Measuring the Tropical Year in Chinese Astronomy

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An academic exercise presented in partial fulfillment for the degree of Bachelor of Science with Honours in Mathematics

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Lee Suling

Dedicated to ……

GOD

Family

Friends

Especially, Xiao Hui, Xiao Juan.

Thank you. It’s been a crazy year, u made it bearable.

Summary

The objective of this project is to write a clear mathematical supplement to the journal "Accuracy of Pre-Modern Determinations of Tropical Year Length" by Shigeru Nakayama. It aims to provide clear explanations and proofs on certain theories and calculated values stated under the first three sections in Nakayama's journal: Solstitial Observation, Kuo-Pei Shu and Equinoctial Observation.

To understand how the shadow paths of a gnomon changes throughout the year at different latitudes, we will need the knowledge of conic sections. This thesis will show a relation between conic sections and shadow paths.

Due to a strange phenomenon observed by our supervisor on his trip to Mexico, this thesis will give the theories and explanations on the Sun's path at Chichen Itza in Mexico and the shadows formed by a gnomon there. These explanations will require some understanding of basic astronomy.

Statement of Author's Contributions

We have done the following in this project:

1. Under Section 3.1, we have shown a clear explanation on how a 1 cm error in measurement of shadow length creates four to five days error in determining the date of winter solstice and eight days error in the case of the summer solstice.
2. Under Section 3.2, we have provided a clearer view of the amount of error produced in Kuo-Pei Shu and corrected certain terms defined in Nakayama's journal. We have also explained how an error of almost nineteen hours was obtained if the equinoxes were chosen as points *x* and *c*, as mentioned in Nakayama's journal.
3. Under Section 3.3, we have provided explanations and a number of clear pictures to illustrate certain topics mentioned in Nakayama's journal under equinoctial observations: how equinox can occur at night, accuracy of the equatorial ring and the comparisons between the accuracy of the equatorial ring and the gnomon.
4. Under Chapter 4, we have come up with explanations of how different shadow paths are formed by a gnomon throughout a year at certain latitude by using the help of conic sections.
5. Under Chapter 5, we have offered detailed observations of the Sun's motion and the gnomon's shadow paths for what we term "The Maya Phenomenon". Explanations and pictures have been provided to help in the understanding of this phenomenon.

Contents

Acknowledgements……………………………………………………………………i

Summary…………………………………………………………………………...ii

Statement of Author's Contributions………………………………………………...iii

Chapter 1 Basics of Astronomy…………………………………….…………….1

1.1 Horizon Reference System……………………….………..…...1

1.2 Equatorial System of Coordinates………………….……………...3

1.3 Altitude vs. Latitude……………………………………………….4

Chapter 2 Solstices and Equinoxes……………………………………………….5

2.1 Heliocentric Model………………………………………………...5

2.2 Sun's Daily Path…………………………………………………...7

Chapter 3 Accuracy of Pre-Modern Determinations of Solstices and Equinoxes……………………………………………….8

3.1 Solstitial Observation……………………………………………...8

3.2 Kuo-Pei Shu………………………………………………………...13

3.3 Equinoctial Observations……………………………………...24

Chapter 4 Conic Sections and Shadows……………………………………....33

4.1 Basics of Conic Sections……………………………………....33

4.2 Shadows cast in the Artic……………………………………...39

4.3 Shadows cast in the Temperate Zone…………………………….57

4.4 Shadows cast in the Tropics…………………………………………...63

Chapter 5 Maya Phenomenon…………………………………………………...74

5.1 Introduction……………………………………………………....74

5.2 The Maya Phenomenon ………………………………………….78

5.3 Detailed Observation of Sun's Motion…………………………...82

5.4 Gnomon Shadow………………………………………………....85

5.5 Transitions Points………………………………………………...87

5.6 Critical Periods…………………………………………………...92

Glossary…...…………………………………………………………………….99

Appendix 1 - Table of SUN & PLANETS AT GMT 0hrs DECEMBER 2000…....104

Appendix 2 - Table of SUN & PLANETS AT GMT 0hrs JUNE 2000…………….105

Appendix 3 – Table of Sun's Azimuth for One Day……………………………..107

Bibliography…………………………………………………………...……109

Chapter 1

Basics of Astronomy

1.1 Horizon Reference System

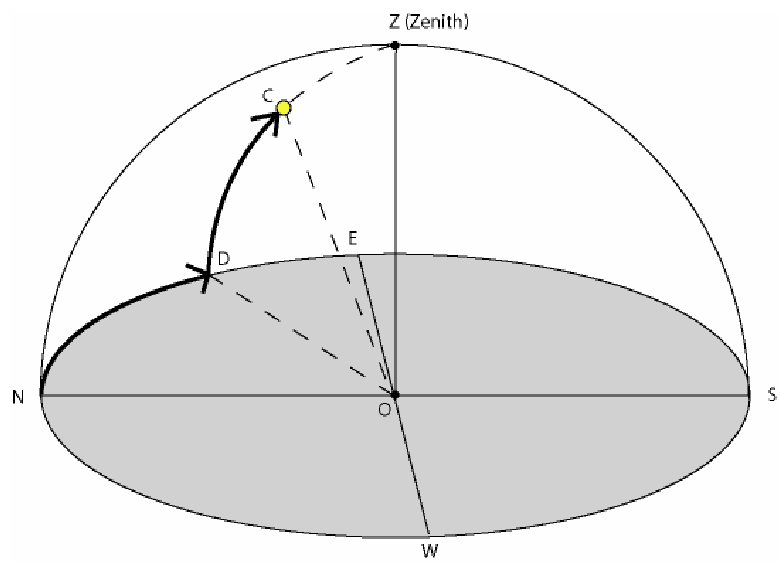

Figure 1.1

In this system, the *celestial sphere* is an infinite sized sphere centered about an observer situated at point O. The NSEW plane in Figure 1.1 is called the observer's *horizon*. The point Z which marks the overhead position of the observer is called the *zenith*. We can define the position of a celestial body, C, by stating its *azimuth* and *altitude*. The azimuth is the angle measured from the NS line to its position on the horizon, D, in the direction marked above. In this diagram, the celestial body's azimuth is marked by an angle NOD. Its altitude is the angular distance measured vertically upwards from horizon, marked by the arc DC.

1.2 Equatorial System of Coordinates

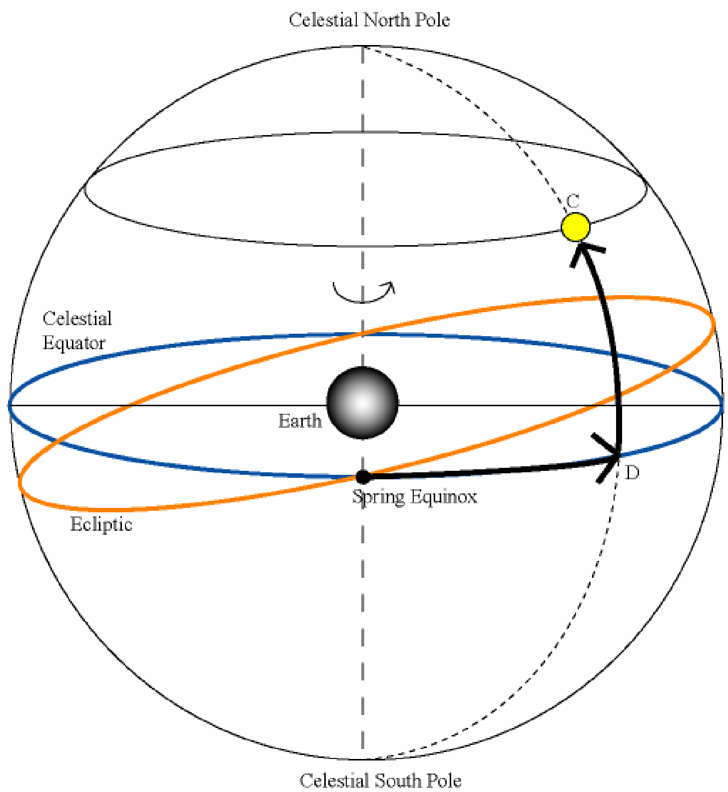

Figure 1.2

To indicate the position of a celestial body C on the celestial sphere about the earth, we need a reference point on the celestial sphere, called the spring equinox, where the *ecliptic* crosses the *celestial equator*. Drop the position of C onto the celestial equator to the point D. The angular distance between spring equinox and point D is the *right ascension* of C. From D, the angular distance measured upwards along a *meridian line* from the celestial equator to the celestial body C is its *declination*.

1.3 Altitude vs. Latitude

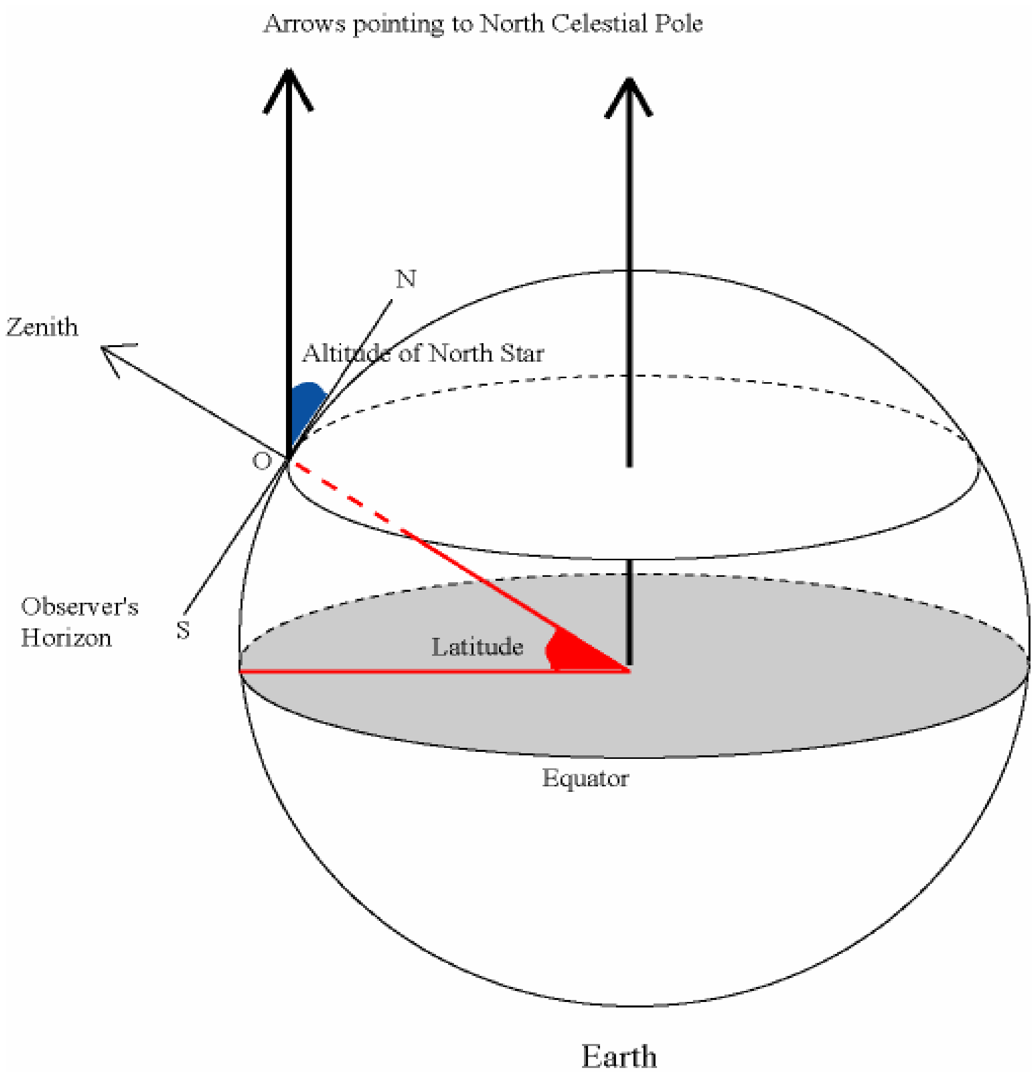

Figure 1.3

The *latitude* of an observer standing at point O is equal to the altitude of the North Star.

Chapter 2

Solstices and Equinoxes

2.1 *Heliocentric* Model

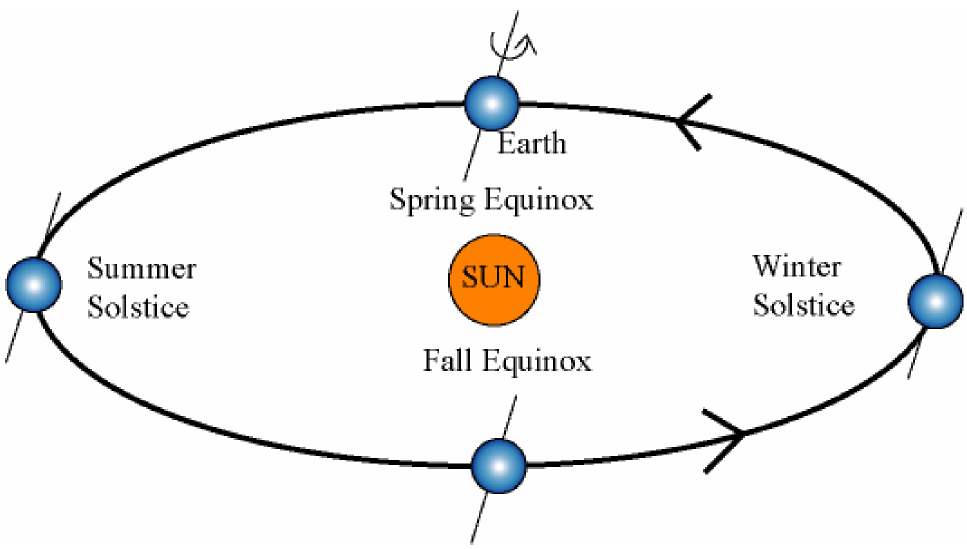

Figure 2.1

The earth rotates about its axis which points to the *North celestial pole* and it orbits the sun in a plane called the ecliptic. Alternatively, the ecliptic may be defined as the apparent path of the sun on the celestial sphere. During the summer solstice, the earth's axis is tilted towards the sun while during the winter solstice, it tilts away from the sun. Thus, an observer situated at the Northern hemisphere will receive more sun during the summer solstice than the winter solstice. However, during the *equinoxes*, everywhere on the earth will experience equal amount of day and night.

2.2 Sun's Daily Path

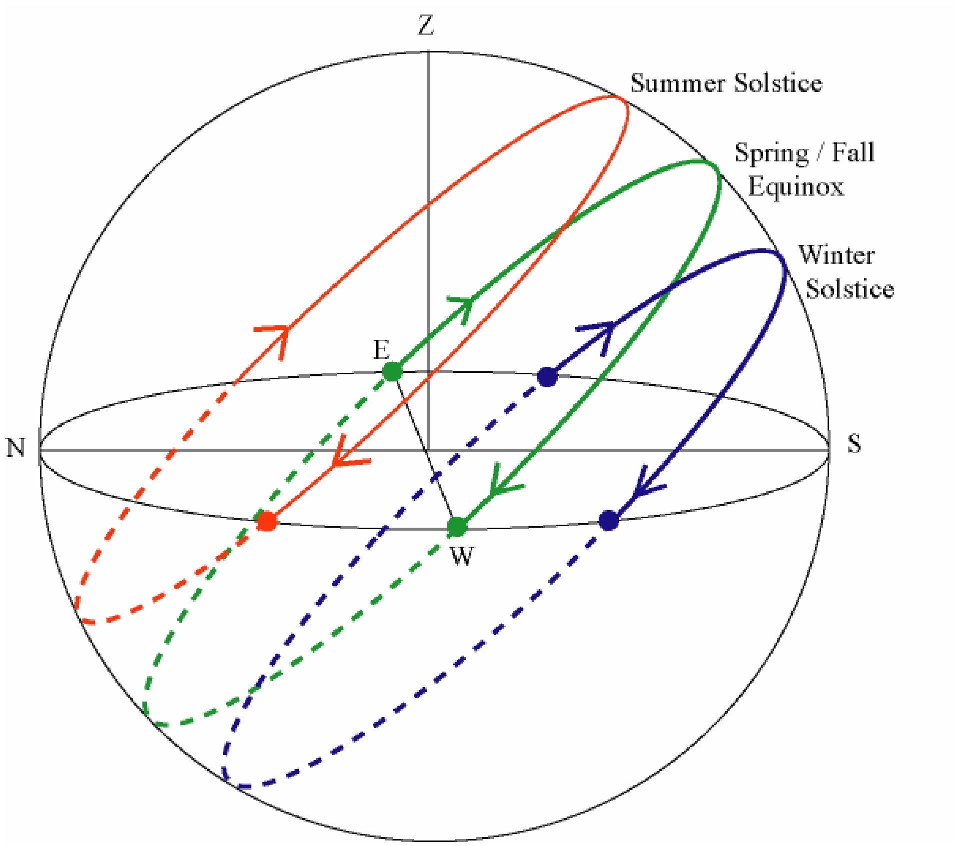

Figure 2.2

For an observer in the Northern hemisphere at latitude of about $40°$N, the paths of the sun during the *solstices* and the equinoxes are illustrated in Figure 2.2. During the summer solstice, the sun rises somewhere in the North-east and sets in the North-west. During the spring and autumnal equinoxes, it rises exactly from the east and sets exactly west. During the *winter solstice*, it rises in the South-east and sets in the South-west.

Chapter 3

Accuracy of Pre-Modern Determinations of Solstices and Equinoxes

3.1 Solstitial Observation

In this section, we will show how an error in measurement of shadow lengths leads to an error in determining the date of summer solstice or winter solstice and an example which indicates that an error of 1 cm in measurement of shadow length creates four to five days error in determining the date of the winter solstice and eight days error in the case of the summer solstice.

The *gnomon* is the most primitive astronomical instrument and its shadow length was used to determine the day of winter solstice.

Winter solstice is the moment when the Sun reaches its minimum declination and on the day that it occurs, the midday gnomon shadow length will be the longest as compared to other days.

For simplicity, we will assume the gnomon length to be 200 cm. In China, most observations were made at Yang Cheng whose latitude is $34.6°$. We will adopt this latitude in our calculations.

In Figure 3.1,

l : the length of gnomon (= 200 cm)

s : midday shadow length of gnomon

h : the altitude of the Sun

φ : latitude of observational site ($= 34.6°$)

δ : the Sun's declination at the time of observation

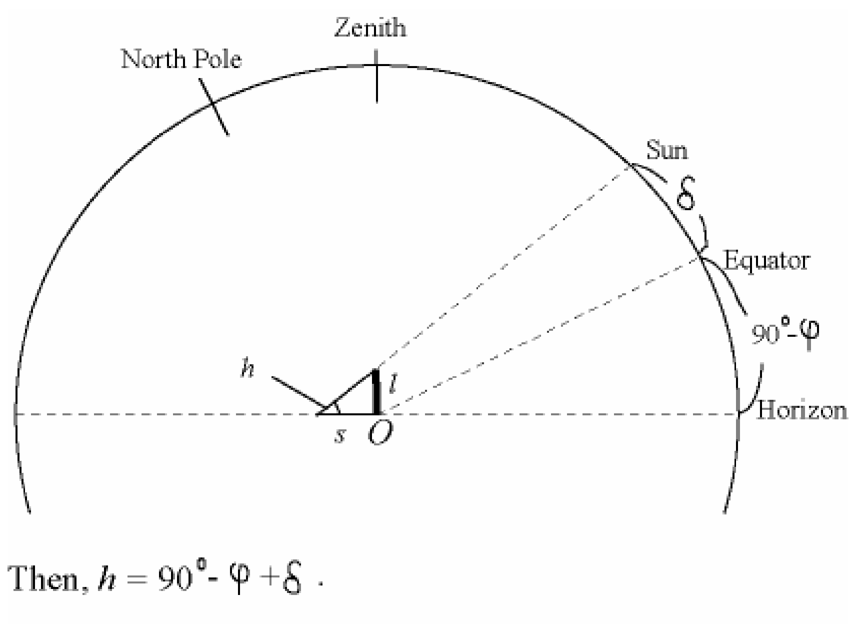

Figure 3.1

Since the variation of s depends solely on the variation of solar declination,

(1) $$\Delta h = \Delta \delta .$$

From the triangle in the above figure,

$$\tan h = \frac{l}{s},$$
$$s = l \cot h.$$

Taking the derivative of s with respect to h,

$$\frac{ds}{dh} = -l \operatorname{cosec}^2 h,$$

(2) $$\Delta s = -l \operatorname{cosec}^2 h \,\Delta h.$$

The error in measurement of shadow length Δs will produce an error in determining the altitude of the Sun and this is indicated by Δh. By (1), $\Delta\delta = \Delta h$. Thus, this difference of solar declination leads to an error in determining the date of the winter solstice.

Shigeru Nakayama [1] compared the accuracy in determining the winter solstice and summer solstice in the use of a gnomon by commenting that a 1 cm error in measurement of shadow length creates four to five days error in determining the date of the winter solstice and eight days error in the case of the summer solstice. We will explain how this error was derived.

In the case of the winter solstice, we will assume the solar declination at that time to be $-23.5°$. Thus, using the following:

$$\delta = -23.5°,$$

$$h = 90° - \varphi + \delta = 90° - 34.6° + (-23.5°) = 31.9° = 0.557\,\text{rad},$$

$$l = 200\,\text{cm},$$

$$\Delta s = 1\text{cm},$$

and substituting them into (2),

$$1 = -200\,\text{cosec}^2(0.557)\Delta h,$$

$$\Delta h = -1.397 \times 10^{-3}.$$

Since $1° = 60'$, we obtain 1 rad $= (360/2\pi \times 60)'$.

Therefore, the difference of solar declination converted to minutes gives

$$1.397 \times 10^{-3} \times (360/2\pi \times 60) = 4.8 \approx 5'.$$

This $5'$ difference of solar declination causes four to five days error in determining the date of the winter solstice. This will be explained in Appendix 1.

In the case of the summer solstice, we will assume the solar declination at that time to be $+23.5°$. Thus, using the following:

$$\delta = +23.5°,$$

$$h = 90° - \varphi + \delta = 90° - 34.6° + 23.5° = 78.9° = 1.37\ \text{rad},$$

$$l = 200\,\text{cm},$$

$$\Delta s = 1\,\text{cm},$$

and substituting them into (2),

$$1 = -200\,\text{cosec}^2(1.377)\Delta h,$$

$$\Delta h = -4.815 \times 10^{-3}.$$

Therefore, the difference of solar declination converted to minutes gives

$$4.815 \times 10^{-3} \times (360/2\pi \times 60) = 16.6 \approx 17'.$$

This $17'$ difference of solar declination causes eight days error in determining the date of the summer solstice. Similarly, this will be explained in Appendix 2.

3.2 勾配术 (Kuo-Pei Shu)

In this section, we will look into one of the methods used in ancient China to determine the day of winter solstice called 勾配术 (Kuo-Pei Shu) and the amount of error in this method at certain situations.

