, distortion occurs along the boundary of the hemispheres, where the analemma is furthest from the viewing centre. This is as a result of the optical distortion of objects far from the point of focus. Although modern cameras correct for such distortions, they are evident in pinhole photographs such as figure 12 shown below. The spheres and cylinders become increasingly distorted as they become further away from the focal point.

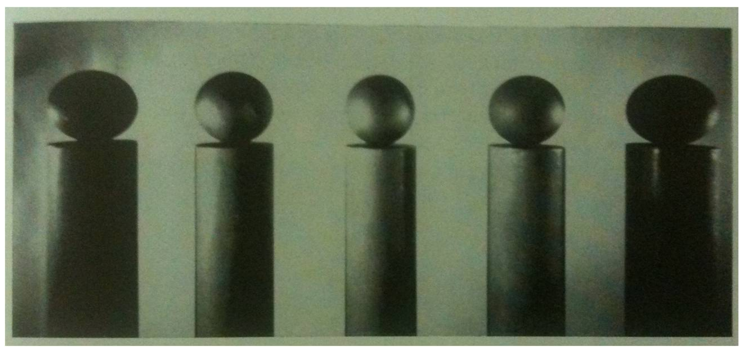


Figure 12: Five spheres atop five cylinders[6]

The only way to correctly view an object is to look directly at it. By directly looking at the analemma shown in figure 11(b), it appears as shown in figure 13. Although this view point is still inaccurate, because it is at a location on the celestial sphere and not on the earth, it significantly improves the appearance of the shape of the analemma. The analemma appears relatively straight and the distortion evident in figure 11(b) becomes corrected.

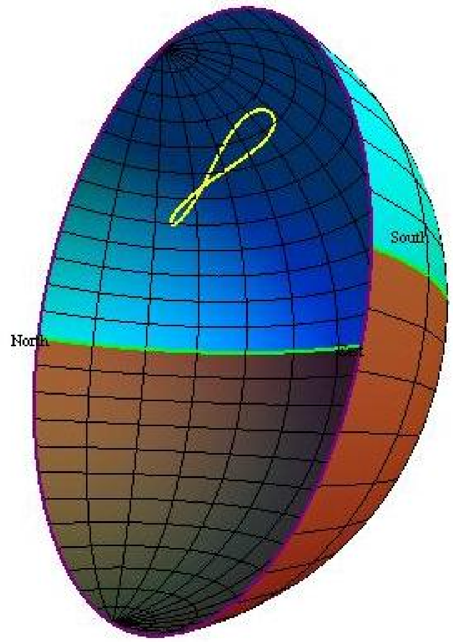


Figure 13: Rotated view of figure 11(a)

4.4 Centre of the analemma

In modelling the analemma, it is computationally practical to determine where the centre of the analemma is located. This can be used to determine where the line of sight should be directed in order to make view point corrections by looking directly at the analemma. Both an average of two points on the analemma approximately six months apart as well as the position of the mean sun at a given time provide a point near the centre of the analemma. As the path of the mean sun has been involved in previous calculations, this is a simple method to pursue.

The mean sun travels at a constant speed of $\frac{360^\circ}{24} = 15$ degrees per hour across the sky, so the change in angle of the mean sun between noon and any t number of hours after noon is given by $\frac{360^\circ}{24}t$. The path of the mean sun is along the celestial equator and the parametrisation for this path is provided below.

$$x = -\cos\left(\frac{360^\circ}{24}t\right)$$

$$y = \sin\left(\frac{360°}{24}t\right)$$
$$z = 0$$

Where $t \in [-12, 12]$.

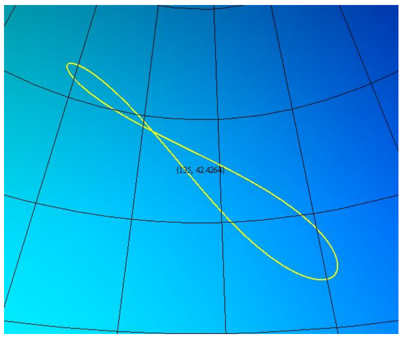
The azimuth and altitude angles of positions on the celestial equator are dependent on latitude. On the equator, the celestial equator passes directly overhead from east to west and undergoes a rotation of 90 degrees around the east-west axis. At the poles, the celestial equator follows the horizon and so undergoes a rotation of 0 degrees. The rotation matrix is given by $(90 - \phi)\text{RotationMatrix}\{0, -1, 0\}$, where ϕ is the latitude of the observer and $\{0, -1, 0\}$ is a rotation matrix about the y (east-west) axis. The angular position of the mean sun at at time t for a viewer at latitude ϕ is given by:

$$\{-\cos\left(\frac{360°}{24}t\right), \sin\left(\frac{360°}{24}t\right), 0\} \times (90 - \phi)\text{RotationMatrix}\{0, -1, 0\}$$