For the determination of the moment of the winter solstice, mere observations of midday gnomon shadow length are not enough to give the time to a fraction of a day. Furthermore, it is difficult to determine the day of the winter solstice as the daily variation of the solar declination is at its minimum at the solstices. Thus, a Chinese astronomer of the fifth century, 祖冲之 (Tsu Ch'ung-chih) (430-501) invented a method called 勾配术 (Kuo-Pei Shu) which is illustrated below.

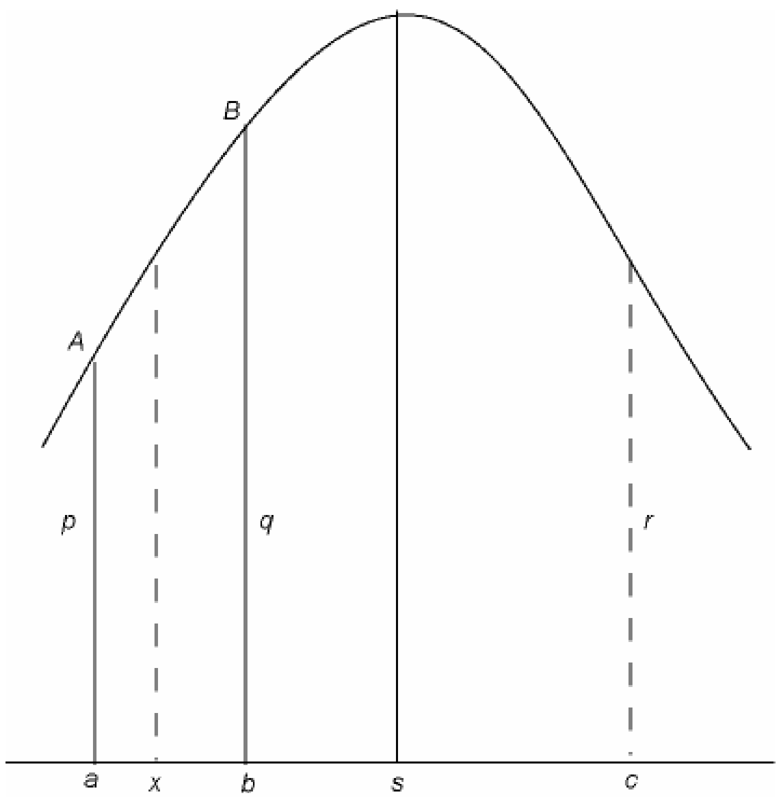

Figure 3.2

In Figure 3.2, p and q are gnomon shadow lengths on two consecutive days a and b before the winter solstice and r is the gnomon shadow length on day c after the winter solstice, where the relation $p < r < q$ holds. Since the distance AB is small, it may be regarded as a straight line. Then, the following relation holds:

$$\frac{q-r}{r-p} = \frac{b-x}{x-a}.$$

The curve was assumed to be symmetrical for the winter solstice. Therefore, the date s is given by

$$\frac{c-x}{2} = s.$$

Thus, not only the day but also the hour of the winter solstice could be determined. This method is also known as the method of gradation.

On page 103 of [1], Shigeru Nakayama discussed the amount of error in 勾配术 (Kuo-Pei Shu). Due to the solar equation of centre, the mean of the times of the same solar declination before and after a winter solstice does not necessarily coincide with the time of the solstice itself. In this paper, we shall give a clear explanation of this error. However, before we discuss this error, we will explain the solar equation of centre.

To understand the solar equation of center, we require the knowledge of *mean anomaly* and *true anomaly*.

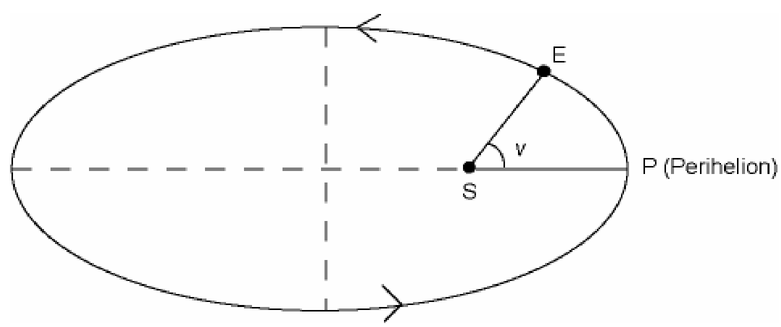

Figure 3.3

Figure 3.3 shows the Earth's orbit around the Sun (at point S). If the Earth is at point E on its orbit, its true anomaly is the angle v that the line SE makes with the line SP.

Suppose the Earth orbits about the Sun with constant angular velocity *n* and the time taken from perihelion is *t*, the mean anomaly is equal to *nt*.

Using Smart's Textbook on Spherical Astronomy [2], we extract the following solar equation of center:

(1) $$v - M = (2e - \frac{1}{4}e^3)\sin M + \frac{5}{4}e^2 \sin 2M + \frac{13}{12}e^3 \sin 3M \ ,$$

where *v* stands for true anomaly, *M* stands for mean anomaly and *e* stands for eccentricity of the Earth's orbit around the Sun.

Eliminating higher order terms in *e*, we obtain

(2) $$v - M = 2e\sin M \ .$$

At the same time, we shall make use of these relevant equations from Smart:

(3) $$L = \varpi + v$$

(4) $$l = M + \varpi$$

where *L* stands for *true longitude* of a planet, *l* stands for *mean longitude* of a planet and ϖ stands for longitude of *perihelion*.

From (3) and (4), we obtain

(5) $$v = L - \varpi$$ and

(6) $$M = l - \varpi \ .$$

Substituting these two equations into (2), we get

(7) $$L = l + 2e\sin M .$$

If we define λ as the true longitude of the Sun and λ_0 as the mean longitude of the Sun, we can redefine (6) and (7) as

(8) $$\lambda = \lambda_0 + 2e\sin M$$ and

(9) $$M = \lambda_0 - \varpi .$$

Thus, using (8), (9) and the fact that $\lambda = 3\pi/2$ at the winter solstice, we get

(10) $$3\pi/2 = \lambda_0 + 2e\sin(\lambda_0 - \varpi) .$$

Since $\sin\lambda \approx \sin\lambda_0$, we can substitute $\lambda_0 = 3\pi/2$ in the sine function above.

(11) $$3\pi/2 = \lambda_0 + 2e\sin(3\pi/2 - \varpi) .$$

In order to avoid confusion, we define λ_0^s as the mean longitude of the winter solstice and hence

(12) $$3\pi/2 = \lambda_0^s + 2e\sin(3\pi/2 - \varpi) .$$

Since $\sin(3\pi/2 - \varpi) = -\cos\varpi$,

(13) $$3\pi/2 = \lambda_0^s - 2e\cos\varpi .$$

If we define x and c to be the days before and after the winter solstice on which the shadow lengths are equal, then the solar altitudes at x and c are the same and therefore the solar declinations at x and c are the same, $\delta^x = \delta^c$.

Next, we observe the position of the Sun in the equatorial system.

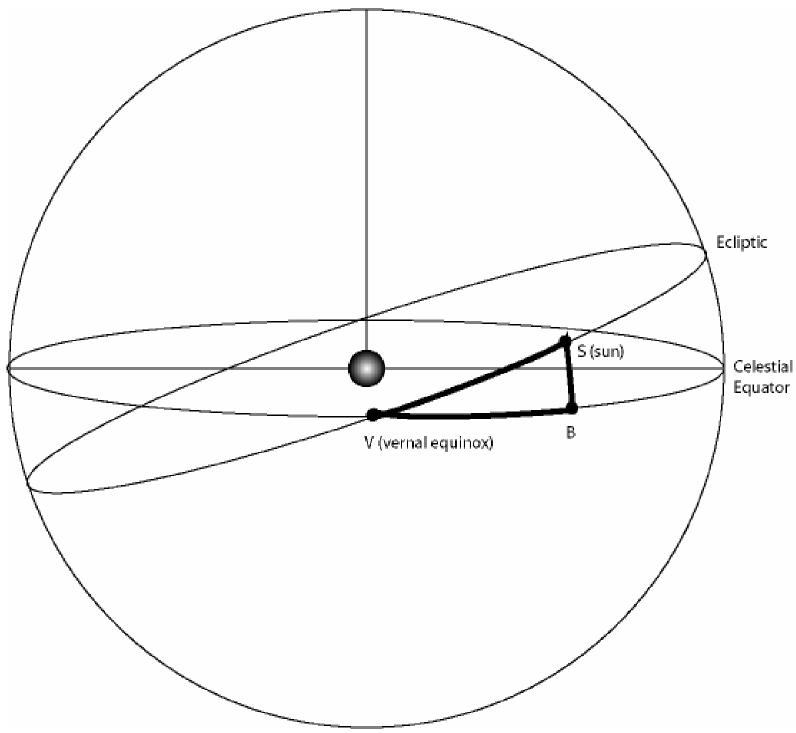

Figure 3.4

$VS = \lambda$ Longitude of the Sun

$BS = \delta$ Declination of the Sun

$\angle BVS = \varepsilon$ *Obliquity of the ecliptic* (Note: $\varepsilon = 23.5°$)

$\angle VBS = 90°$

Since ΔVBS is a spherical triangle, we have the following equation:

$$\frac{\sin\delta}{\sin\varepsilon} = \frac{\sin\lambda}{\sin 90°},$$

(14) $$\sin\delta = \sin\lambda\sin\varepsilon\,.$$

Now we define δ^x, δ^c as the declinations of the Sun at *x* and *c*, λ^x, λ^c as the longitude of the Sun at *x* and *c*, and M^x, M^c as the mean anomaly of the Sun at *x* and *c*.

Using (14), we can further establish these equations:

(15) $$\sin\delta^x = \sin\lambda^x \sin\varepsilon\,.$$

(16) $$\sin\delta^c = \sin\lambda^c \sin\varepsilon\,.$$

Since $\delta^x = \delta^c$, we have $\sin\delta^x = \sin\delta^c$.

Therefore, looking back at (15) and (16), we obtain

$$\sin\lambda^x \sin\varepsilon = \sin\lambda^c \sin\varepsilon\,,$$

(17) $$\sin\lambda^x = \sin\lambda^c\,.$$

Solving this equation, we get two general solutions for λ^x:

(18) $$\lambda^x = \lambda^c + 2n\pi \qquad \text{or} \qquad \lambda^x = (2n+1)\pi - \lambda^c$$

If we use the latter solution in (18) and $n = 1$, we get:

$$\lambda^x = 3\pi - \lambda^c\,,$$

(19) $\lambda^x + \lambda^c = 3\pi$.

Looking back at (8), we obtain:

(20) $\lambda_0^x = \lambda^x - 2e\sin M^x$

and $\lambda_0^c = \lambda^c - 2e\sin M^c$.

Therefore, the mean of the mean longitudes at *x* and *c* is

(21)

$$\begin{aligned}
\frac{\lambda_0^x + \lambda_0^c}{2} &= \frac{\lambda^x - 2e\sin M^x + \lambda^c - 2e\sin M^c}{2} \\
&= \frac{(\lambda^x + \lambda^c) - 2e(\sin M^x + \sin M^c)}{2} \\
&= \frac{3\pi - 2e(\sin M^x + \sin M^c)}{2} && (\because \lambda^x + \lambda^c = 3\pi) \\
&= 3\pi/2 - e(\sin(\lambda^x - \varpi) + \sin(\lambda^c - \varpi)) && (\because \lambda_0 = M + \varpi \Rightarrow \sin M^x = \sin(\lambda_0^x - \varpi) \\
& && \approx \sin(\lambda^x - \varpi)) \\
&= 3\pi/2 - e(\sin(\lambda^x - \varpi) + \sin(\lambda^x + \varpi)) && (\because \lambda^c = 3\pi - \lambda^x \Rightarrow \lambda^c - \varpi = 3\pi - (\lambda^x + \varpi) \\
& && \Rightarrow \sin(\lambda^c - \varpi) = \sin(3\pi - (\lambda^x + \varpi)) \\
& && = \sin(\lambda^x + \varpi)) \\
&= 3\pi/2 - e(2\sin\lambda^x \cos\varpi) \\
&= 3\pi/2 - 2e\sin\lambda^x \cos\varpi \ .
\end{aligned}$$

Earlier, we have the mean longitude of winter solstice,

(22) $\lambda_0^s = 3\pi/2 + 2e\cos\varpi$.

Calculating the difference between (21) and (22), we have

(23) $$\lambda_0^s - \left(\frac{\lambda_0^x + \lambda_0^c}{2}\right) = 3\pi/2 + 2e\cos\varpi - 3\pi/2 + 2e\sin\lambda^x \cos\varpi$$

$$= 2e\cos\varpi(1+\sin\lambda^x)\,.$$

There was a typing mistake when Nakayama stated this difference as the difference between the mean and the true winter solstice. It should be the difference between the mean longitude of the winter solstice and the mean of the mean longitudes at days before and after the winter solstice with the same solar declination.

When we divide this difference of mean longitudes by the mean solar motion, we will be able to get the error in day units.

During 郭守敬 (Kuo Shou-Ching)'s time in the 1270's, the perihelion coincided with the winter solstice. Thus, the longitude of perihelion, ϖ, equals $3\pi/2$.

Using this value of ϖ, the difference in (23) is zero since $\cos(3\pi/2) = 0$.

In other words, 郭守敬 (Kuo Shou-Ching) was able to apply 勾配术 (Kuo-Pei Shu) without any error and the curve was in fact symmetrical at that time.

However, in the case of the older observations in the 460's, a large error occurred when the summer solstice was determined using 勾配术 (Kuo-Pei Shu). There was another typing mistake when Shigeru Nakayama pointed out that the longitude of the perihelion at that time was equal to $156.5°$.

The Earth's axis revolves around in a circle with a period of about 25 800 years. This makes the spring equinox move clockwise by $50''$ with respect to the stars each year. This phenomenon is known as the precession of the equinoxes. The Earth's orbit rotates counterclockwise in the ecliptic plane with a period of about 110 000 years. This means that the orbit rotates in the opposite direction of the precession of the equinoxes. The net effect is that while it takes the spring equinox 25 800 years to complete one clockwise revolution with respect to the stars, it only takes about 21 000 years to complete one clockwise revolution with respect to the orbit. Hence, it indicates that the longitude of the perihelion takes approximately 21 000 years to increase by $360°$ in the orbit.

Thus, comparing with the longitude of the perihelion in 1270's, we get the difference in the longitudes as $\frac{1270-460}{21000} \times 360°$. Using this difference and the fact that the longitude of the perihelion in the 1270's was $3\pi/2$, which equals to $270°$, we get the longitude of the perihelion in the 460's, $\varpi = 270° - \left(\frac{1270-460}{21000}\right) \times 360° = 256.11° = 4.47\,\text{rad}$.

Using this corrected ϖ, we seek to explain how an error of almost nineteen hours was obtained if the equinoxes were chosen as points *x* and *c*, as mentioned in Nakayama's paper.

Combining the equations in (8) and (9), we obtain

$$\lambda = \lambda_0 + 2e\sin(\lambda_0 - \varpi).$$

In a similar way as before, we define λ^J, λ_0^J as the true and mean longitude of the summer solstice respectively. Thus,

$$(24)\qquad \begin{aligned}\lambda_0^J &= \lambda^J - 2e\sin(\lambda_0^J - \varpi)\\ &= \pi/2 - 2(0.0167)\sin(\pi/2 - 4.47)\\ &= 1.5788\end{aligned}$$

Using (21), we obtain the mean of the mean longitudes at day *x* and day *c*.

$$(25)\qquad \text{Mean} = \frac{\lambda_0^x + \lambda_0^c}{2}$$

$$\begin{aligned}&= \frac{\lambda^x - 2e\sin M^x + \lambda^c - 2e\sin M^c}{2}\\ &= \frac{(\lambda^x + \lambda^c) - 2e(\sin M^x + \sin M^c)}{2}\\ &= \frac{\pi}{2} - e(\sin(\lambda^x - \varpi) + \sin(\lambda^c - \varpi))\\ &= \frac{\pi}{2} - 0.0167(\sin(0 - 4.47) + \sin(\pi - 4.47))\\ &= 1.5708\end{aligned}$$

Since the difference = 1.5788 – 1.5708 = 0.008, we convert this difference into day units by dividing it by the mean solar motion. Thus, we obtain the error = $0.008 \div \frac{2\pi}{365.25}$ = 0.465 day $\approx$ 11 hours. Though it was indicated in Nakayama's paper that the error would amount to 19 hours, we believe that the difference of 8 hours could be due to other observations done and/or other methods of calculation with higher accuracy used.

3.3 Equinoctial Observation

During the day of the equinoxes, the Sun rises exactly in the East and sets exactly in the West. In this section, we would like to look at how the equinoxes are determined and the accuracy when determining them.

(a) Introduction to the instruments for determining the Equinoxes

In the West since olden times, armillary spheres had been used to determine the moment of equinox. However, we will only look at equatorial ring which is the simplest form of armillary sphere.

Figure 3.5: Armillary sphere

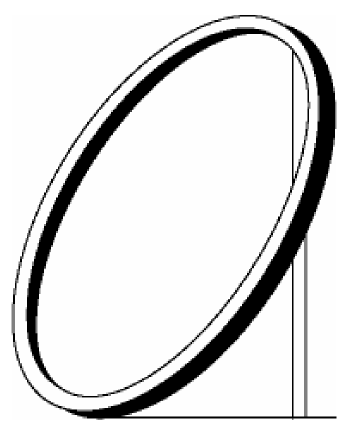

Figure 3.6: Equatorial ring

(b) Using the equatorial ring

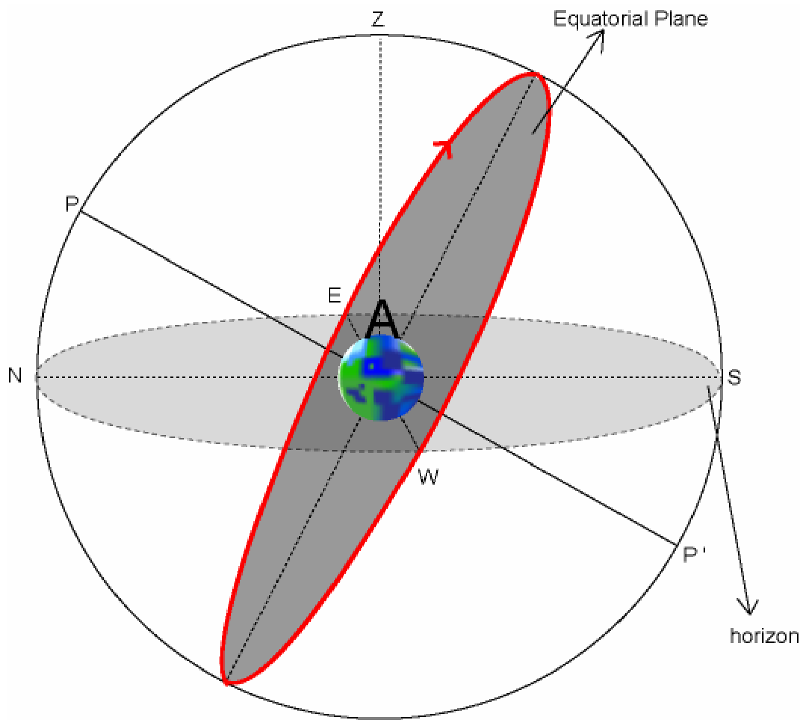

Figure 3.7

The plane that encompasses the celestial equator is called the equatorial plane as shown in Figure 3.7. The equatorial ring is called as such because it is placed in the equatorial plane. Recall that *latitude* is measured from the horizon to the north celestial pole. Let latitude be angle θ°, so the equatorial ring is inclined at angle θ from the normal of the horizon as shown in Figure 3.8.

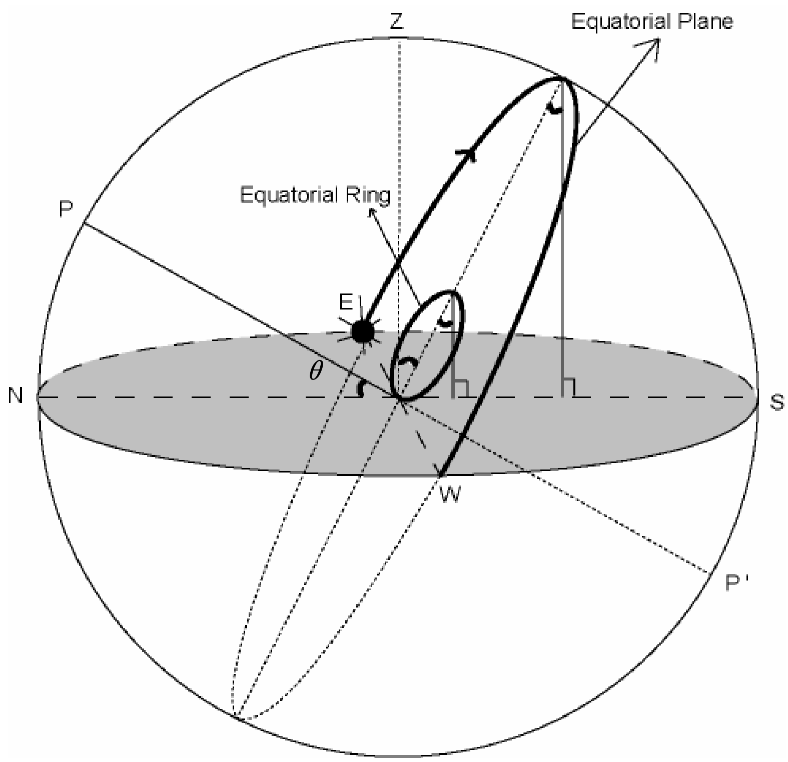

Figure 3.8

Looking at Figure 3.7, imagine standing on the Earth at A, looking at the sky. Z stands for zenith and the plane shaded is A's horizon in the celestial sphere. During equinox, the Sun travels on the path on the equatorial plane as indicated by the red line in Figure 3.7. So the Sun makes a great circle along the ring at the equinox.

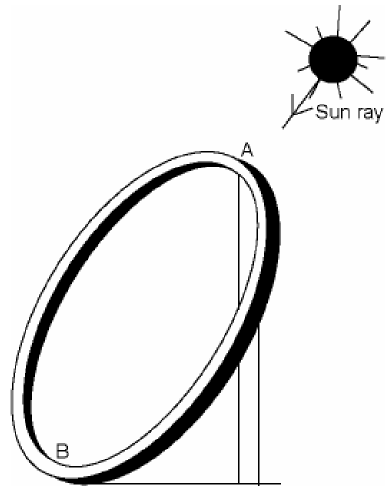

Figure 3.9

So at the moment of equinox, the shadow of A on the equatorial ring falls exactly on B as shown in Figure 3.9. Thus unless the moment of equinox occurs at night, it can be measured directly.

When person B is able to observe the moment of equinox in the day, the equinox occurs at night for person A, as illustrated in Figure 3.10.

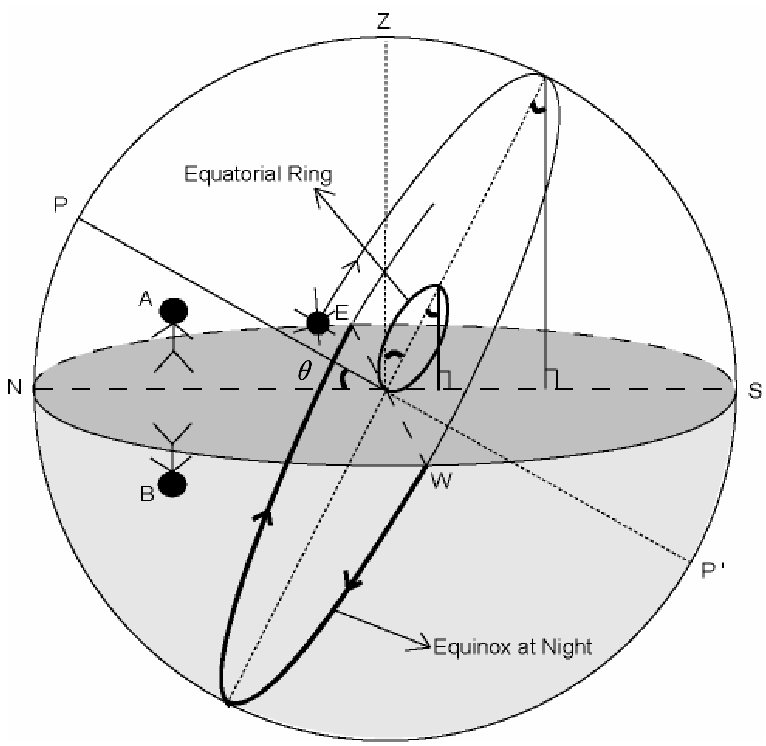

Figure 3.10

(c) Accuracy of the instrument

The accuracy of the instrument is proportional to the diameter of the ring. That is, the bigger the ring, the larger the difference of the shadow length for a small change in the position of the Sun of angle α.

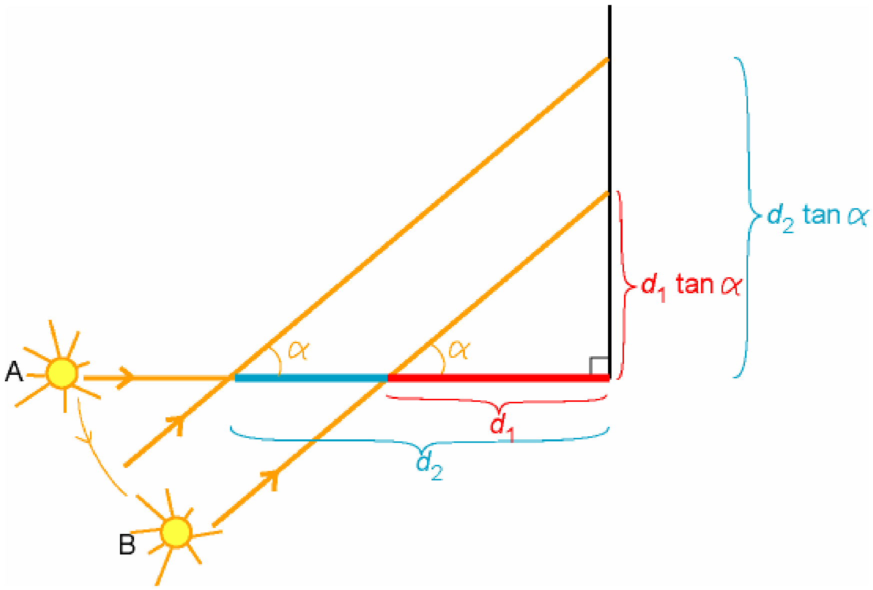

Figure 3.11

Suppose the equatorial rings are viewed sideways as shown in Figure 3.11.

Let d_1 be the diameter of the smaller equatorial ring X_1, and

Let d_2 be the diameter of the bigger equatorial ring X_2.

When the Sun travels from position A to B over an angle of α, the shadow length of X_1 and X_2 increases by $d_1 \tan \alpha$ and $d_2 \tan \alpha$ respectively.

Since $d_1 < d_2$,

$$d_1 \tan \alpha < d_2 \tan \alpha.$$

That is, for a small change in the position the Sun, a larger equatorial ring will give a larger difference in shadow length as compared to using a smaller equatorial ring. With a larger difference in shadow length, it is easier to conclude that there is an error in the determining of the day of equinox. Hence using a larger ring will be more accurate.

In addition, we assume that the equatorial ring does not deviate from the real equatorial plane, so error in determining the latitude of observational site is a cause of error in determining the moment of equinox.

From Skywatcher [3] pg 64, the daily variation of the solar declination around the days of the equinoxes is approximately 23.7'.

So, $24h \approx 24'$

(*) $6' \approx 6h$ error

This means a 6' error in latitude results in approximately 6 hours of error in determining equinoctial time.

Also note that at the time of the equinoxes, the daily variation of the solar declination is at a maximum. So it is possible that we miss the day of the equinox especially if equinox happens at night.

Moreover, when the altitude of the Sun is low, refraction causes significant error. So it is difficult to determine if the positions of sunrise and sunset are exactly in the East and West respectively.

(d) Comparisons of using gnomon and equatorial ring

In a single observation, determining equinox using an equatorial ring gives much better results than determining solstice using a gnomon.

The two main reasons are as follow. Firstly, around the time of the equinoxes, the daily variation of the solar declination is at a maximum, while around the time of the solstices, the daily variation of the solar declination is at a minimum. The second reason is the difference in instrument used. That is, we can determine the exact moment of equinoxes by using the equatorial ring as shown in Figure 3.9, but we have to calculate the moment of solstices after getting a few readings from the gnomon as described in the section on Solsticial Observations.

For example, to show that an equatorial ring gives less error than a gnomon, suppose the diameter of an equatorial ring and the height of a gnomon are both two meters and both instruments have an error of 1cm.

We will show that an error of one centimeter in the tilt of the equatorial ring, results in 18h error in determining the day of equinox.

Recall that the equatorial ring is to be placed on the equatorial plane. So looking at Figure 3.12, the correct position of the equatorial ring should be on the red line. For an error of 1cm, that is the equatorial ring is 1cm away from the correct position, will result in an angular error α.

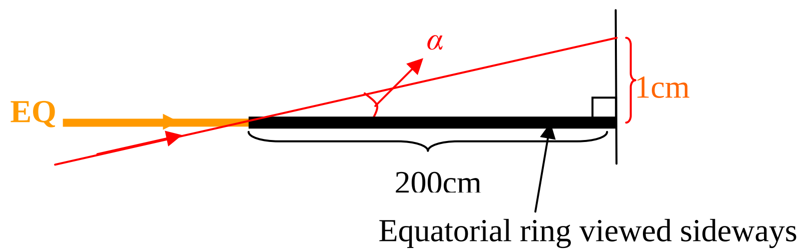

Figure 3.12

$\tan \alpha = 1/200$

$\alpha = 0.286^{o} \approx 0.3^{o} = 18'$ (since $1^{o} = 60'$)

And by (*), $18' \approx 18$ hour error

In actual practice, the error of measurement could not exceed three millimetres, which means a six hours precision when using equatorial ring could be easily obtained.

However, an error of one centimetre when using gnomon on the winter solstice results in approximately four days difference. Please refer to the section on Solsticial Observations for the explanation.

Chapter 4

Conic Sections and Shadows

4.1 Basics of Conic Sections

In this section we will discuss how the shape of the shadow of a gnomon changes at different parts on the Earth throughout the year. We can view the paths of the shadows are conic sections made by the sunlight on the horizon. The sunlight can be visualized as a spotlight with a conical beam. The conic sections are obtained by changing the position of the beam which is due to the changes in latitude. So before we can further explain the shape of the shadow, we have to look at the basics of conic sections.

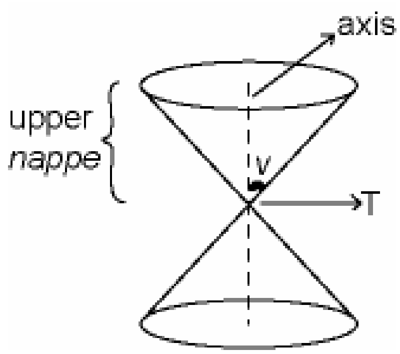

Figure 4.1

Consider a cone in the three-dimensional Euclidean space with vertical axis and vertex T. The vertex T divides the cone into 2 parts called nappes where the upper nappe looks like an upright ice-cream cone. The angle v away from the axis is called the *angular aperture* and $0 < v < \pi/2$.

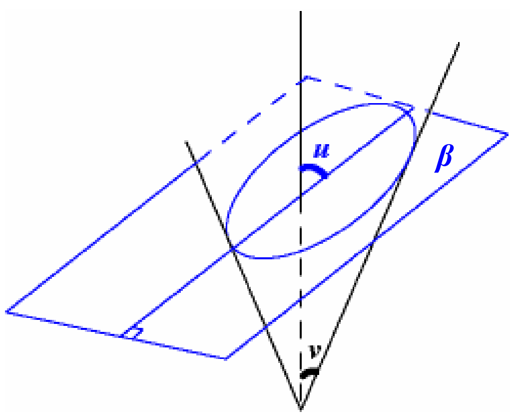

Figure 4.2

Imagine a plane β cutting the nappe at an angle u with the axis of the cone, with $0 < u \le \pi/2$. The boundary curve produced is called a conic section. So the shape of the conic section depends on the angle u.

The shape of the curve for

I. $u > v$, is an ellipse.

II. $u = v$, is a parabola.

III. $u < v$, is a hyperbola which has 2 separate pieces known as branches.

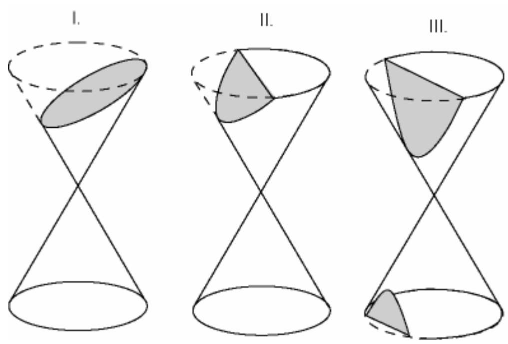

Figure 4.3

Proof of the 3 cases:

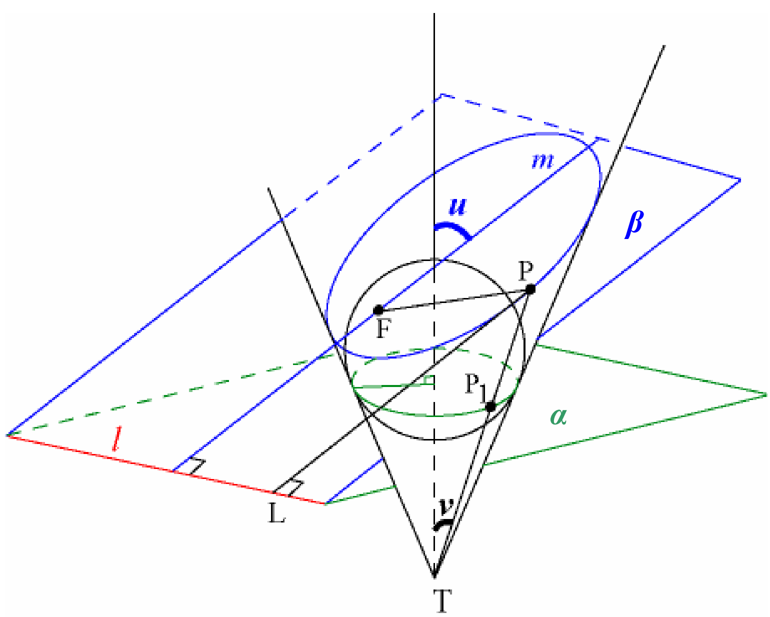

Figure 4.4

Consider a sphere in the inside of the cone that touches the cone along a circle lying in a horizontal plane α. Let a plane β touches the sphere at a point F. The line l is the line of intersection between the planes α and β.

Consider an arbitrary point P on the conic section and let P_1 be the point in α in which the line through P and T intersects the circle of tangency between the sphere and the cone. Since the line segments PF and PP_1 are both tangents from P to the sphere we get

$| PF | = | PP_1 |$.

Also, the line segments PL and PP_1 have the same right-angle projection onto the axis of the cone, since the points L and P_1 both lie in the plane α, and this plane is perpendicular to axis of the cone.

Therefore,

$$| PF | \cos(v) = | PP_1| \cos(v) = |PL| \cos(u) = |Pl| \cos(u),$$

$$|PF| = \frac{\cos(u)}{\cos(v)} |Pl|.$$

Let $$e = \frac{\cos(u)}{\cos(v)}.$$

where e is called the *eccentricity* of the conic section,

F is the *focal point*,

L is a *directrix* and

Pl is the line from point P perpendicular to the line l.

So for $\pi/2 > u > v$, $0 < e < 1$.

When $u = \pi/2$, $e = 0$, we get a circle.

So, for the three cases,

I. $u > v$, is an ellipse, $0 \leq e < 1$.

II. $u = v$, is a parabola, $e = 1$.

III. $u < v$, is a hyperbola, $e > 1$.

More on conic sections:

(1) The curve called a conic section in the Euclidean plane is a set of ordinary points whose homogeneous coordinates satisfy an equation of the second degree.

(2) While there are 3 different kinds of conic sections in the Euclidean plane, namely, the ellipse, parabola, and hyperbola, there is only one kind of point conic in the real projective plane.

(3) The ellipse is a closed curve; the parabola is an open curve with only one branch; the hyperbola is an open curve with 2 branches.

(4) However, a parabola maybe related to an ellipse in the following way. In any ellipse, draw the longest diameter. The end points of this diameter are called the vertices of the ellipse. Keep one vertex of the ellipse fixed and make the ellipse

longer and longer, so that the other vertex moves off to infinity. Then the ellipse can be shown to approach a parabola as a limit.

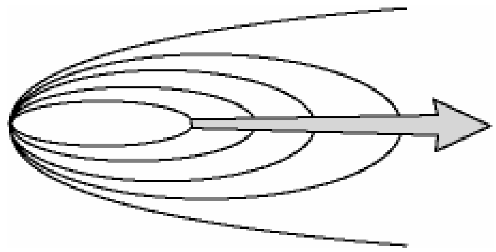

Figure 4.5

(5) Every hyperbola has a pair of asymptotes. If a point moves in either direction along a branch of the hyperbola, as it goes off to infinity it comes closer and closer to one of the asymptotes. So each branch may be thought of as meeting each asymptote at its ideal point. If we add these 2 ideal points to the hyperbola, its 2 branches are joined at these points, both of which are on the line at infinity. The resulting point conic becomes a closed curve, like an ellipse, with 2 point on the line at infinity.

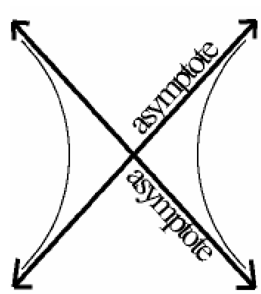

Figure 4.6

4.2 Shadows cast in the Artic

To understand the path of a gnomon's shadow on the ground, we need to understand its relationship with conic sections. The diagram below shows the observer's horizon and the upper hemisphere of the celestial sphere situated about the observer. A gnomon is positioned at the center of horizon, O. The tip of the gnomon is where the vertex of two cones lies. Upper nappe of the cone produces the Sun's path in the sky while the intersection of the horizon with the lower nappe is a conic section. This conic section is the path of the gnomon's shadow.

We will introduce two important angles, u and v, as their relationship will indicate the kind of conic section produced. As indicated in Figure 4.7, angle u is the angle between the axis of the cones and the horizon (intersecting plane) while angle v is the angle between the same axis and the lower nappe.

It is essential to take note that angle v ranges from 66.5 to 90 degrees. The angle of 66.5 degrees is achieved when we observe at solstices and the latter degree is achieved when we observe at the equinoxes. In addition, we can flatten the cones to understand the transition of the Sun's path and the shadow's path throughout a year at fixed latitude. We will only explain the transition from summer solstice to winter solstice as the transition from winter solstice to summer solstice can be seen as the reverse.

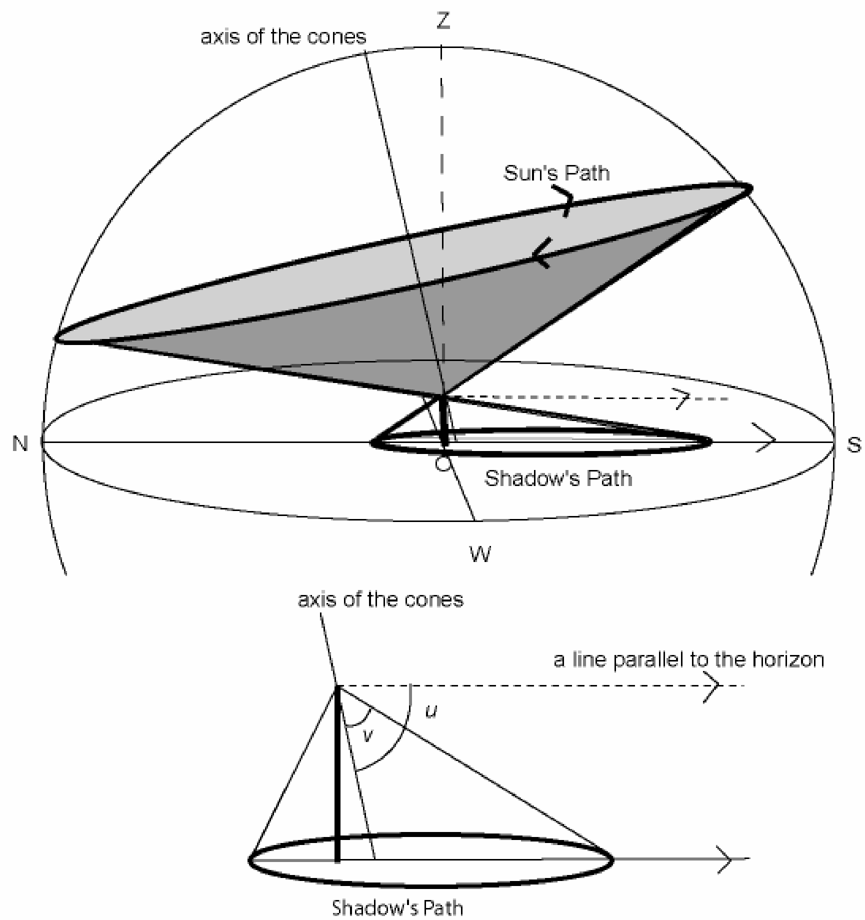
axis of the cones
Z
Sun's Path
N
S
O
Shadow's Path
W
axis of the cones
a line parallel to the horizon
u
v
Shadow's Path

Figure 4.7

As for angle *u*, it is the latitude of the observer.

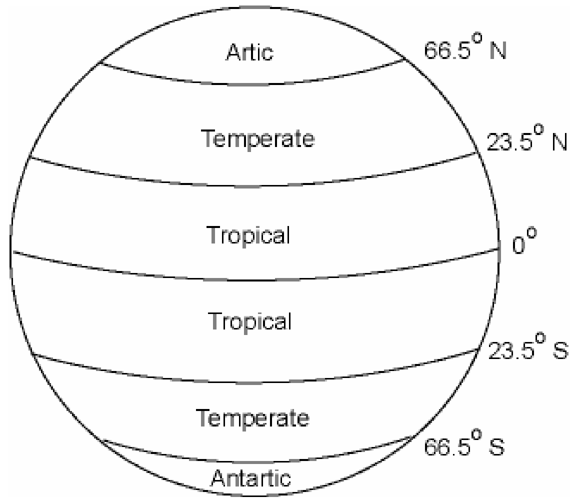

Figure 4.8

It is essential to take note that angle *u* ranges from 66.5 to 90 degrees. At the Artic Circle, angle *u* is equal to 66.5 degrees while at the North Pole, it is equal to 90 degrees.

(a) North Pole (or South Pole)

At the North Pole, since its latitude is 90 degrees north, angle *u* is equal to 90 degrees.

The diagram below shows the path of the Sun and shadow at summer solstice.

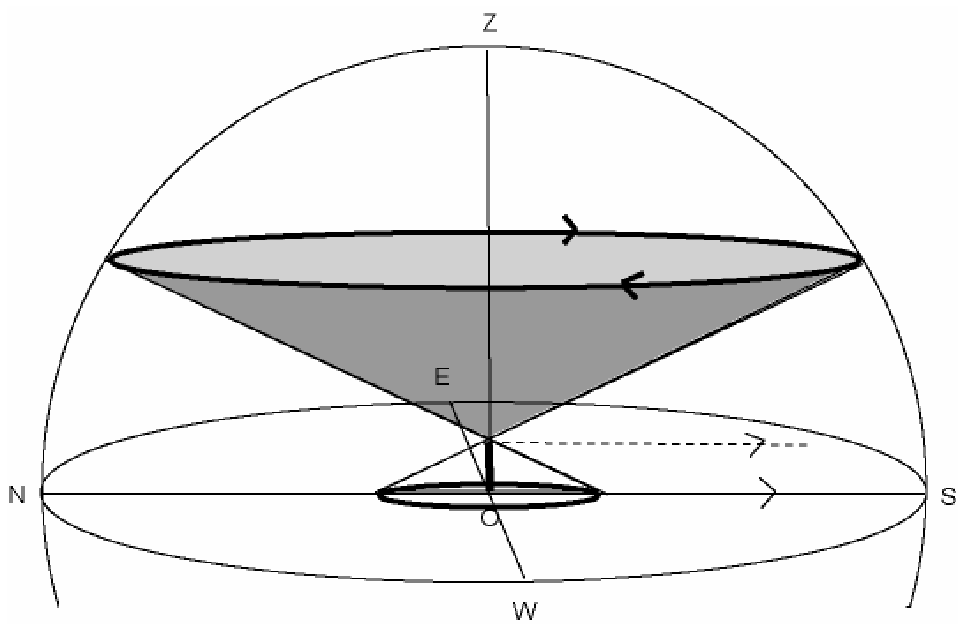

Figure 4.9

Since angle *u* is equal to 90 degrees, we will get a circle as the conic section. Thus, the shadow marks out a circle on the ground.

As mentioned earlier, the transition throughout a year can be observed as the cones are flattened. Figure 4.10 shows that when the cones are flattened, we get a larger circle on the ground.