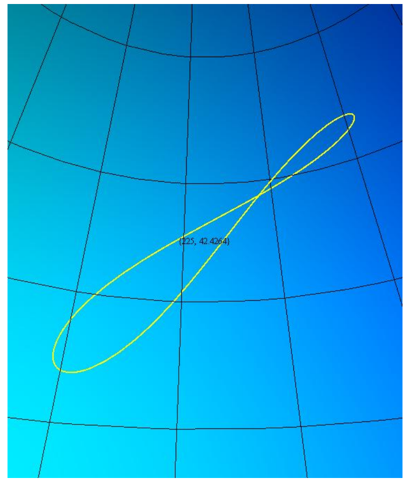
Once the position of the mean sun at a certain time and latitude is known, it becomes possible to computationally direct the line of sight towards the centre of the analemma. This alleviates optical distortions caused by viewing the analemma indirectly.

4.5 Viewpoint correction

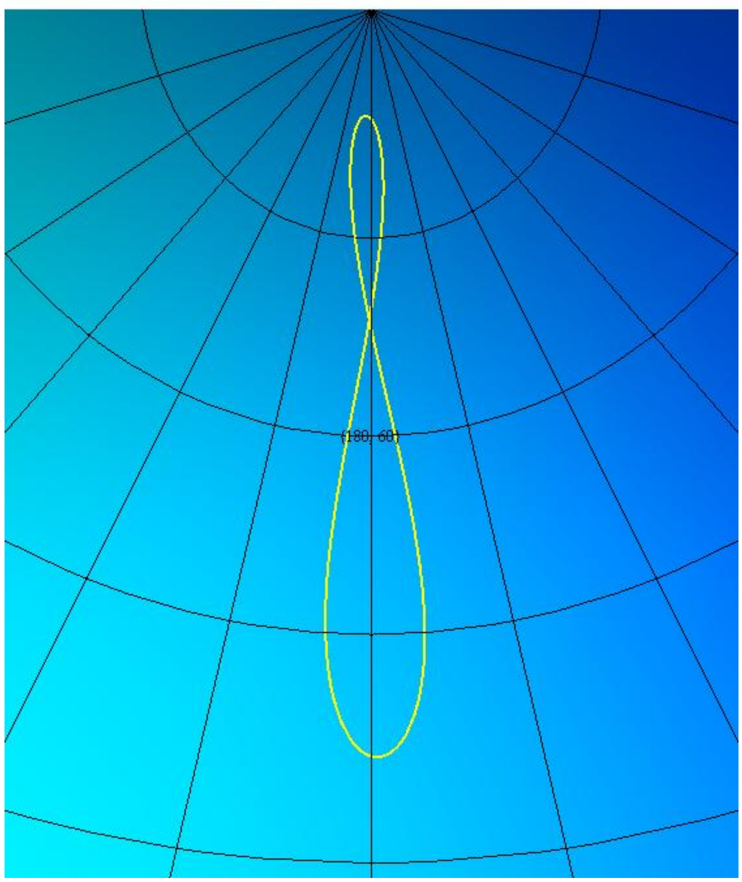
The following series of images show analemmas being viewed as they would be on earth. The view vector is from the centre of the sphere (the earth) to the centre of the analemma (position of the mean sun). These graphics of the analemma bear the strongest resemblance to photographs of the analemma taken from earth and are both realistic and accurate. The azimuth and altitude angles of the mean sun are marked in brackets at the location of the mean sun at the given time and latitude.