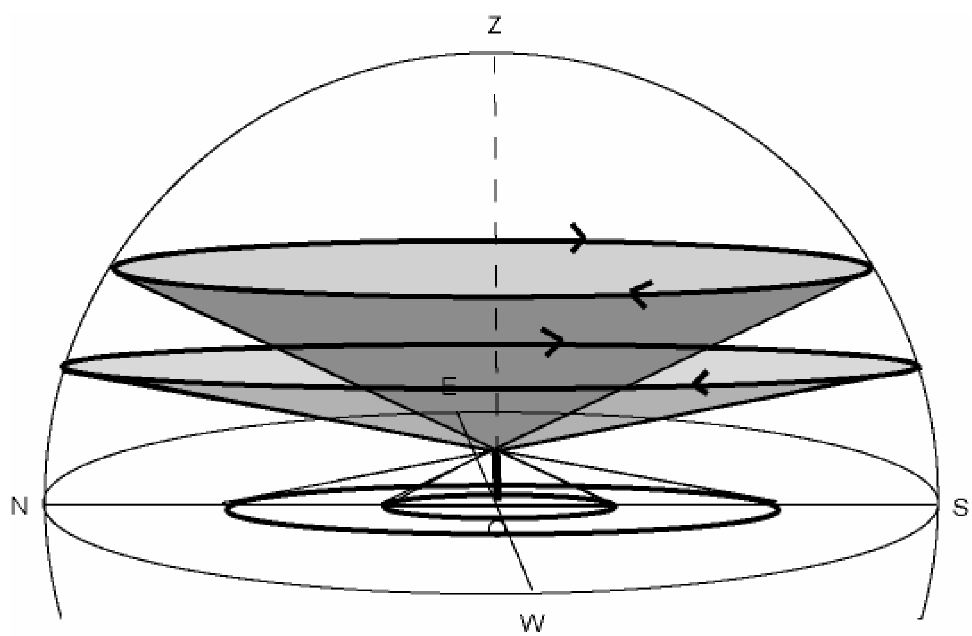

Figure 4.10

This circle will extend till fall equinox when the cone flattens into a plane which is non-identical and parallel to the horizon. In projective geometry, we can consider this intersection between 2 non-identical, parallel planes as a line. Thus, the shadow is a line known as a line at infinity.

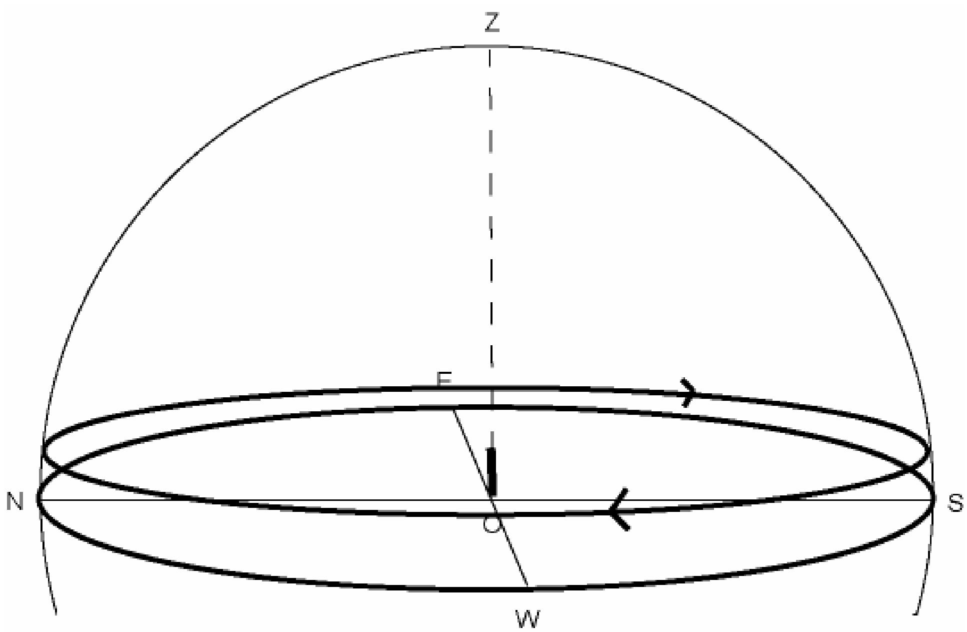

Figure 4.11

In all the previous diagrams, we must keep in mind that the gnomon has been enlarged many times. In actual fact, the size of the gnomon is insignificant as compared to the celestial sphere about an observer at point O of the horizon. Thus, we can consider the gnomon stick as a point at O. Referring back to Figure 4.11, we will be able to observe that the Sun's path is around the horizon plane at this stage.

Once the cones are flattened and flipped over, the lower nappe changes into the upper nappe. From this stage onwards, the Sun will be below the horizon and *midnight sun* occurs.

Thus, we can conclude that the transition of shadow paths formed at the North Pole is as follows: Starting from the summer solstice, an ellipse is formed on the ground as shadow path which gradually widens till it encompasses the whole horizon at the fall equinox and total darkness occurs from then which results in no shadows formed till the winter solstice. The transition from the winter solstice to the next summer solstice can be obtained by reversing the above.

(b) Within Artic (or Antarctic)

Within the artic, using the earlier notation of angle u and the latitudes of the artic, we have the following inequality, $66.5° < u < 90°$.

Figure 4.12 below shows the path of the Sun and shadow at the summer solstice within the artic. Since we are observing at fixed latitude, the angle u remains constant. However, things are different for angle v. As the cones are flattened, angle v increases.

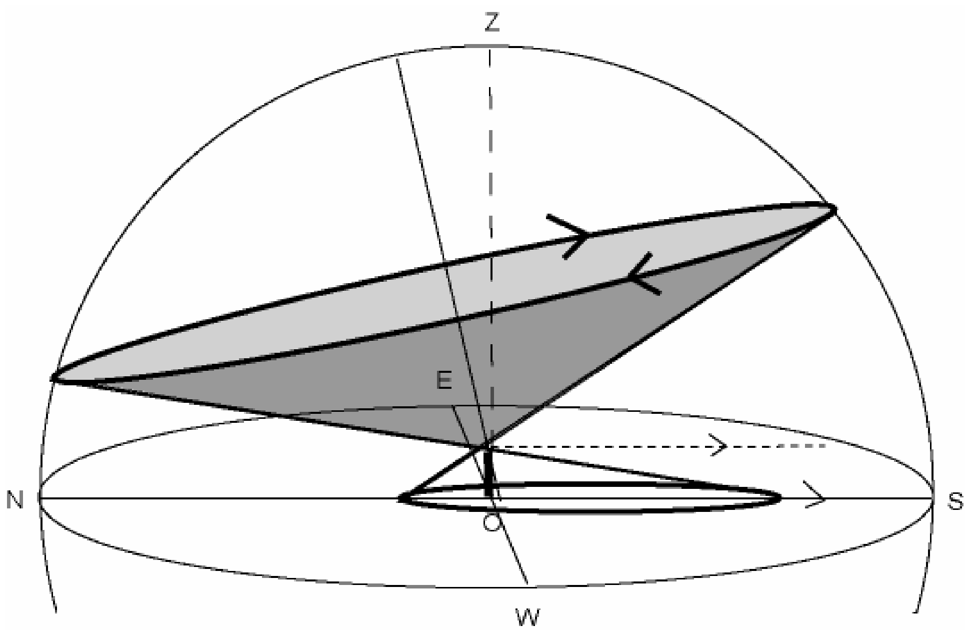

Figure 4.12

At the summer solstice, angle *v* is equal to 66.5 degrees which is less than angle *u* since angle *u* is more then 66.5 degrees. Thus, we obtained an ellipse as a conic section which will be reflected on the ground as the shadow's path.

The ellipse enlarges with the flattening of the cones and angle *v* increases to a point when it is equal to angle *u*. When this stage is reached, we observe that a line pointing north on the lower nappe becomes parallel to the horizon. As a result, the earlier ellipse breaks off into a parabola which fits in nicely with the fact that a parabola is produced as a conic section when *u* is equal to *v*.

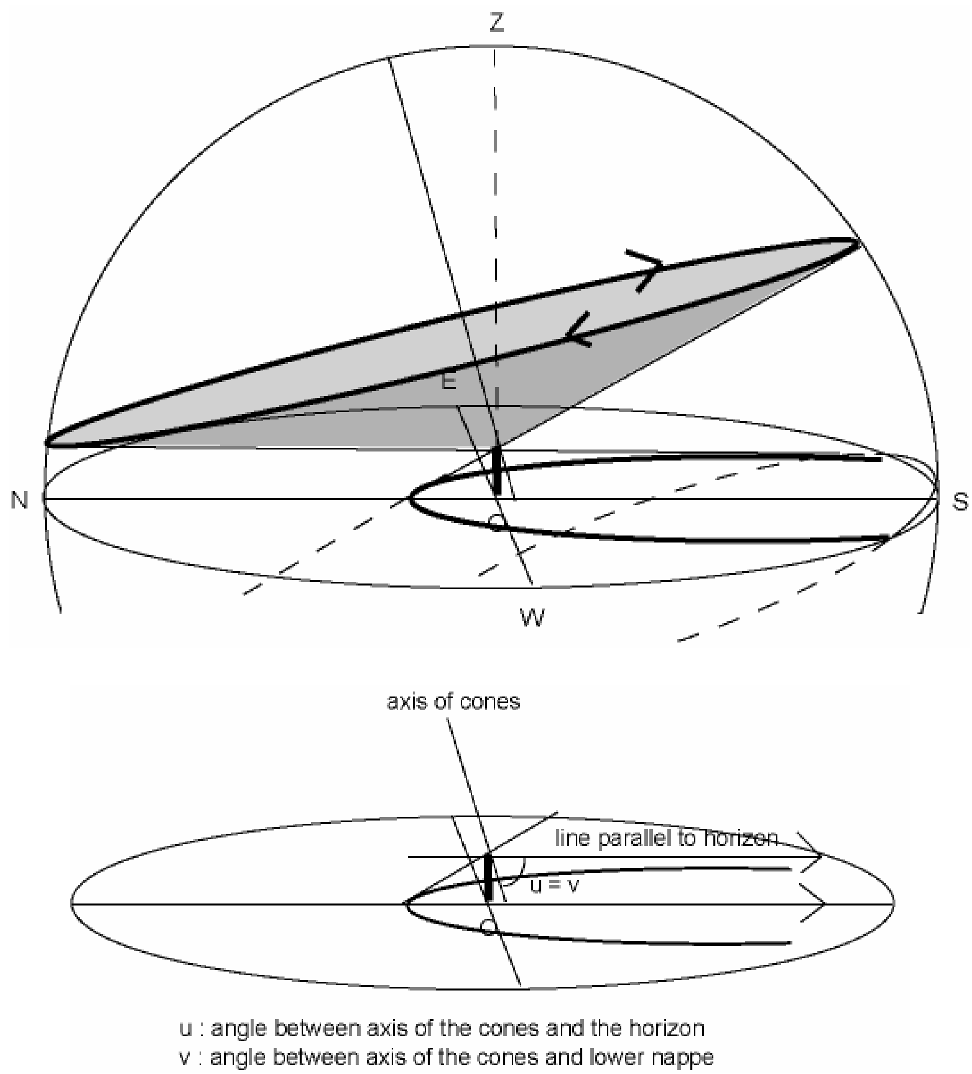

Figure 4.13

Once again, we take note that the gnomon is of insignificant size. Thus, the Sun will be rising and setting at the same point on this day and this day will mark the end of midnight Sun.

Immediately after this stage, angle *v* becomes larger than angle *u* and thus, a hyperbola is produced. This is shown in the diagram below.

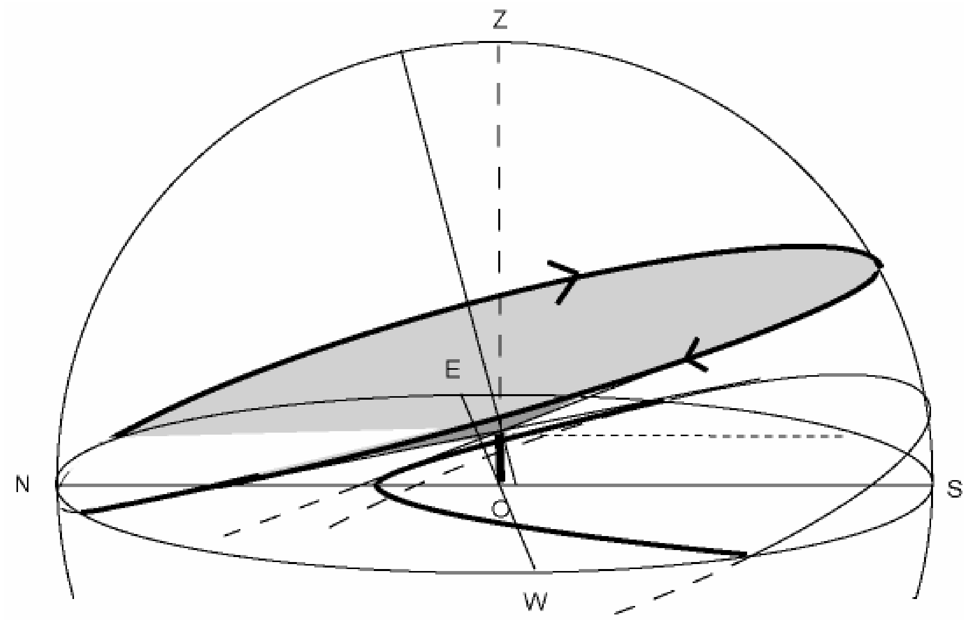

Figure 4.14

This hyperbola will continue to flatten till it reaches a line when the cones are flattened into a plane at the fall equinox.

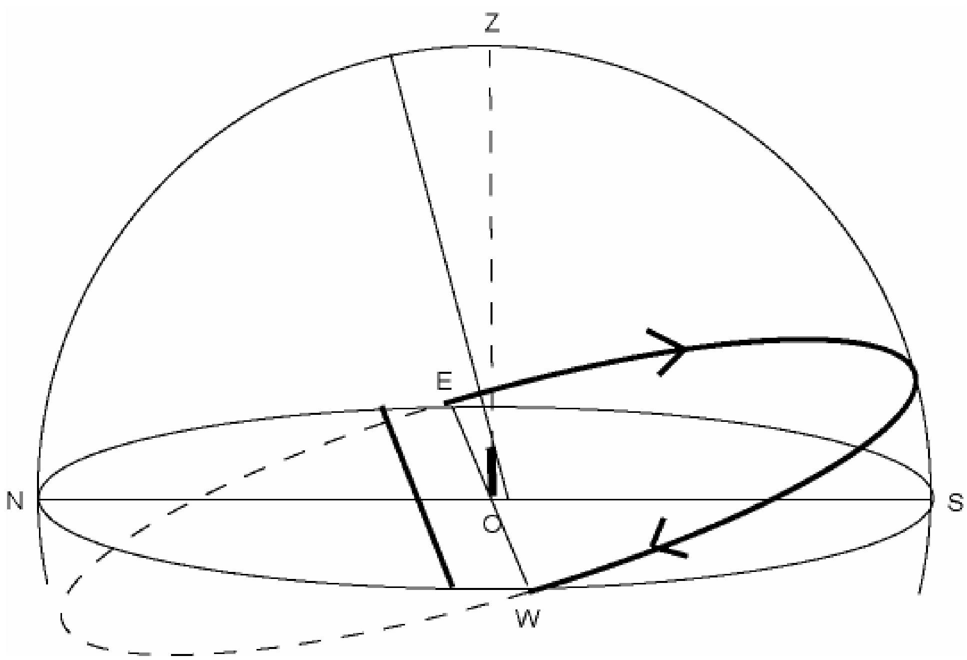

Figure 4.15

When we flipped the cones and we can consider viewing the upper nappe in earlier diagrams as lower nappe. Thus, after the fall equinox, the intersection of a plane parallel to the horizon with the upper nappe will produce a conic section that represents the shadow’s path.

Since angle *v* is still larger than *u*, we continue getting a hyperbola after a line is obtained at the fall equinox.

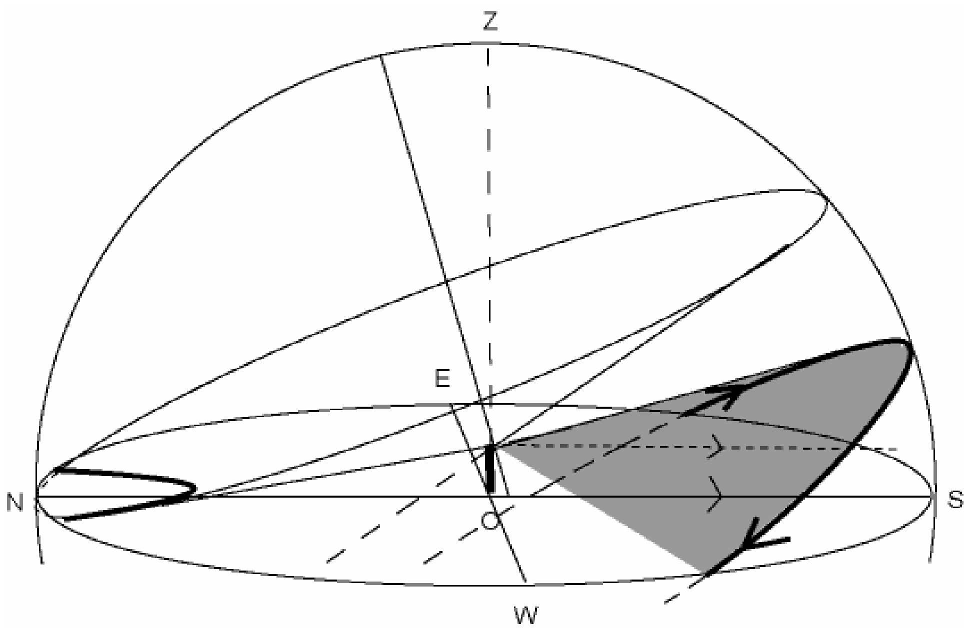

Figure 4.16

A reverse pattern seems to be observed as angle *v* decreases. However, we would not obtain a parabola as that would require a line on the upper nappe to be parallel to the horizon which is not possible. Thus, the hyperbola will just disappear as midnight sun occurs.

Thus, we can conclude that the transition of shadow paths formed at the North Pole is as follows: Starting from the summer solstice, we get an ellipse which eventually breaks off into a parabola and widens into a hyperbola. The hyperbola continues to widen till it reaches a line at the fall equinox. After the fall equinox, hyperbola occurs again and starts to sharpen till total darkness occurs which results in no shadows formed till the winter solstice. The transition from the winter solstice to the next summer solstice can be obtained by reversing the above.

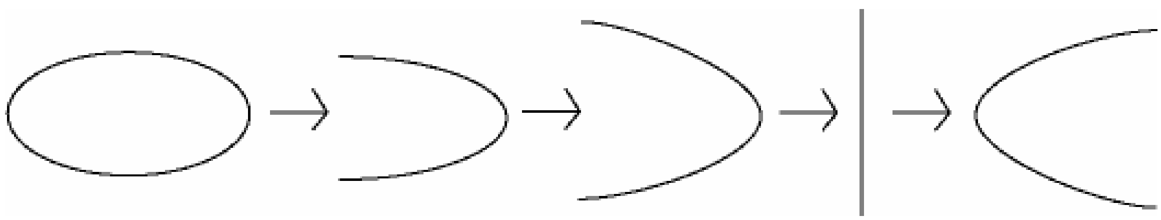

Figure 4.17

(c) At the Artic circle (or Antarctic circle)

Since the latitude of Artic is 66.5 degrees north, the angle *u* is equal to 66.5 degrees. At summer solstice, angle *v* is equal to 66.5 degrees too. As angle *u* is equal to angle *v*, we understand that a parabola will be obtained. The diagram below shows the Sun's Path at this stage. Once again, if we consider the gnomon stick to be of insignificant size as compared to the celestial sphere, we get the Sun's path as the sunrise and sunset points are the same.

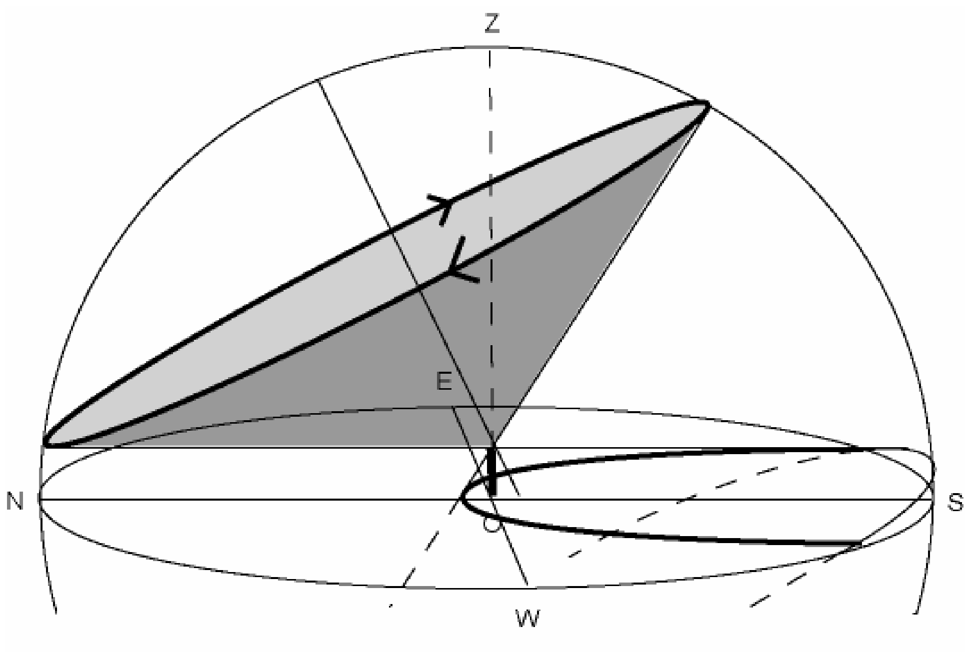

Figure 4.18

As mentioned before, due to the size of the gnomon, the Sun is actually seen to be rising and setting at the same point on this day. After this day, there will not be midnight sun and the setting of the Sun below the horizon will be observed.

The cones start to flatten after the summer solstice occurs and angle *v* increases. Since angle *v* is larger than angle *u* from this stage onwards, we obtain hyperbolas as the shadow's paths.

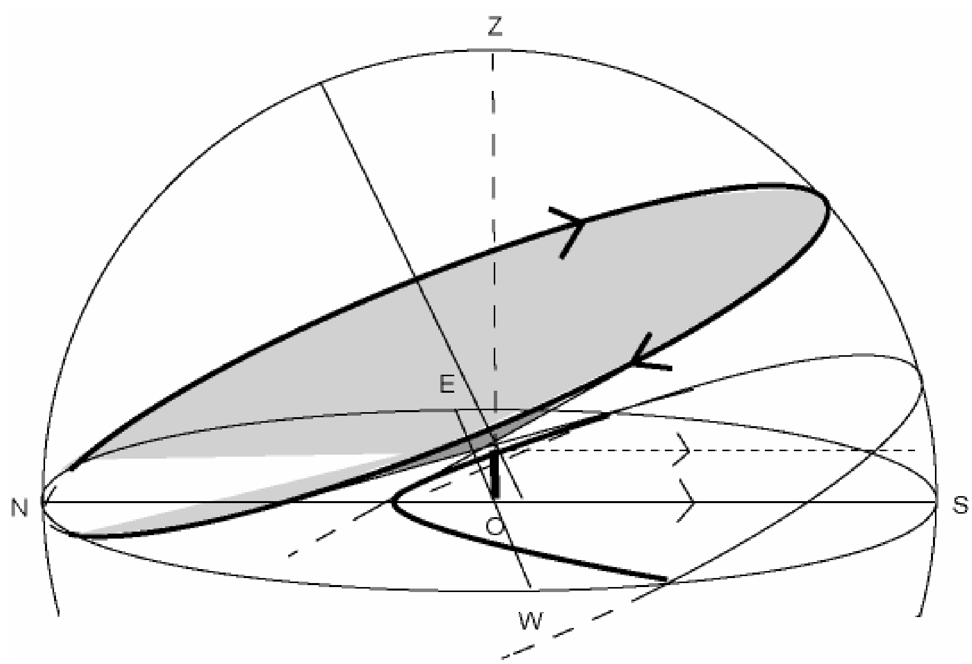

Figure 4.19

This continues till the occurrence of the fall equinox. At this point, the cones are flattened into a plane and similar to the previous cases, we get a line as the intersection and as the shadow's path.

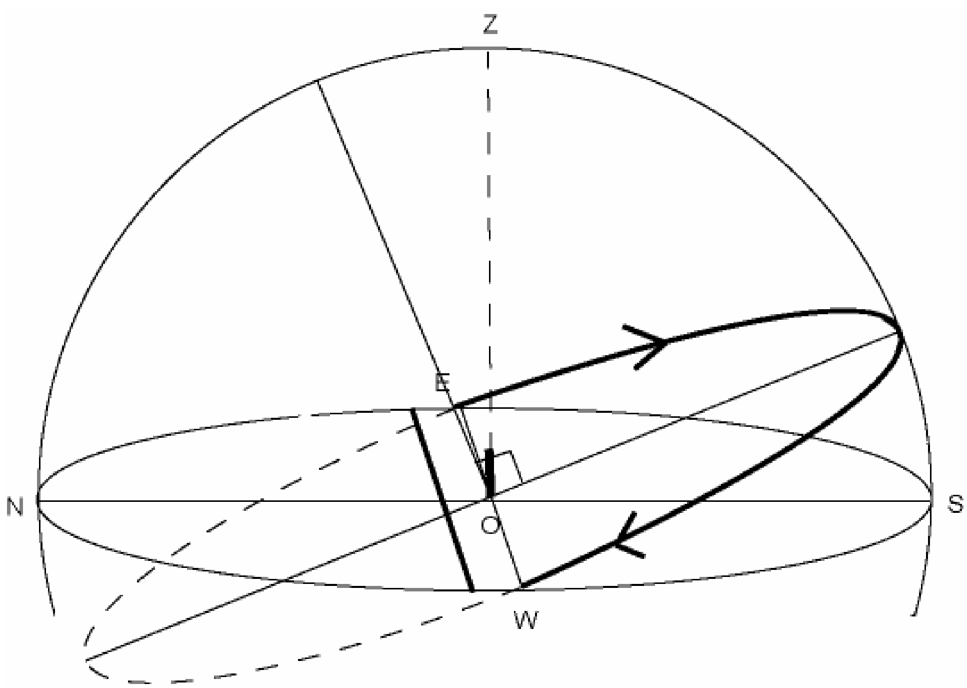

Figure 4.20

After the fall equinox, the cones are once again flipped over and the Sun's path is now at the lower nappe. Angle *v* starts to decrease as the cones begin to sharpen. However, it remains to be larger than *u* and thus, we obtain hyperbolas as conic sections.

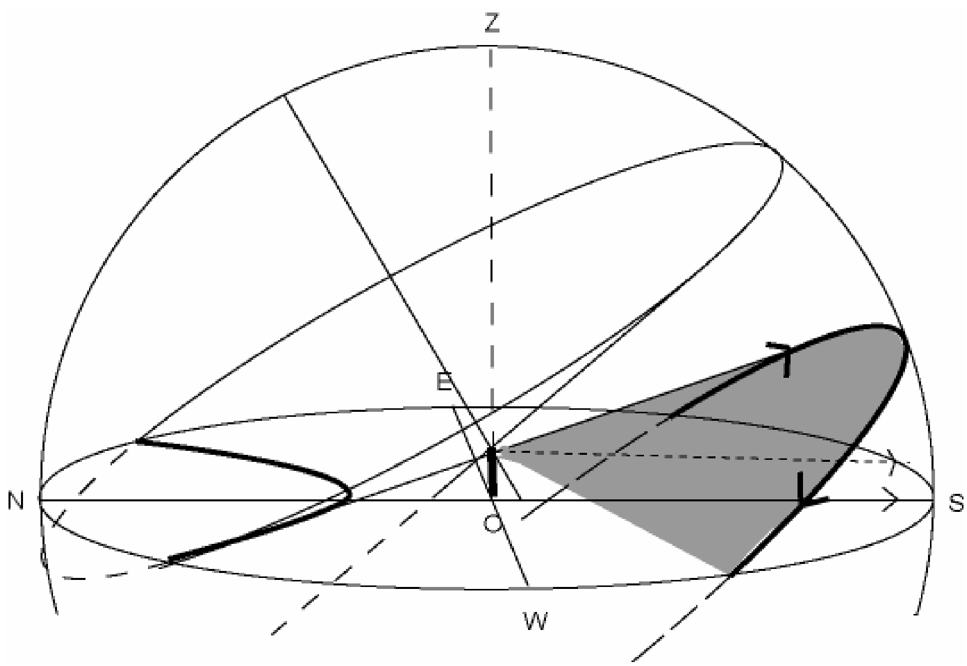

Figure 4.21

The shadow's path will continue to appear as a hyperbola till angle *v* decreases to the same value as *u*, 66.5 degrees, when the winter solstice occurs. At this stage, the Sun's path seems to be just a point on the horizon. That is, the Sun rises and sets immediately from an observer's point of view.

Thus, we can conclude that the transition of shadow paths formed at the North Pole is as follows: Starting from the summer solstice, a parabola is formed which is followed by a hyperbola. The hyperbola will continue to widen into a line at the fall equinox. After the fall equinox, hyperbola is formed and starts to sharpen till winter solstice. The transition from the winter solstice to the next summer solstice can be obtained by reversing the above.

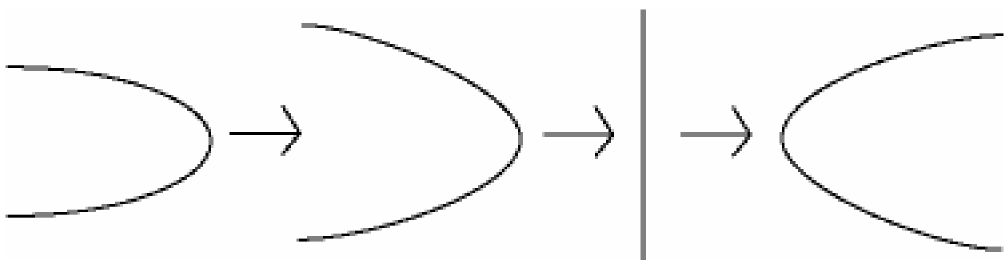

Figure 4.22

4.3 Shadows cast in the Temperate Zone

Before we explain the changes of the shape of a gnomon's shadow on the ground from the summer solstice to the winter solstice, we need to recall some points in the horizon reference system.

Within the temperate zones, latitude u is between 23.5 and 66.5°. Recall that the acute angle between the line joining the *celestial poles* and the horizon plane is equal to latitude u. Thus, angle between the celestial equator and the equinoxes is equal to the angle u between the axis and the horizon. However, the angle between the summer solstice and the equinoxes is 23.5^{o}, which is less than u as shown in Figure 4.21. Since the Sun reaches the highest point in the sky at the summer solstice, the highest point of the Sun's daily path will not pass through zenith at any time of the year. So the shadow of the gnomon will always stay on one side of the East – West line because the Sun will not pass through zenith.

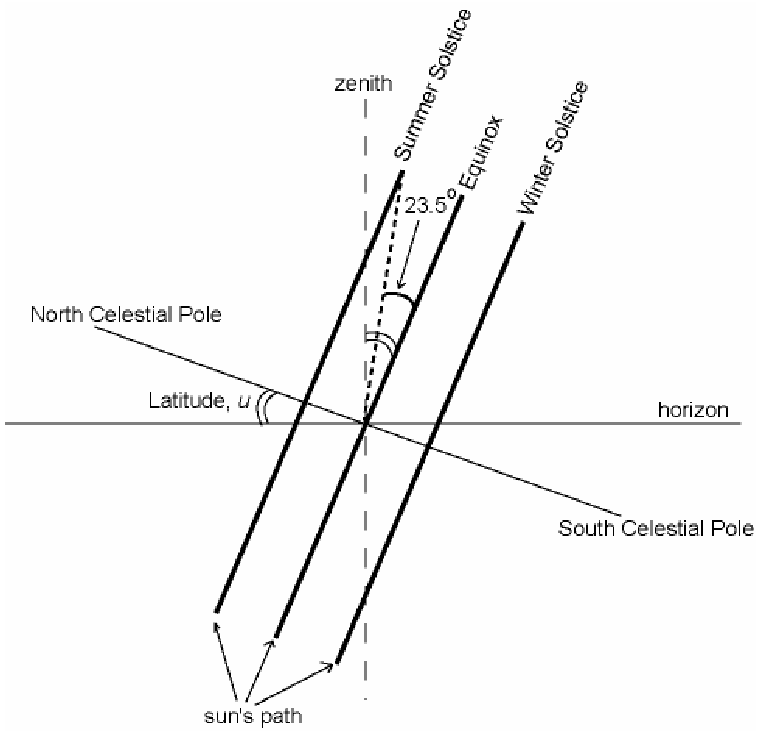

Figure 4.23

(a) Starting from the Summer Solstice

Figure 4.22 below illustrates an observer at latitude *u* on the day of summer solstice. A gnomon is placed at the center of horizon, O. The Sun's path is a cross section of the upper nappe of the double cone with its vertex at the tip of the gnomon while the intersection of the horizon with the lower nappe is a conic section. This conic section is the path of the gnomon's shadow.

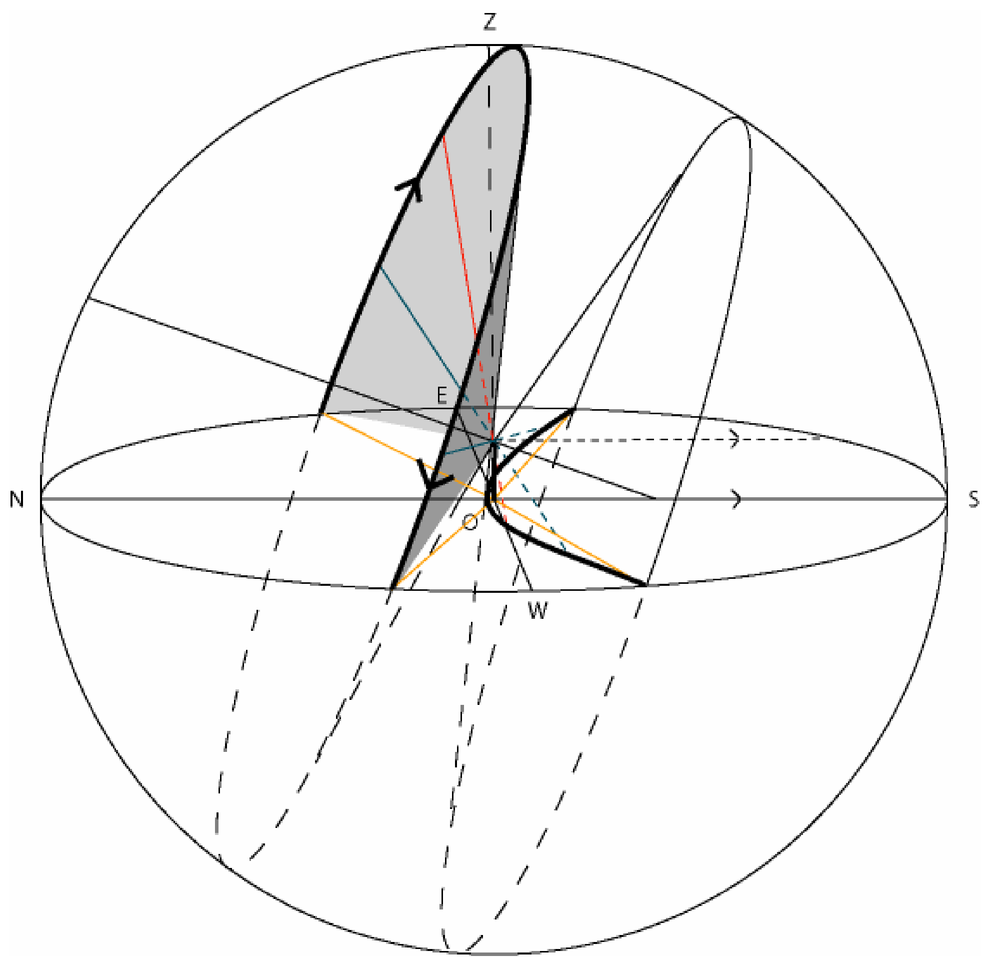

Figure 4.24

Note that the axis of the cones is also the axis of the celestial sphere.

Let *v* be the angle away from the axis. Recall that angle *u* is the latitude which is also the angle between the axis of the cones and the horizon. Thus we can see that $u < v$. So the path of the shadow, which is the conic section, is a hyperbola.

We get another explanation for the shape of the shadow as a hyperbola, when we view the Sun's path as part of a circle and the horizon as a line of infinity. Then the line at infinity cuts the circle at two points on the summer solstice. Hence the shape of the shadow is a hyperbola.

(b) Between the Summer Solstice and the Equinoxes

The shape of the shadow is a hyperbola, as the year passes on from the summer solstice to the equinoxes, while the cones flatten. So the hyperbola shape of the shadow flattens, near to being a straight line.

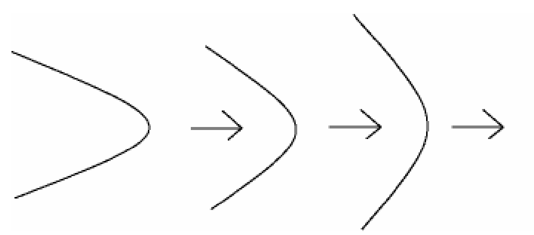

Figure 4.25

(c) At the Equinoxes

At the equinox, the cones flattened into a plane so the shadow formed is a line. However the line does not pass through the base of the gnomon because the Sun's path does not pass through zenith.

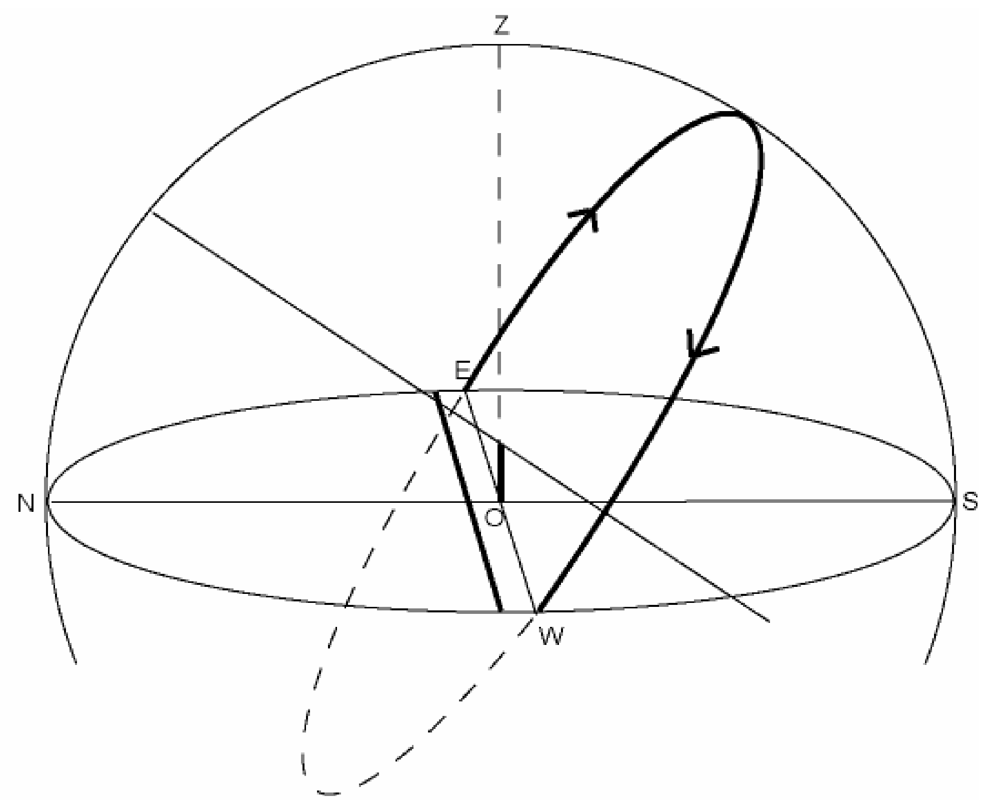

Figure 4.26

(d) From the Equinoxes to the Winter Solstice

After the equinox, the plane inflates to a cone. The Sun's path is once again a cross section of the upper nappe of the double cones where its vertex is on the tip of the gnomon. The intersection of the horizon with the lower nappe is a conic section which is the path of the gnomon's shadow. The shape of the shadow is hyperbola again because $u < v$.

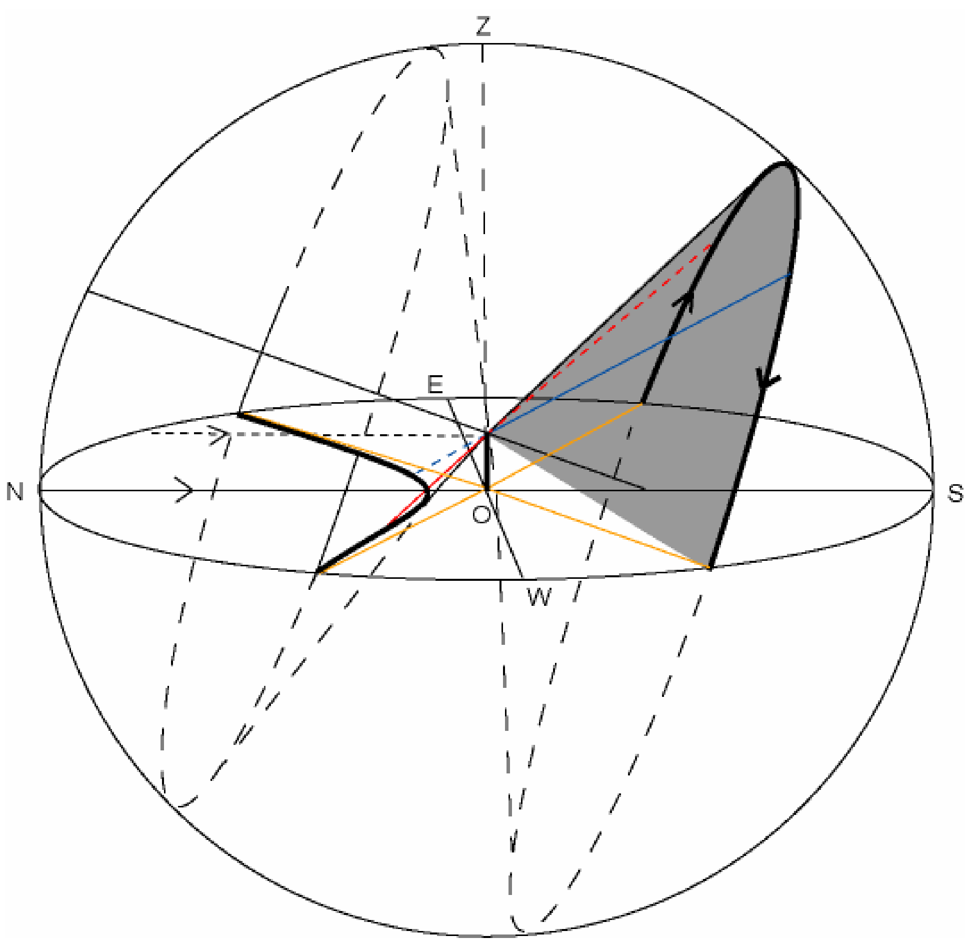

Figure 4.27

As the cone inflates, that is angle *v* increases, the acute angle between the asymptotes of hyperbola decreases.

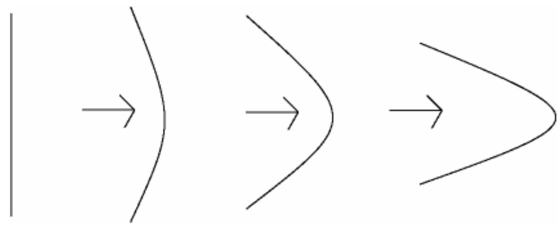

Figure 4.28

4.4 Shadows cast in the Tropics

Tropics are the regions of the Earth where the Sun will shine directly overhead at some time of the year. Two imaginary circles, the Tropic of Cancer and the Tropic of Capricorn, form the boundaries of the tropics. The Tropic of Cancer is about 23.5° North of the equator, and the Tropic of Capricorn is 23.5° South of the equator.

In a year, as the Sun makes a complete circle around the Earth, the angle between the Sun and the equator varies between +23.5° to –23.5. Hence the nappe of each cone of the double cone makes an angle of 66.5° (90° - 23.5°) to the axis of the double cone. The double cone is tilt at an angle equivalent to the latitude of the observer. At the equator, where it is latitude 0°, there is no tilt; the axis of the double cone is parallel to the horizon.

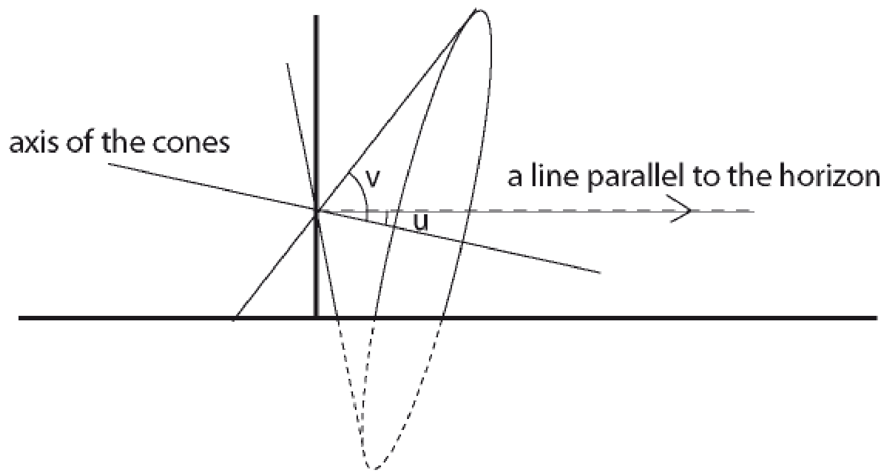

Figure 4.29

In the tropics, the path of the shadow of the gnomon takes only the curve of a hyperbola or a straight line. On the equinoxes, the conic section is a straight line, since the double cones flattens to a plane on those two days in a year. On other days, the conic section is a hyperbola. This is as, in the tropics, the horizon plane always cut through both nappes of the cone. Recalling earlier, this means that $u < v$ in the tropics and hyperbolas are obtained.

The Sun is furthest from the overhead position at the solstices; hence the path of the shadow curves the most then. It gradually flattens to a straight line at the equinoxes. The day where the Sun passes directly overhead is the *zenith passage*.

(a) Equator

The equator which is the line dividing the North and South hemispheres lies within the tropics. At the equator, the Sun's path is always perpendicular to the horizon plane. The shadow cast by the gnomon on the equinox is a straight line that falls on the base of the gnomon. On other days, the shadow lies on either side of the East – West line.