(a) 9am



(b) 3pm



(c) 12pm

Figure 14: Corrected viewpoint at 30°N

4.5.1 Sunrise on the equator

Correct and realistic viewing of the analemma assists in accurately explaining related phenomena such as the varying sunrise times in Singapore. As Singapore lies near the equator (1°22'N), an early morning analemma appears horizontal. Sunrise times in Singapore range from around 6:45am to 7:15am, although there is only about eight minutes variation between the length of the shortest and longest day.[1] Figure 15 shows the shape of the analemma at 7am in Singapore. Azimuth and altitude angles are marked in order to estimate the width and the height of the analemma.

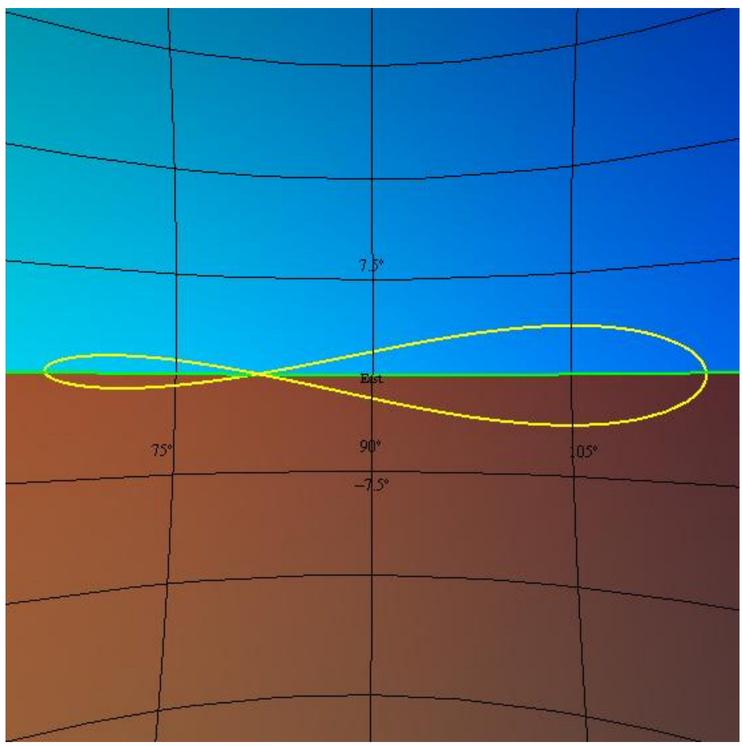
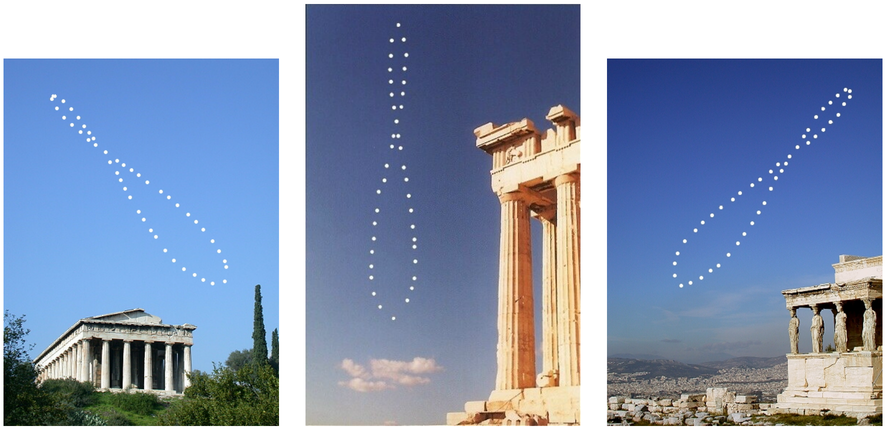


Figure 15: Sunrise in Singapore

As predicted, the angular distance between the 'top' and 'bottom' of the figure eight is approximately 47 degrees (23.45°+23.45°). The angular distance between the top and bottom of the analemma is approximately 7.5°. The sun travels 15 degrees every hour, so the 7.5 degree variation between the highest and lowest altitude angle on the analemma implies a 30 minute variation in sunrise time. The points of earliest and latest sunrise on the analemma correspond to around November 3 and February 10 respectively.

5 The Analemma in Athens



(a) Hephaisteion, 10am (b) Parthenon, 12:28.16pm (c) Erechtheion, 3pm

Figure 16: The Analemma in Athens[2]

Image Data

Photographer: Anthony Ayiomamitis

Latitude: 38.2997°N

Longitude: 23.7430°N

Time zone: UTC+2

Dates recorded: January 7, 2003 to Decemeber 20, 2003 (10am and 3pm analemma), January 12, 2002 to December 21, 2002 (12:28.16pm analemma)

5.1 Photographing the analemma

In order to photograph the analemma, is is essential to know where the mean sun would be located in the sky and to point the camera in that region. The camera's frame must also be wide enough to capture all possible locations of the sun at a particular time of day. To determine the position of the mean sun at a specific time of day at a specific location, corrections need to be made due to time zones and daylight savings time. In the photograph of the noon analemma in figure 16(b), Ayiomamitis attempted to capture a perfectly straight analemma occuring at mean solar noon. Unlike in Greenwich, mean solar noon does not occur at around 1200, rather at some time between 1200 and 1300, as Athens lies within two time zone boundaries.

The earth rotates at an average velocity of $\frac{24}{360}$ hours per degree. The photographs are recorded to have been taken at 23.7340 degrees longitude, so the actual time zone of Athens should be $+\frac{24}{360} \times 23.7340 = +1.583$ hours. Therefore, $2 - 1.583 = 0.417$ hours or 25.054 minutes should be subtracted from times given under the time zone UTC+2. This is especially important when generating models using Calendrica. Using the package AstronomicalData, the longitude and time zone are input values and therefore no corrections need to be made.

Interestingly, Ayiomamitis' photograph of the noon analemma was taken at 12:28.16pm. Using the longitudinal data provided, mean solar time should however occur at 12:25.02pm. The reason behind this is that the position of the analemma in the sky varies annually. The Gregorian calendar year is an approximation to the time taken for the earth to revolve around the sun and every four years, a correction must be made in the form of a leap year. As a result, there is also a marginal change in the sun's declination from year to year on a particular date, which is corrected each leap year. In order to calculate the time needed to capture a vertical analemma in 2002, Ayiomamitis calculated the time of day where the sun at the summer solstice and winter solstice would have the same azimuth angle. He found this to be 12:28.16pm, which is also the time of day where the analemma can be seen perfectly vertically in the sky.

Photographs of the analemma are a composition of photographs of the sun which are superimposed on a foreground image. In Ayiomamitis' solar photography, he uses photographs of Athens as the foreground image. It is hence impossible to judge the azimuth and altitude angle from the position of the horizon and also impossible to determine the direction which the camera is pointing. It is a fair assumption to presume that the camera was positioned on a flat surface, as the equivalent noon analemma in Athens is vertical. Presumably, the camera was also aimed in the region of the mean sun.

5.2 A comparison of images

After corrections are made for the time zone in which the images were taken, models made using Calendrica should strongly resemble the photographs of the analemma. Specifically, the shape and angle of the analemma in the models and the photographs should be the same, if the camera in the photographs is approximately pointing towards the position of the mean sun. It is difficult to judge whether the analemma in the models is the correct angular distance above the horizon; however, the example of the analemma at sunrise in Singapore implies that this aspect of the models is correct.