On the days between the summer solstice and the equinoxes, the shadow lies to the South of the East – West line.

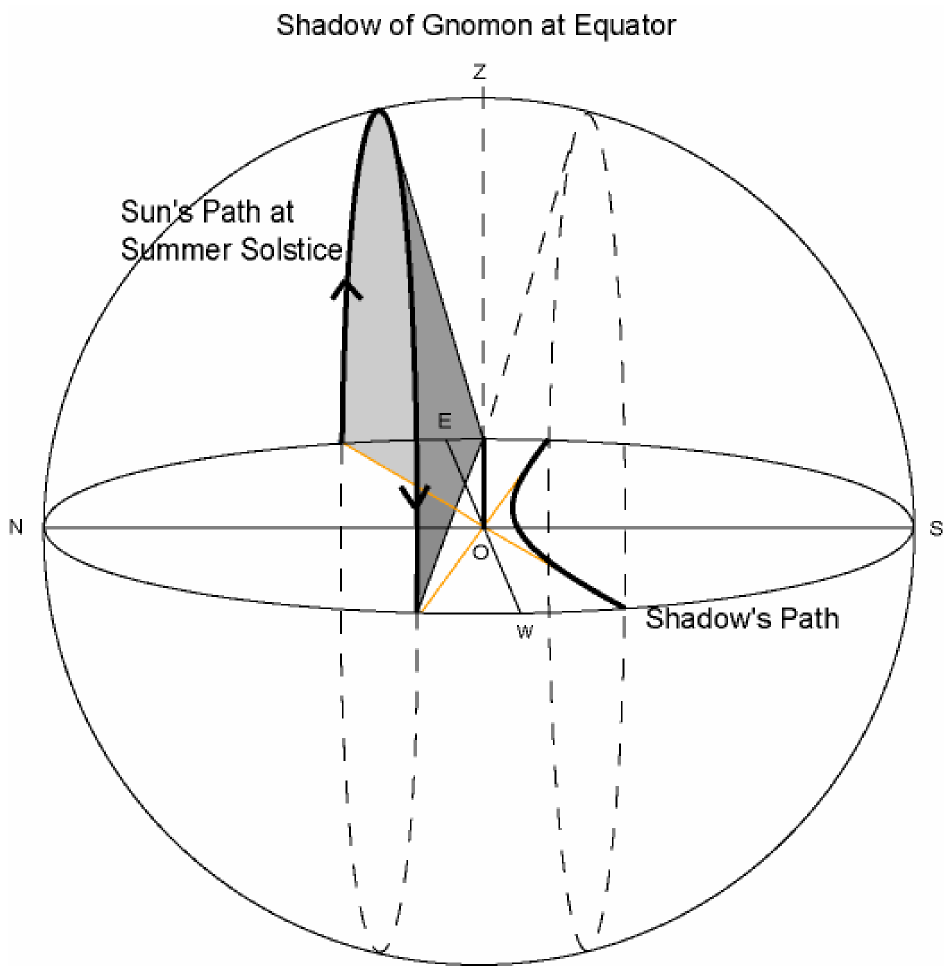

Figure 4.30

On the days between the winter solstice and the equinoxes, the path of the shadow lies to the North of the East–West line.

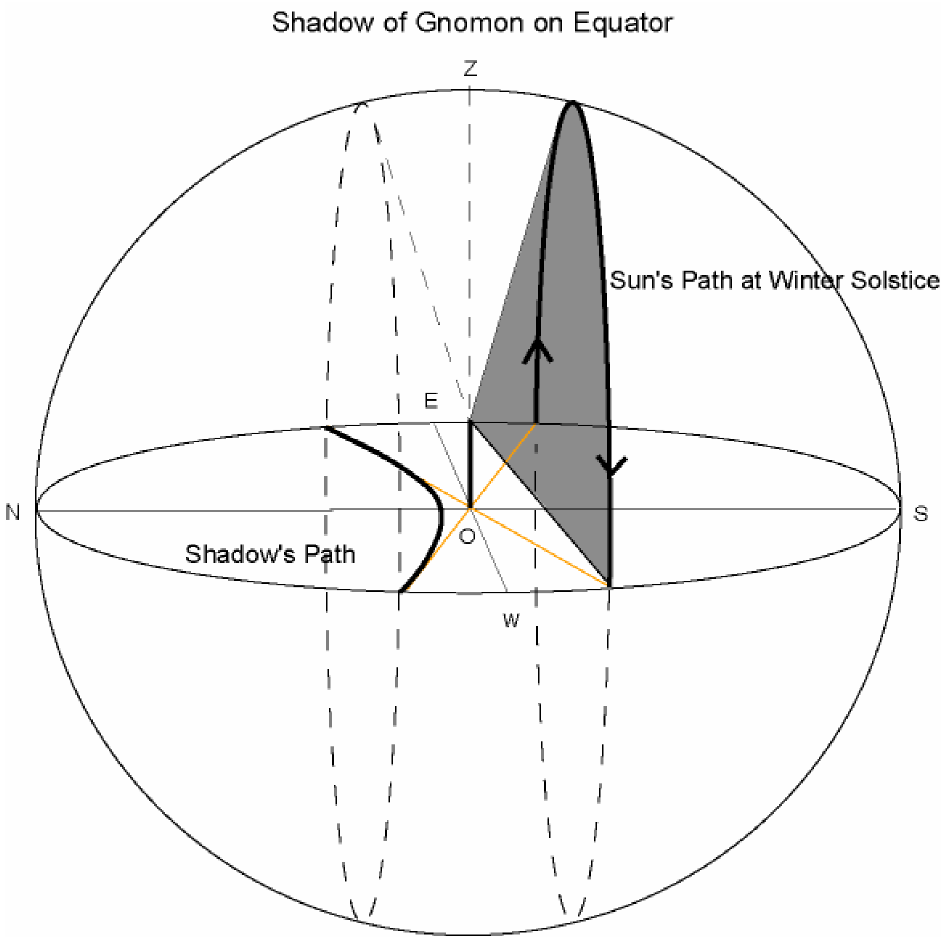

Figure 4.31

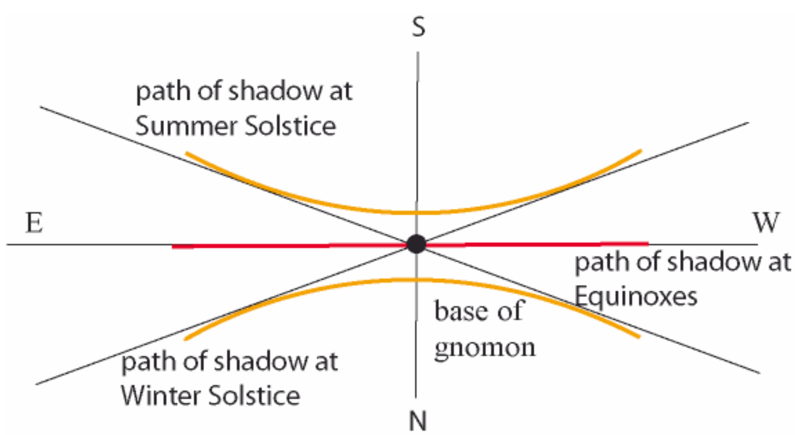

Figure 4.32

(b) Within the Tropics

Between the Tropic of Cancer and the Tropic of Capricorn, the tilt of the double cone, which depends on the latitude of the observer, ranges between 23.5 °and 0°.

Between the summer solstice and the zenith passage, the path of the shadow lies to the South of the East – West line.

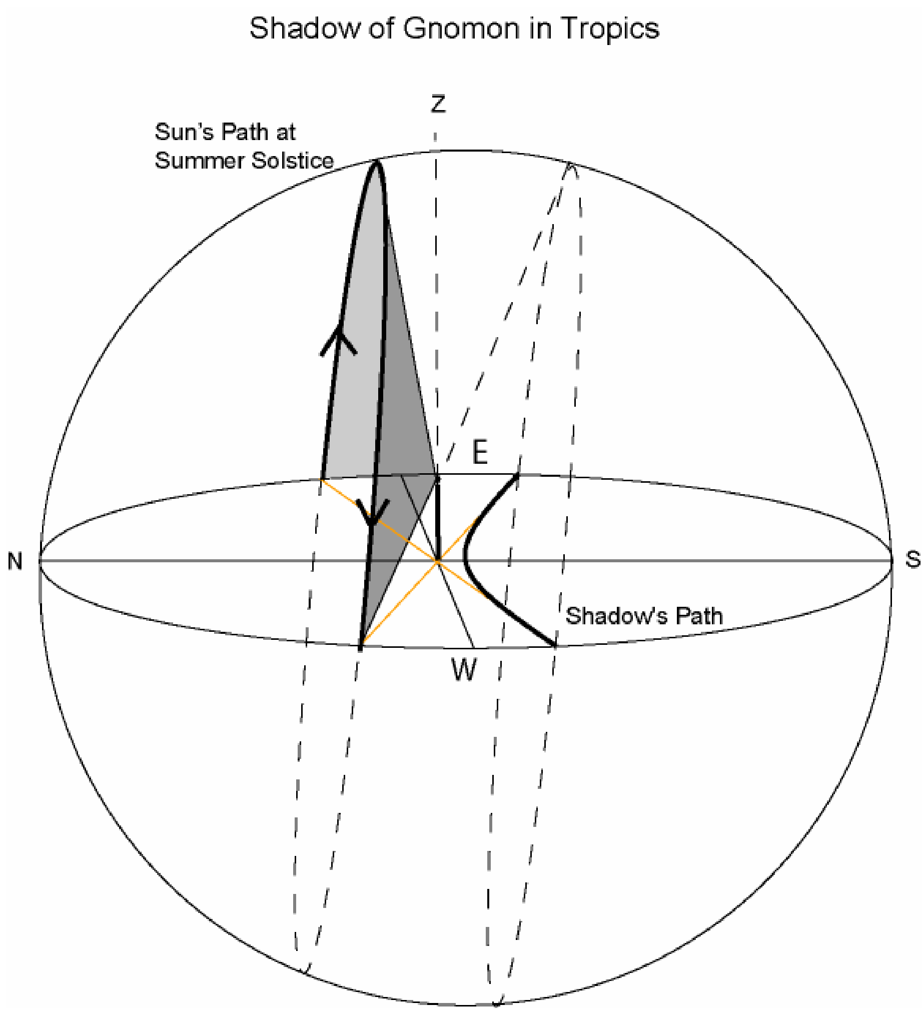

Figure 4.33

On the days between the zenith passage and the winter solstice, the path of the shadow lies to the North of the East – West line.

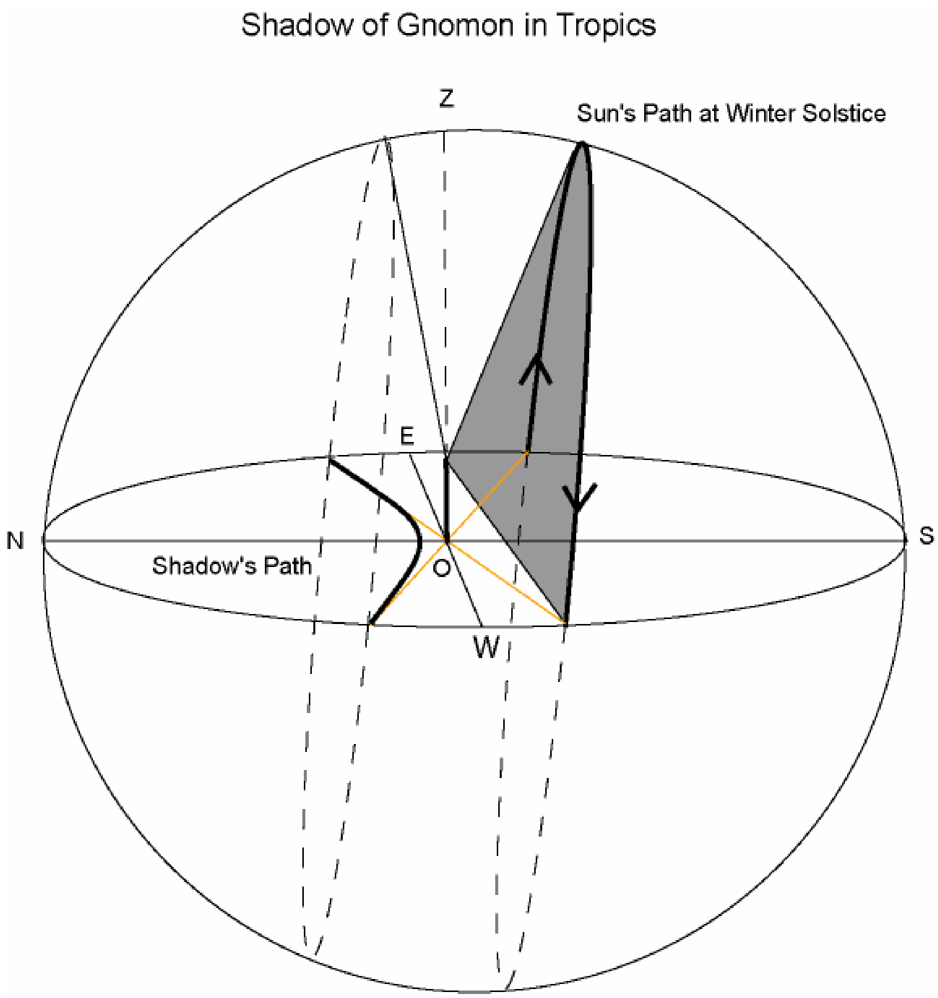

Figure 4.34

As the axis of the cone is tilted by and angle equal to the latitude of the observer, the shadow cast on the day of equinox (a straight line) will not fall on the base of the gnomon.

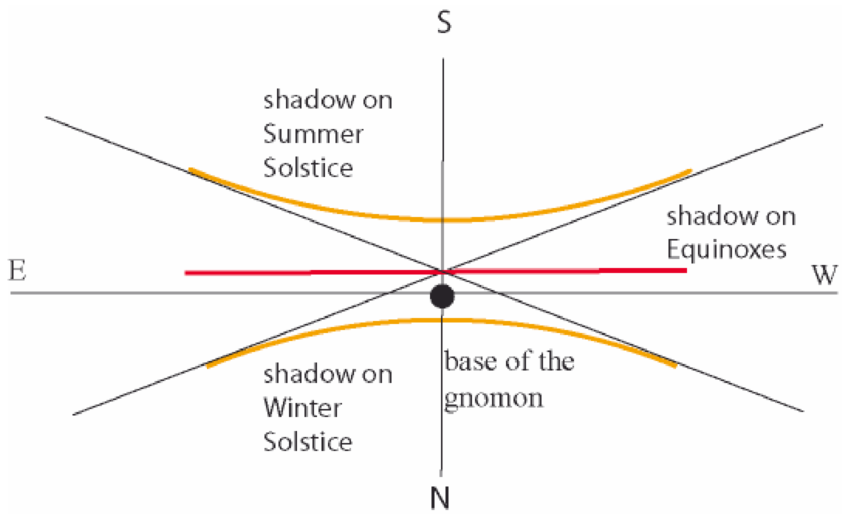

Figure 4.35

(c) Tropic of Cancer

The Tropic of Cancer is of latitude 23.5° North. Hence the axis of the double cone is tilted 23.5° to the horizon.

Due to the tilt of the double cone, the sun only passes overhead on the day of the summer solstice. The Sun will pass overhead at the winter solstice for an observer on the Tropic of Capricorn.

Consequently, the shadow formed on the summer solstice will fall on the base of the gnomon.

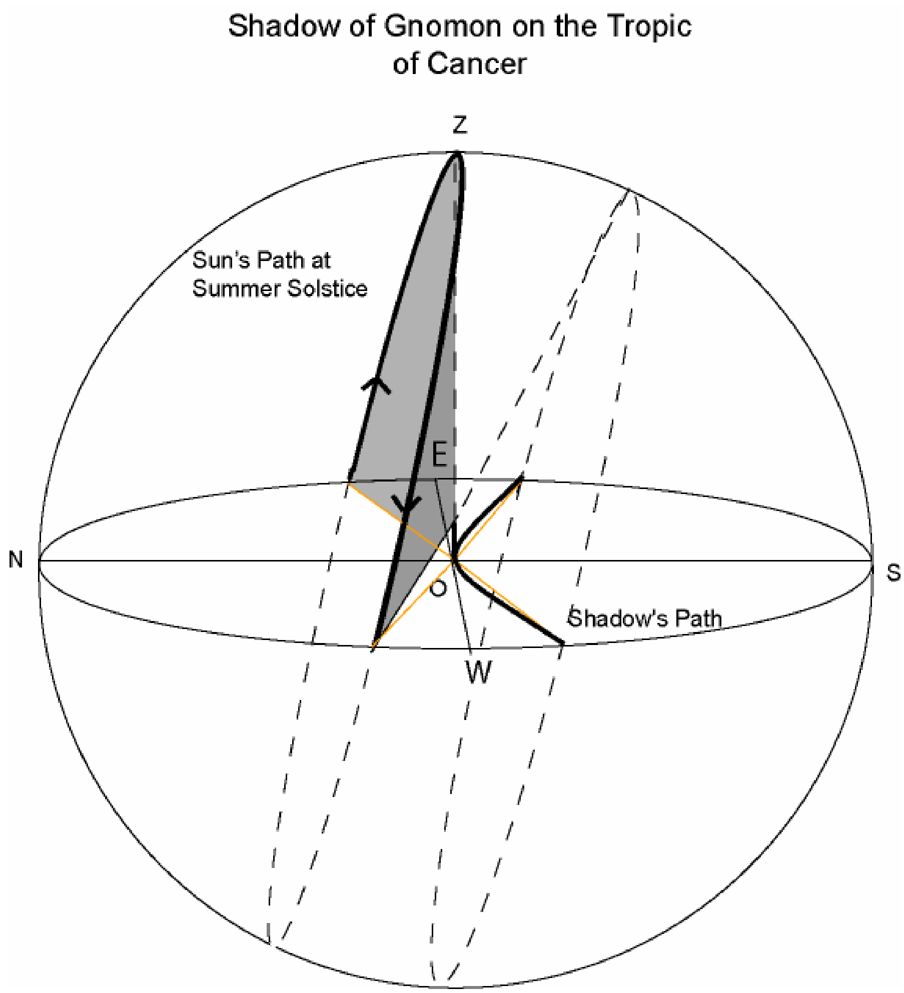
Shadow of Gnomon on the Tropic of Cancer
Z
Sun's Path at
Summer Solstice
E
N
S
O
Shadow's Path
W

Figure 4.36

Likewise for the winter solstice,

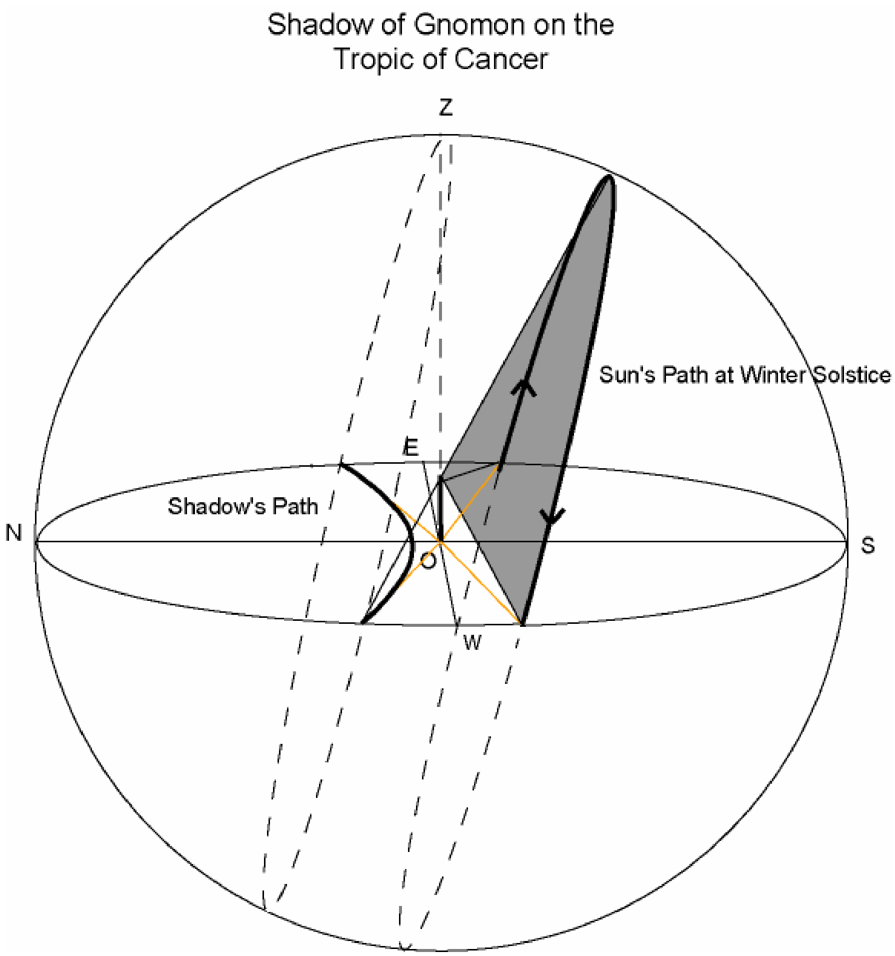

Figure 4.37

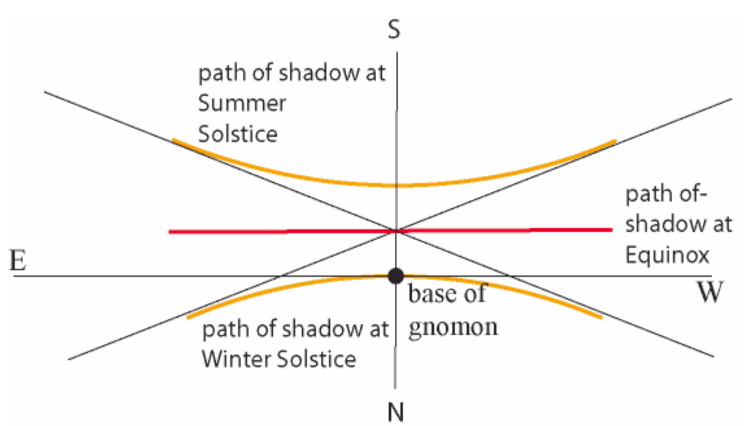

Figure 4.38

We see that the shadow paths formed are either hyperbolas or straight lines which flattens approaching the equinoxes.

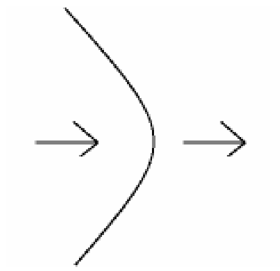

Figure 4.39

Hyperbolas then sharpen on approaching the solstices.

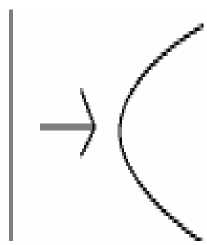

Figure 4.40

Moreover in the northern hemisphere, the shadow formed in the equinoxes shifts southwards with increasing latitude north and in the southern hemisphere, the shadow formed on the equinoxes shifts northwards with increasing latitude south.

Chapter 5

Maya Phenomenon

5.1 Introduction

Before introducing the Maya phenomenon we will go through some basic information required to define it. In this section we will revise some concepts about the daily path of the Sun, which will help us understand when the zenith passage of the Sun takes place at different latitudes.

The daily path of the Sun is roughly a circle parallel to the celestial equator. In the course of a year, the Sun travels in a helical path (corkscrew fashion) across the sky. That is, on the days of the equinoxes, the daily path of the Sun will match the celestial equator and the Sun rises exactly East and sets exactly West. When approaching summer, the Sun will rise and set at positions closer to the North of the celestial sphere. At the day of the summer solstice, the Sun attains its northern most rising and setting points and has a maximum declination of about 23.5°. And on the days following the summer solstice, the Sun will rise and set slightly more south from the previous day’s positions until the day of the winter solstice where the Sun reaches its greatest southern most declination of about 23.5°. Approaching the fall equinox, the Sun rises and sets slightly more north each day. This cycle repeats itself every tropical year.

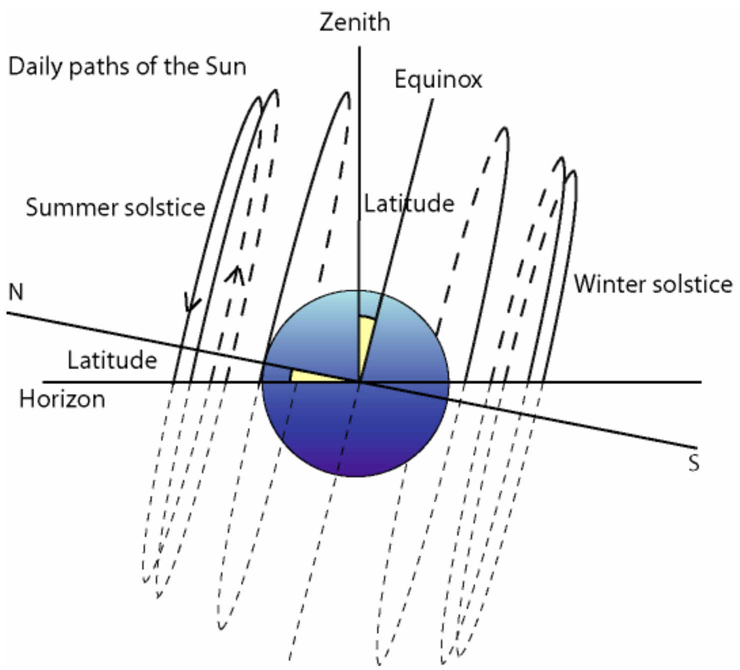

Figure 5.1: Daily Path of the Sun in a Year

The tropics is the region on the Earth where the Sun passes the zenith at some time of the year. The day where the Sun passes the zenith is called the zenith passage. Two imaginary circles, the Tropic of Cancer and the Tropic of Capricorn, bound it. They lie at about 23.5° North and 23.5° South of the equator respectively.

At the Tropic of Cancer, the zenith passage of the Sun will take place on the summer solstice. In the northern hemisphere, moving towards the equator, the days of the zenith passages will move further from the summer solstice as latitude decreases. Finally at the equator, the zenith passages will coincide with the equinoxes.

This is similar for the regions in the Tropics in the southern hemisphere. From now on we will concentrate only on the regions in the Tropics within the northern hemisphere.

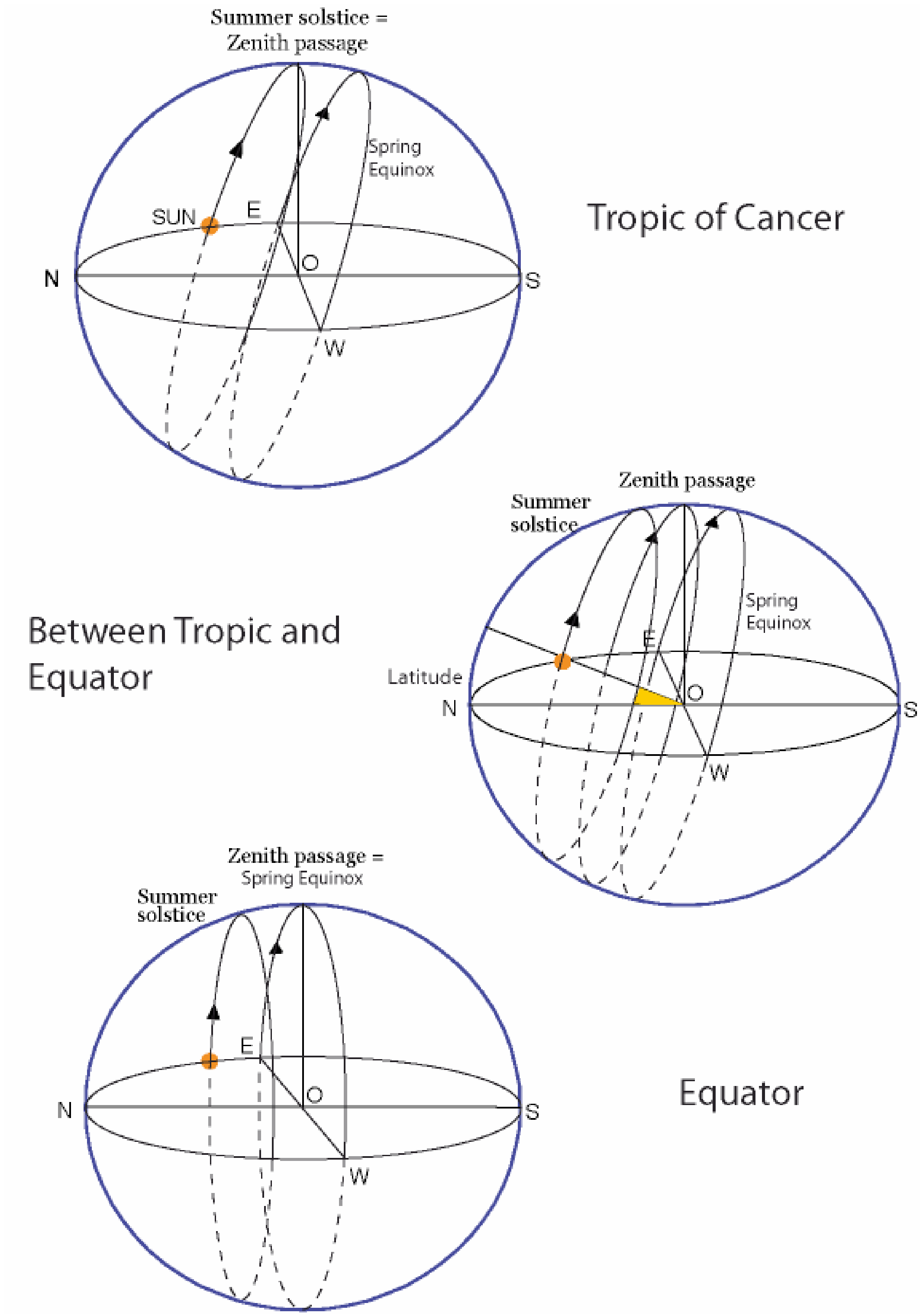

Figure 5.2: Latitude and Zenith Passage

5.2 The Maya Phenomenon

In this section, we give a description of the Maya phenomenon based in the increase and decrease in azimuth, which is defined as the angular distance measured from north to the projection of the Sun on the horizon in the clockwise direction.

In the northern hemisphere of the tropics, on the days between the summer solstice and the zenith passage inclusive of the summer solstice, the Sun never passes directly overhead. (In the southern hemisphere of the tropics, the Sun never passes directly overhead in the period between the winter solstice and the zenith passage inclusive of the winter solstice.) On those days, the azimuth of the Sun first increases, as it rises. To the observer, the Sun appears to travel clockwise then. However, as the Sun rises beyond a certain point, T1, the azimuth decreases. At that moment, the Sun's motion appears anti-clockwise in the sky as the Sun avoids passing overhead. Finally, beyond another point, T2, the Sun's azimuth increases once again and it resumes its clockwise motion until sunset. (Refer to Appendix 3 for data showing azimuth of the Sun in a day.) The increase and decrease in azimuth is due to the sharp inclination of the plane of the Sun's path towards the zenith. We call this the Maya Phenomenon since the Mayans were the first astronomers who observed this.

From now on we will concentrate only on the regions in the Tropics within the northern hemisphere, as the phenomenon is similar in the southern hemisphere in the period between the winter solstice and the zenith passage.

5.2.1 Existence of Turning Points T1 and T2

In the previous section, we note that the azimuth of the Sun first increases, decreases then later increases again. Hence in this section we will define and prove the existence of the transition points, which marks the change from an increase to decrease and vice-versa.

In the northern hemisphere, at sunrise the azimuth of the Sun is increasing. When the Sun crosses the *meridian* in the North, the azimuth is decreasing. Hence there exist a point, T1 where the azimuth changes direction. In addition, the azimuth at sunset is increasing hence there is another point T2 where the azimuth changes direction again. T1 and T2 are the turning points of the azimuth.

5.2.2 Sun’s Path at Chichen Itza in Mexico

We will now give another description of the Maya phenomenon based on the Sun’s motion during the critical period. For simplicity we will refer to Chichen Itza, which was the ancient city of the Maya civilization between 800 and 1200 AD.

Chichen Itza lies at latitude 20.7° North of the equator. We shall station the observer at this latitude to better understand this phenomenon. For now on we will refer to the period between the summer solstice and the zenith passage including the summer solstice, as the critical period.

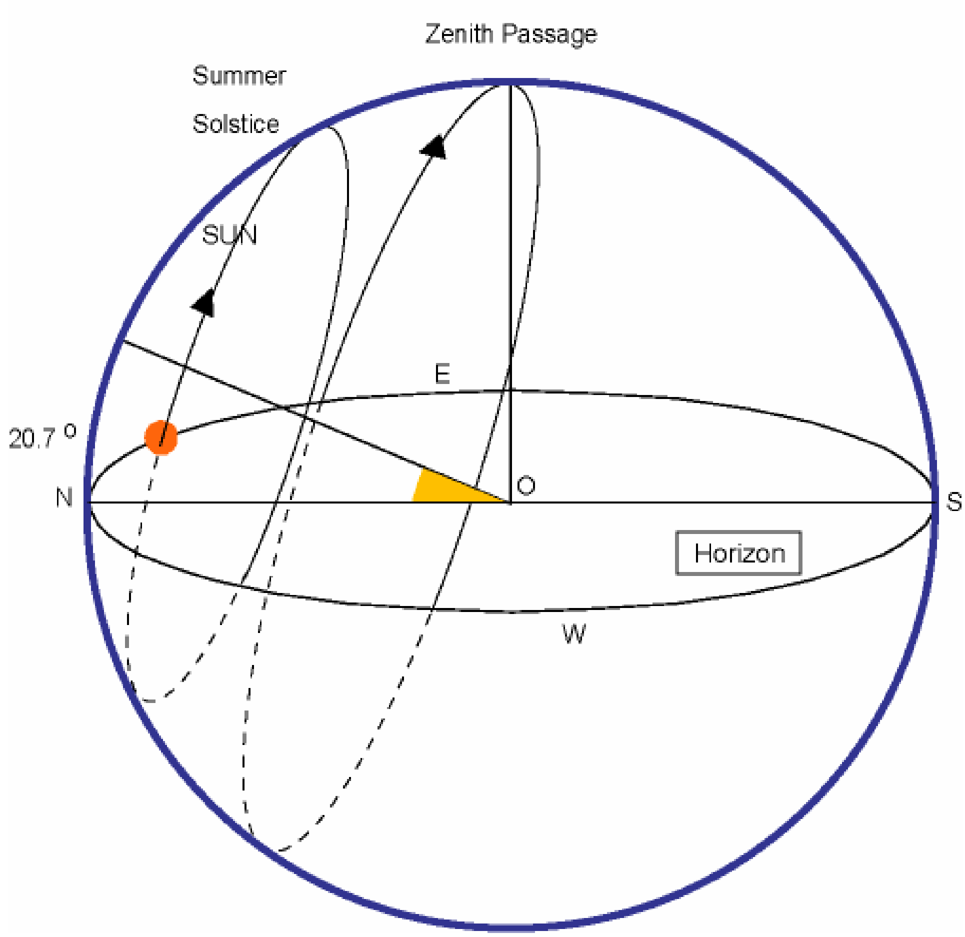

Figure 5.3: Critical Period: Days between Summer Solstice and Zenith Passage

During the critical period, the Sun makes an angle of about 69.2° (= 90° – 20.7°) from the horizon as it rises. During that period the Sun never passes directly overhead. Hence within the critical period, the Sun will still lie north of the East-west line.

We define clockwise motion as an increase in azimuth of the Sun and likewise anti-clockwise motion is the decrease in azimuth of the Sun. As the Sun rises it approaches the zenith and appears to be traveling on a path, which will pass overhead. Its azimuth increases. This gives the impression that the Sun is traveling in a clockwise direction. Later as the Sun passes in front of the East-west line, it becomes clear to the observer that the Sun will not pass overhead and its azimuth decreases. Hence the Sun appears to travel in an anti-clockwise direction then. As the Sun descends, it moves further from the zenith and its azimuth decreases. This gives the impression that it is once again traveling in a clockwise direction.

5.3 Detailed Observation of Sun's Motion

In this section we will give a definition and prove the existence of the transition points for the description of the Maya phenomenon based on the Sun's motion which involves looking at the tangents to the Sun's path.

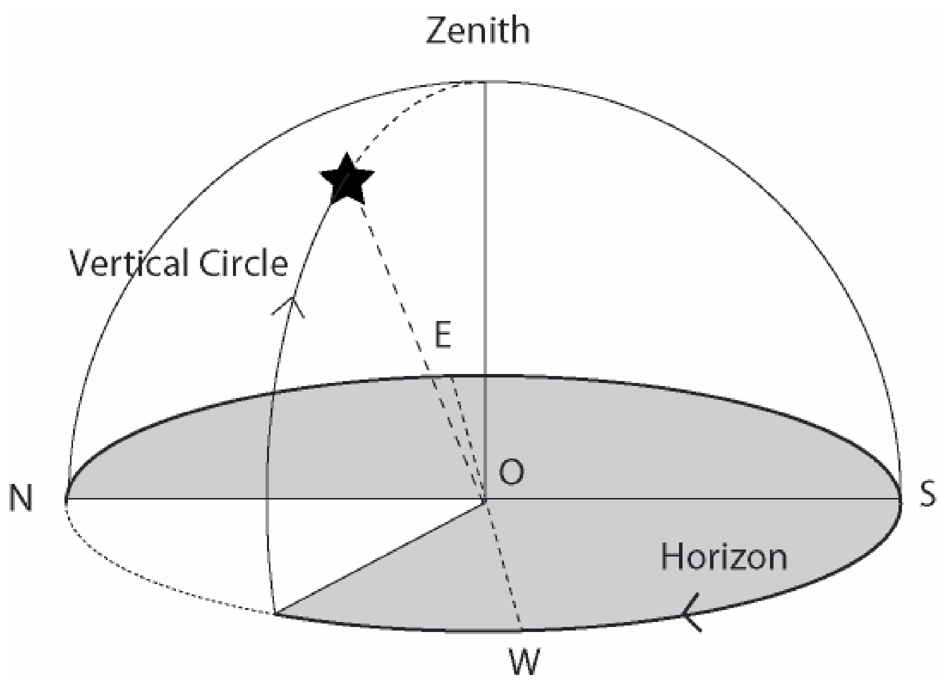

Figure 5.4: Azimuth

The azimuth is defined as the angular distance measured from north to the projection of the celestial body (the Sun) on the horizon in the clockwise direction. Both the celestial body and the zenith lie on a vertical circle on the Celestial Sphere. That vertical circle intersects the horizon at the azimuth of the celestial body.

Vertical circles are the great circles joining the zenith to the *nadir* of an observer. The transition from clockwise to anti-clockwise motion or from anti-clockwise to clockwise motion takes place when a vertical circle is the tangent line to the Sun's path (Figure 5.5), i.e, when the vertical circle of the Sun intersects the Sun's path at only one point. There are two such points. We call the point closer to sunrise, T1 and the point closer to sunset, T2. At all points other than T1 and T2, the vertical circle of the Sun will intersect the Sun's path twice (we include the potion of the path below the horizon), (Figure 5.5). Points T1 and T2 lie on the same *almucantar* as the path of the Sun is assumed to be symmetric.

Between sunrise and T1, the azimuth increases as the Sun rises steeply from the horizon. As it increases, the projection of the Sun on the horizon moves in a clockwise direction. Hence the Sun travels in a clockwise direction in the sky.

When the Sun rises to position T1, the vertical circle of the Sun becomes tangent to the Sun's path. Beyond T1, the Sun travels in front of the zenith. Thus the vertical circle of the Sun rotates anti-clockwise. As a result, the azimuth decreases after T1. Hence the azimuth had reached its East most position on the horizon at T1. Beyond T1, the Sun travels in an anti-clockwise direction as the projection of the Sun moves in an anti-clockwise direction.

When the Sun reaches T2, the vertical circle of the Sun becomes tangent to the Sun's path again. As the Sun moves beyond T2, the vertical circle of the Sun rotates in the clockwise direction. Hence the azimuth continues increasing until sunset. The azimuth had reached its West most position on the horizon when the Sun is at T2. In addition, the Sun will move in a clockwise direction as the projection of the Sun moves in a clockwise direction from T2 to sunset.

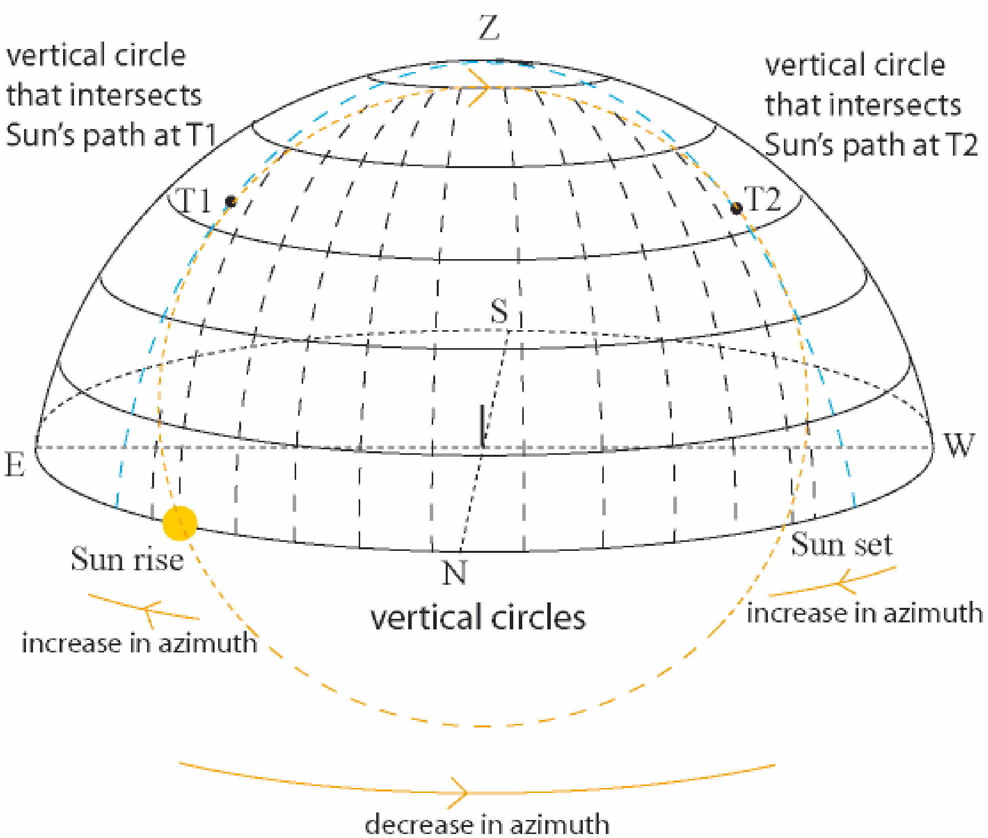

Figure 5.5: Change in Azimuth

5.4 Gnomon Shadow

For this section we give a final description of the Maya phenomenon based on the gnomon shadow cast in a day. As before we will give a definition and prove the existence of the transition points.

Another characteristic of the Maya phenomenon is the behavior of a shadow cast by a gnomon. Taking the definition of clockwise motion as an increase in azimuth of the Sun and likewise anti-clockwise motion is the decrease in azimuth of the Sun as before, during the critical period, a gnomon shadow will appear to first travel clockwise as the shadow curves towards the base of the gnomon but its motion becomes anti-clockwise, as the shadow passes in front of the base of the gnomon. Hence there exist a point T1 on the shadow where the shadow changes direction.

Approaching sunset, the shadow resumes its clockwise motion as it curves away from the gnomon. Hence there exists another point T2 that marks the point on the shadow where the change in direction occurs.

*Diagram is exaggerated to show the direction of the shadow.

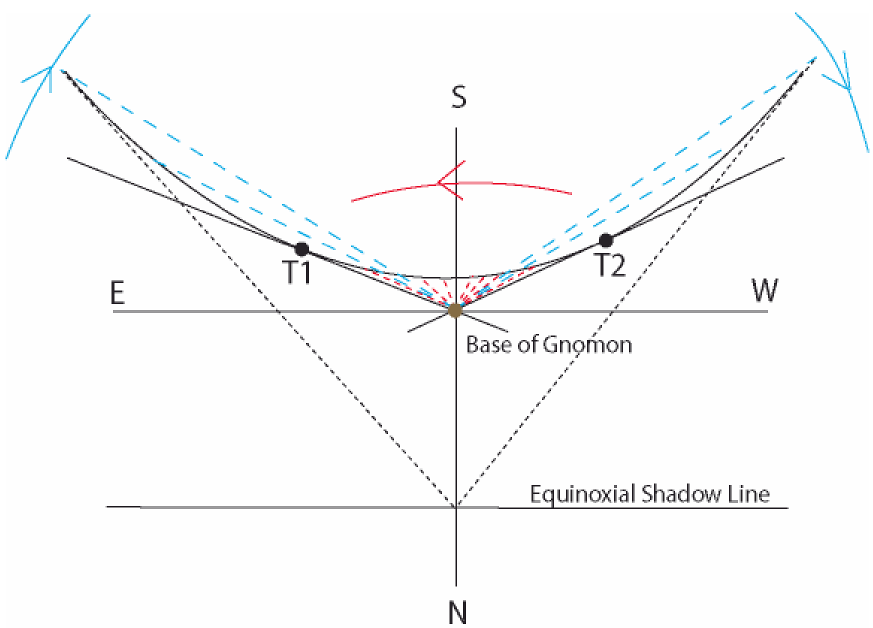

Figure 5.6: Change in Shadow Direction

The shadow first curves towards as the azimuth increases then away from the base of the gnomon as the azimuth decreases. Hence the change in direction occurs at T1, where the tangent to the shadow intersects the base of the gnomon. Likewise, T2 occurs where the next tangent to the shadow intersects the base of the gnomon. The shadow is assumed to be symmetric about the North-south line; hence T1 and T2 are equidistance from the base of the gnomon.