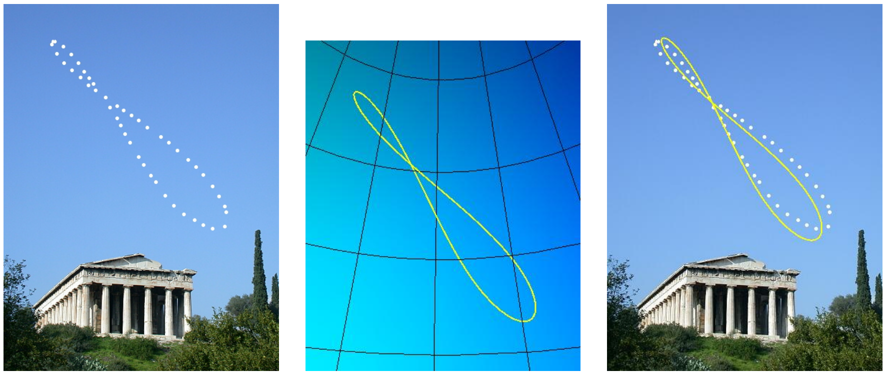


Figure 17: Hephaisteion, 10am

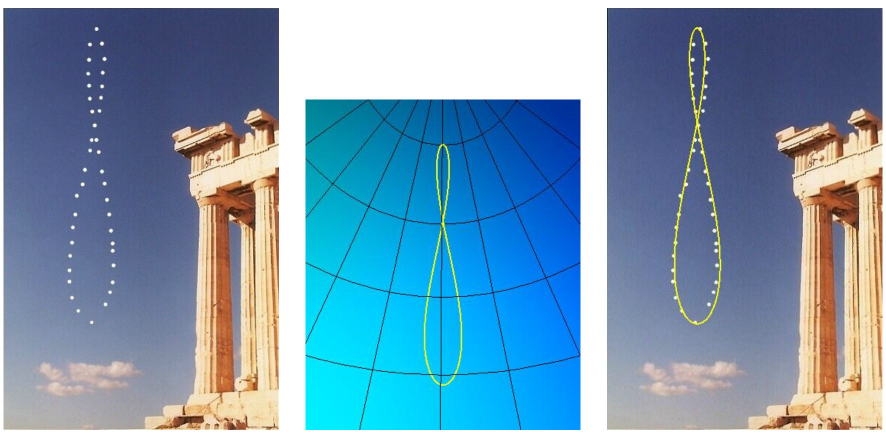


Figure 18: Parthenon, 12:28.16pm

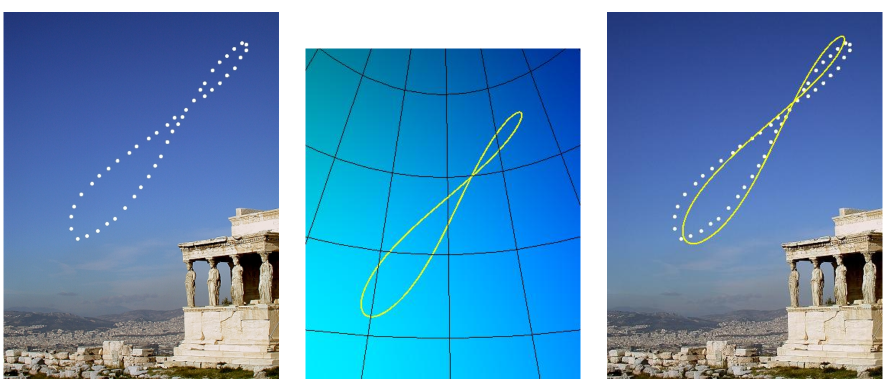


Figure 19: Erechtheion, 3pm

5.3 Discrepancy in results

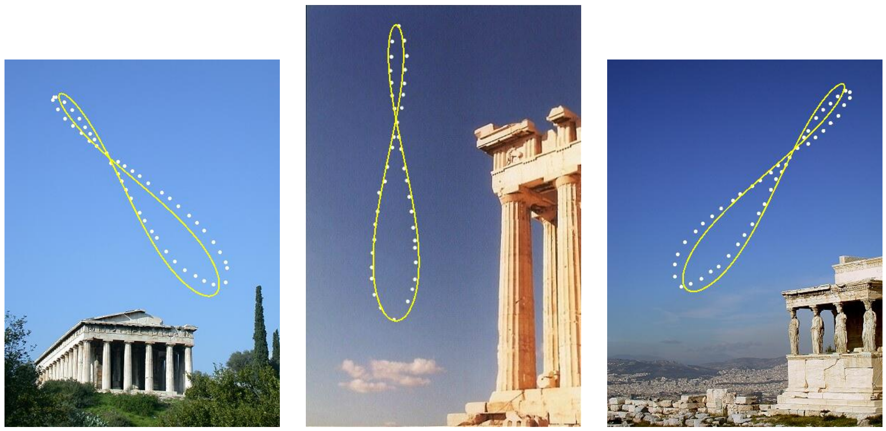


Figure 20: Model Testing

Although the photograph of the vertical analemma is replicated almost perfectly in the model, the morning and afternoon analemma models are both positioned at greater angle to the horizon than the photographs of the analemma at these times. There seems to be some systematic error in the assumptions made when creating the models. The accuracy of the noon analemma implies that the time corrections and the longitude of the photographer are correct. If the photographer were at a higher latitude, then the analemma would possess a greater angle at 10am and 3pm. The latitude provided by the photographer is, however, likely to be correct. A much more likely issue in matching the models to the photographs is the line of sight of the camera. If the camera is not positioned towards the mean sun, then there are numerous visual effects which can affect the shape and angle of the analemma. The camera used presumably corrects for visual distortions by using a rounded lens, but it is difficult to determine how the camera may change the angle of the analemma.

It is possible to change the direction of the viewing vector to look at another point in the sky so that the analemma appears at the correct angle. It is however also possible to change the viewing vector so that the analemma appears in many desired forms. To proceed with modifying the viewing direction in the programmatic figure is not very useful unless more information is acquired about the direction the camera is pointing whilst photographing the analemma and the camera's mechanisms for correcting visual distortions.

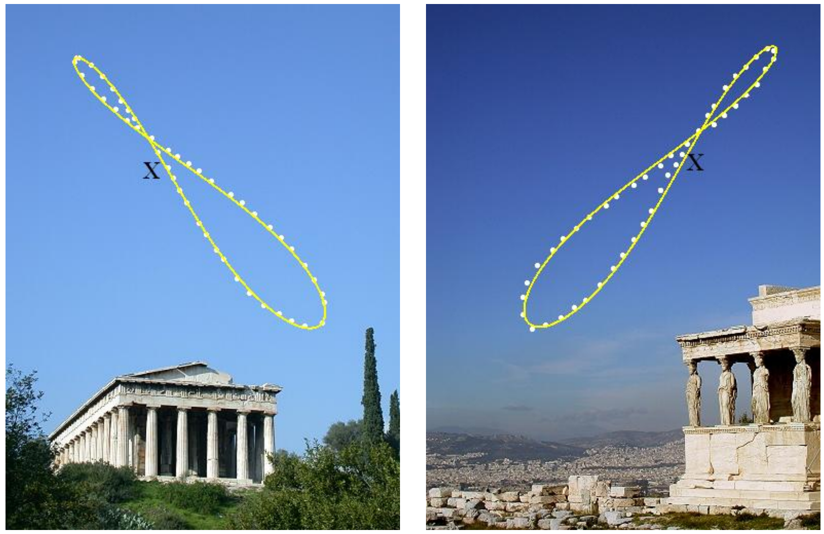


Figure 21: Adjusting the viewing direction

In figure 21, the 'X' marks the location of the direction of the adjusted viewing vector. By shifting the angle of the viewing vector to some point a few degrees east of the mean sun in 21(a) and to some azimuth angle a few degrees west of the mean sun in 21(b), the angle of the analemma changes in such a way that it approximately matches the photographs. This does not necessarily mean that Ayiomamitis' camera was pointed in these directions, as a camera lens may also adjust the angle of the analemma. Figure 5.3 shows how the angle of the analemma in the computer models can be significantly affected by a slight change in direction of line of sight.

6 Conclusion

Many astronomical phenomena can be explained through the use of typical models, such as the celestial sphere as seen from an external perspective or two dimensional projections. An external view of the celestial sphere is useful for understanding orbital motions for example the revolution of the earth around the sun, whereas the declination of the sun at different times throughout the year can be better depicted from inside the celestial sphere. When projecting the celestial sphere onto a plane, different projections should be used depending on whether lengths, area or shape should be conserved. Often it is simpler and more accurate to depict three dimensional phenomena on three dimensional axes. The way in which objects appear depends on the viewer's line of sight. In order to minimise visual distortions and see an object correctly, it is essential to look directly at the object.

By modelling the analemma using programmable figures, it is possible to see how the analemma appears at different parts of the world at different times of the day. There are various ways to depict the analemma, but the most useful way to depict the analemma is realistically and accurately. The analemma is a three dimensional phenomenon which is seen from the earth. Good models of the analemma must therefore be presented on three dimensional axes and be viewed from the centre of the celestial sphere. View point and line of vision can significantly affect how the analemma appears in programmatic figures, even when the model is scientifically correct.

Realistic and accurate models of the analemma should resemble photographs of the analemma. Through comparing models of the analemma in Athens with Ayiomamitis' photography, the shape of the models is shown to be accurate. The angle of the analemma in the models closely matches that in the photographs, where the discrepancy is possibly due to the inability to view the models in the same direction in which the camera is pointing. The appearance of the analemma is dependant on the location and line of sight of the viewer, be it from outside, on or inside the celestial sphere. In this project, the most useful depictions of the analemma were those which viewed the centre of the analemma from a view point on the earth. These models assist in explaining the irregular path of the sun at the same time of each day throughout each year.

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