5.5 Transition Points

In the previous sections we have introduced three different descriptions of the Maya Phenomenon, in this section we claim that all the three transition points, defined differently for each description, occur at the same time since all they all lie on the same vertical circle.

This behavior of the shadow is related to the motion of the Sun in the sky. In a day the Sun's path is approximately a circle. We take the Sun to be a point source in the sky, then a line from the Sun to the tip of the gnomon traces out a cone as the Sun travels in that circle. Extending that same line from the tip of the gnomon to the ground traces out congruent cone. Hence we have a double cone whose vertex rest on the tip of the gnomon. A trace of a gnomon shadow cast in a day would be the intersection between that double cone and the horizon.

Projecting a light ray on the ground, we see that from sunrise to T1, as the azimuth of the Sun increases (Figure 5.5), the shadow travels in a clockwise direction (Figure 5.6). This is as both the projection of the Sun on the horizon and the extension of the shadow to the horizon lies on the same vertical circle (Figure 5.7).

Hence the position T1 on the Sun's path, which is where the vertical circle of the Sun becomes tangent to the Sun's path (refer Figure 5.5) is the same point T1 on the shadow, which is where the tangent to the shadow first intersects the base of the gnomon (Figure 5.6). Likewise for T2, the position on the Sun's path corresponds to the position on the shadow.

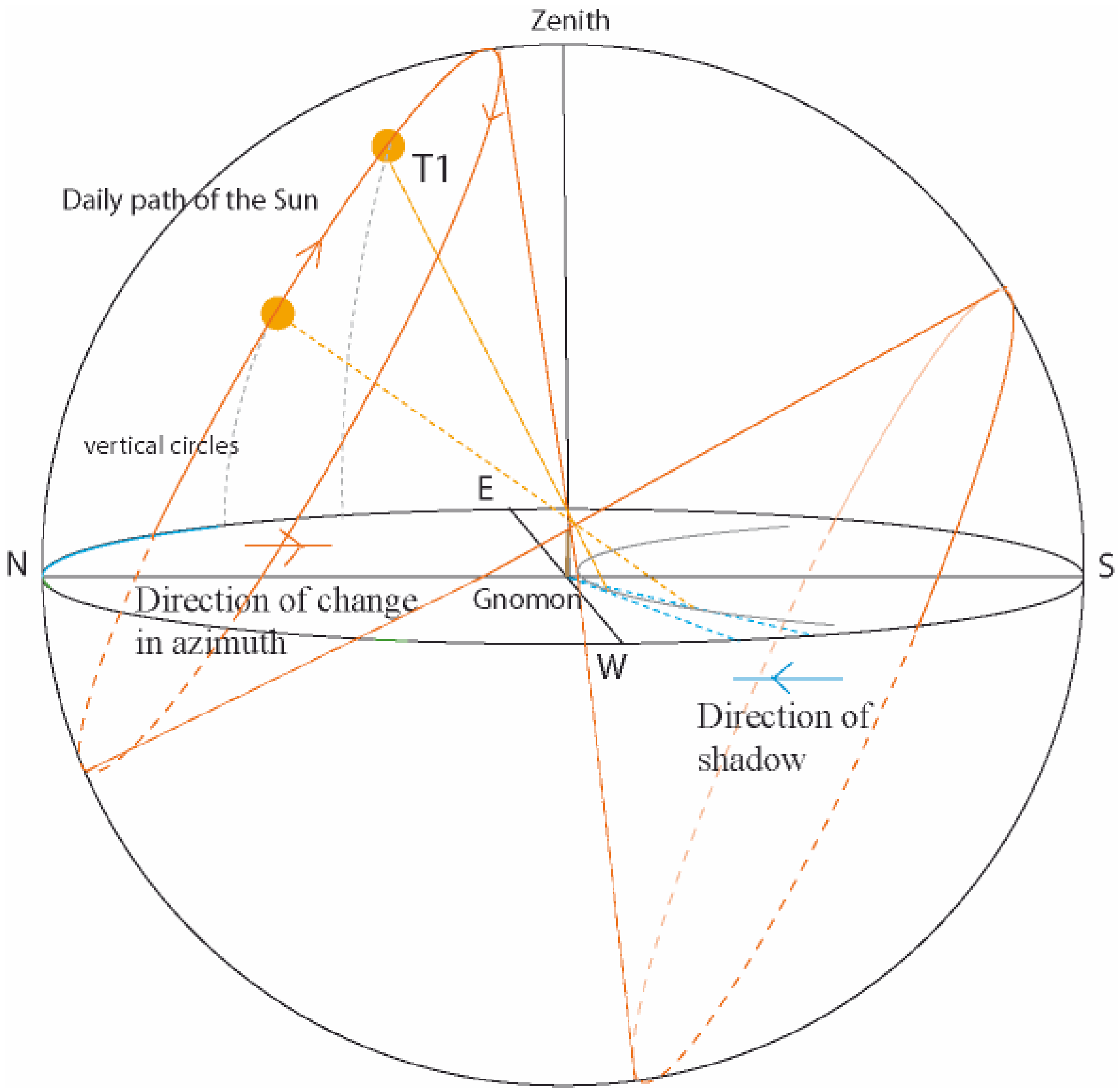

Figure 5.7: Sunrise to T1: Change in Azimuth and Direction of Shadow

Similarly, from T1 to T2, projecting a light ray on the ground, we see that as the azimuth of the Sun decreases, the projection of the Sun on the horizon moves anti-clockwise (Figure 5.5) and thus the shadow travels in an anti-clockwise direction (Figure 5.6).

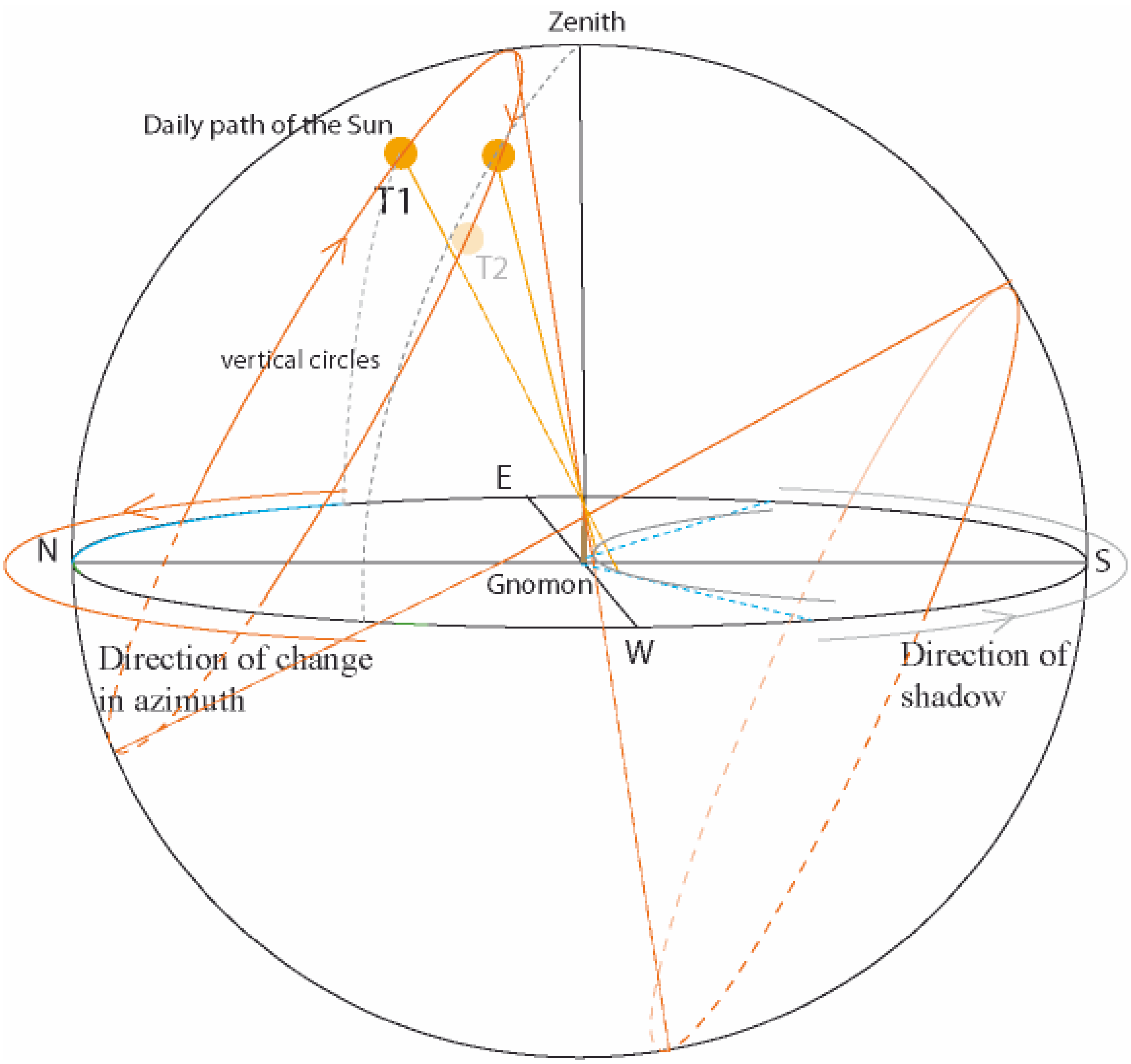

Figure 5.8: T1 to T2: Change in Azimuth and Direction of Shadow

In the same way, from T2 to sunset, the increase in azimuth of the Sun results in a clockwise direction of the shadow.

Shape and Position of Gnomon Shadow

At the solstices, the nappe of the double cone makes an angle of about 66.5° (90° – 23.5°) from the axis of the cone. This angle will be larger than 66.5° on the days between the solstices. At the equinoxes, the double cone flattens into a plane and the angle attains its maximum value of 90°. Likewise, the shadow is the most curved on the solstices and curve less between solstices and gradually flattens into a straight line on the equinoxes.

Besides the gradual flattening of the shadow between solstice and equinox, the distance from the tip of the shadow to the base of the gnomon also varies through out the year. The shadows formed during the critical period will all lies to the South of the East-West line.

Since the base of the gnomon lies on the East-West line, the shadow will touch the base of the gnomon only on the day of the zenith passage. In the Northern hemisphere, the equinoctial shadow line lies to the south of the East-West line. Hence the shadow is the furthest from the base of the gnomon on the winter solstice.

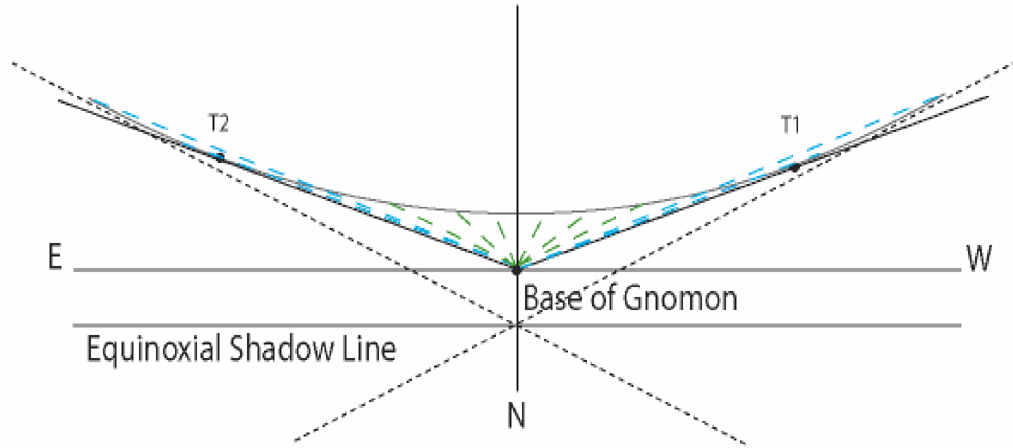

Figure 5.9: Shadow of Gnomon

5.6 Critical Periods

In this section we introduce our criterion for deciding the noticeability of the phenomenon to an observer and later we give the factors affecting noticeability.

The Maya Phenomenon is only observable within the tropics on the days between the summer solstice and the zenith passage.

We base the "noticeabilty' of the phenomenon on when T1 and T2 take place in the day. We note that the change in motion of the Sun is less noticeable when T1 and T2 takes place close to sunrise and sunset respectively. This is as the Sun will be low and close to the horizon. Hence the change in direction is not obvious. There is an analogous case for the shadow cast in a day. We note that the declination of the Sun depends on the date as the declination of the Sun increases on days approaching the summer solstice. Hence, when the declination of the Sun is large, the asymptotes will be far from the East-West line, thus T1 and T2 are far from the base of the gnomon (Figure 5.10). The shadow only moves a very small distance clockwise to reach T1 before traveling anti-clockwise between T1 and T2. Hence the phenomenon is less "noticeable".

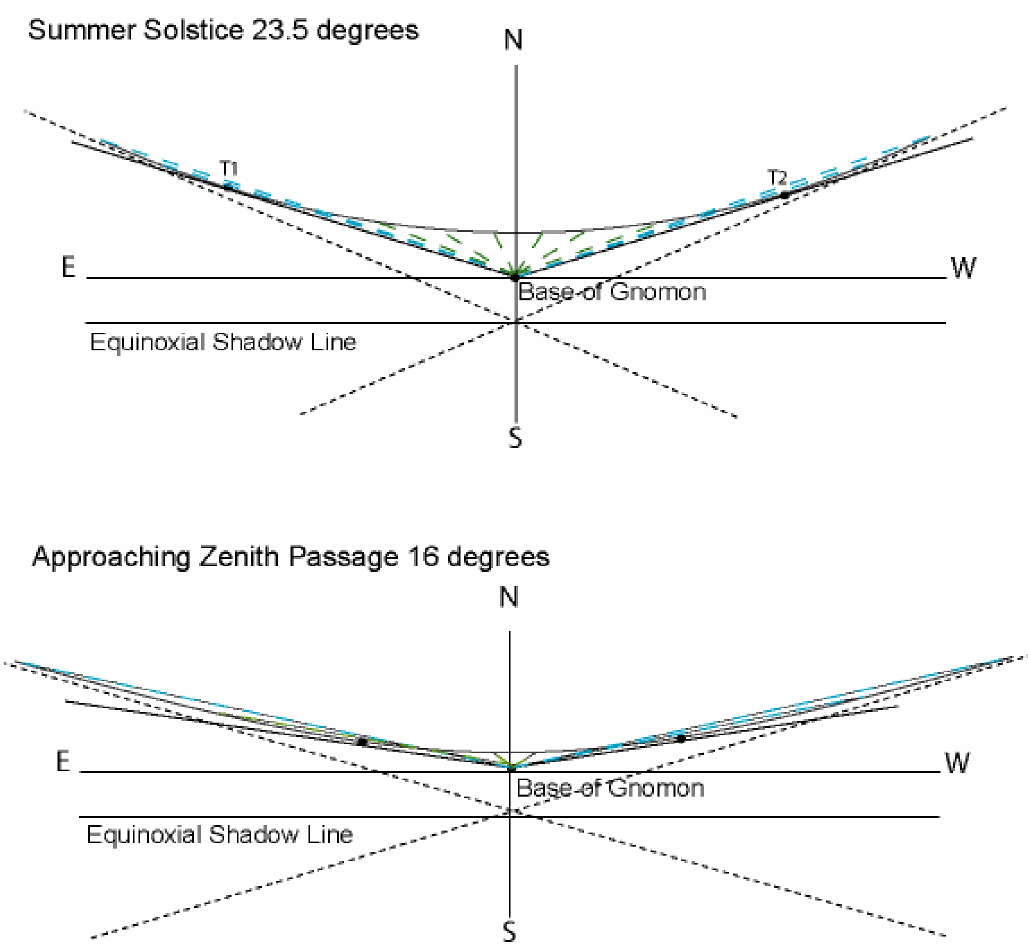

Figure 5.10: Shadow at Summer Solstice and Zenith Passage

We define the critical time as the period between T1 and T2. (Also recall that we had defined the critical period as the period between the summer solstice and the zenith passage.) The critical time depends on two factors: the day during the critical period that the phenomenon occurs and the latitude of the observer.

T1 occur progressively later in the mornings on the days close to the zenith passage. On days closer to the zenith passage, the angle between the asymptotes and the East–West line deceases and the shadow flattens. Hence the shadow has to travel over a further distance clockwise before its tangent points to the base of the gnomon (Figure 5.10). T1 and T2 are points on the shadow closer to the base of the gnomon on days close to the zenith passage, that is the critical time is closer to noon on days approaching the zenith passage.

Day	Period between T1 and T2
Summer solstice	Longest.
Strictly between the summer solstice and the zenith passage.	Gradually decreases nearing the zenith passage.
Zenith passage	Briefly when the Sun is exactly at the zenith. T1 and T2 are the same.
Days beyond the critical period.	T1 and T2 do not exist.

Table 5.1: Critical Times* on Different Days at a Fixed Latitude.

(*The period between T1 and T2)

The critical period spans from one day (at the Tropic of Cancer) to about half a year (at the equator). As the number of days in the critical period varies, the behavior of the azimuth of the Sun during that period also varies.

Range of critical periods in a year	Location of the observer	Azimuth from sunrise to T1	Azimuth from T1 to meridian	Azimuth from meridian to T2	Azimuth from T2 to sunset
Critical period from summer solstice to equinoxes	Equator	No change – azimuth at T1 is azimuth at sunset.	Decreases to 0°.	Decreases from 360°.	No change – azimuth at T2 is azimuth at sunset.
Critical period from summer solstice to zenith passage (before equinoxes).	Within the northern tropics	Increases to azimuth at T1.	Decreases from azimuth at T1 to 0°	Decreases from 360° to azimuth at T2.	Increases from azimuth at T2.
Summer solstice and zenith passage coincides	Tropic of Cancer	Increase to 90° - T1 and T2 are the same points.	-	-	Increases from 270°.
Beyond critical period.	Beyond tropics	Increase.	Increase.	Increase.	Increase.

Table 5.2: Azimuth at Different Critical Periods.

The critical time also depends on the latitude of the observer. Within the tropics the critical time occurs earlier in the days at latitudes approaching the equator (Figure 5.11). At latitudes closer to the equator, the plane of the Sun is inclined more steeply towards the zenith. Hence the tangents to the Sun's path will be closer to the zenith, resulting in T1 occurring later in the day.

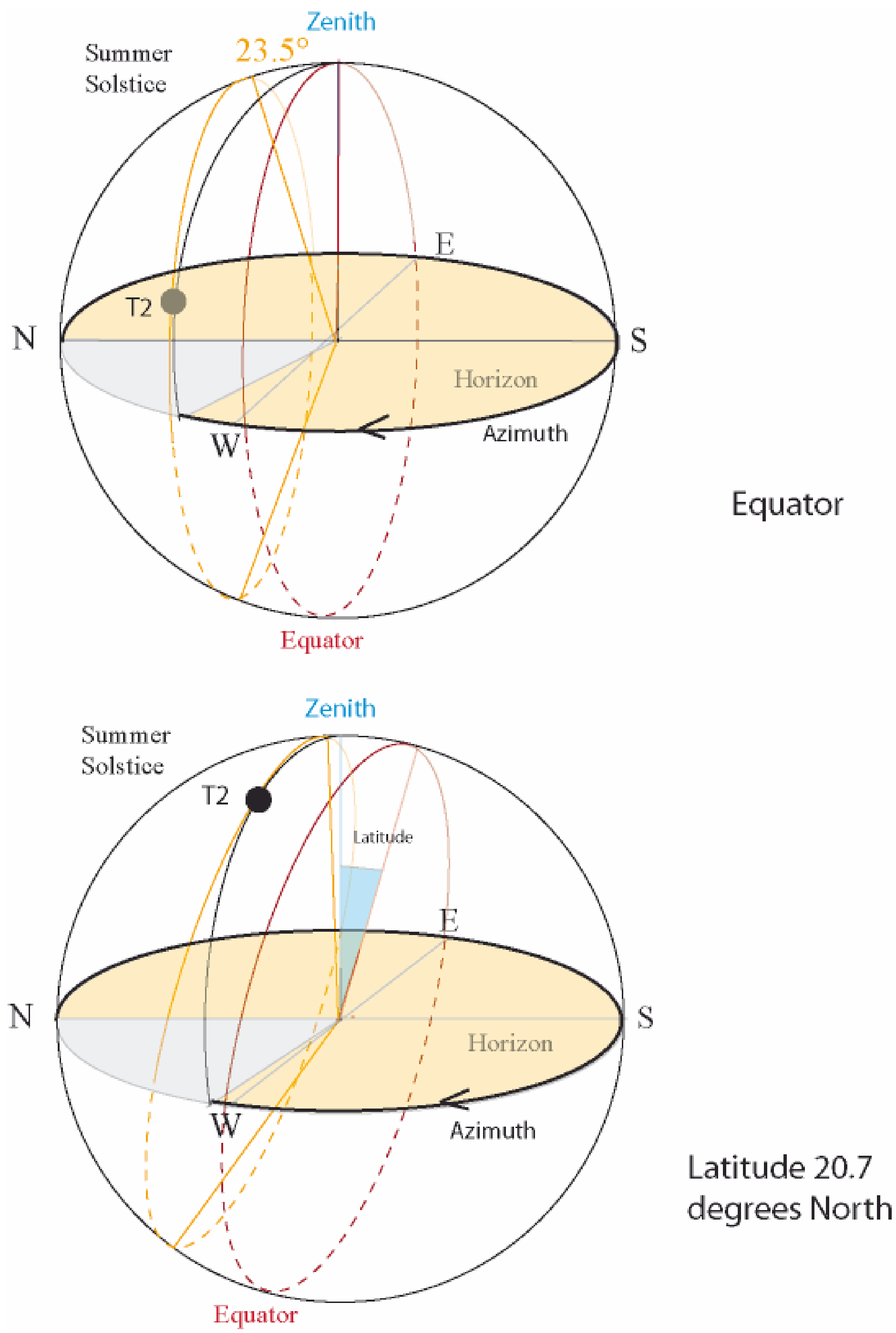

Figure 5.11: T2 at Different Latitude

Latitude (within northern hemisphere)	Azimuth from sunrise to T1	Azimuth from T1 to meridian	Azimuth from meridian to T2	Azimuth from T2 to sunset
0° North – Equator	No change – azimuth at T1 is azimuth at sunset.	Decreases to o°.	Decreases from 360°.	No change – azimuth at T2 is azimuth at sunset.
Strictly between 0° to 23.5°	Increases to azimuth at T1.	Decreases from azimuth at T1 to 0°	Decreases from 360° to azimuth at T2.	Increases from azimuth at T2.
23.5° North – Tropic of Cancer	Increase to 90° - T1 and T2 are the same points.	-	-	Increases from 270°.
Greater than 23.5° North (T1 and T2 does not exists)	Increase.	Increase.	Increase.	Increase.

Table 5.2: Azimuth at Different Latitudes on a Fixed Date*

* Table is similar to “Table 5.2: Azimuth at Different Critical Periods” since the latitude of the observer determines the number of days in the critical period.

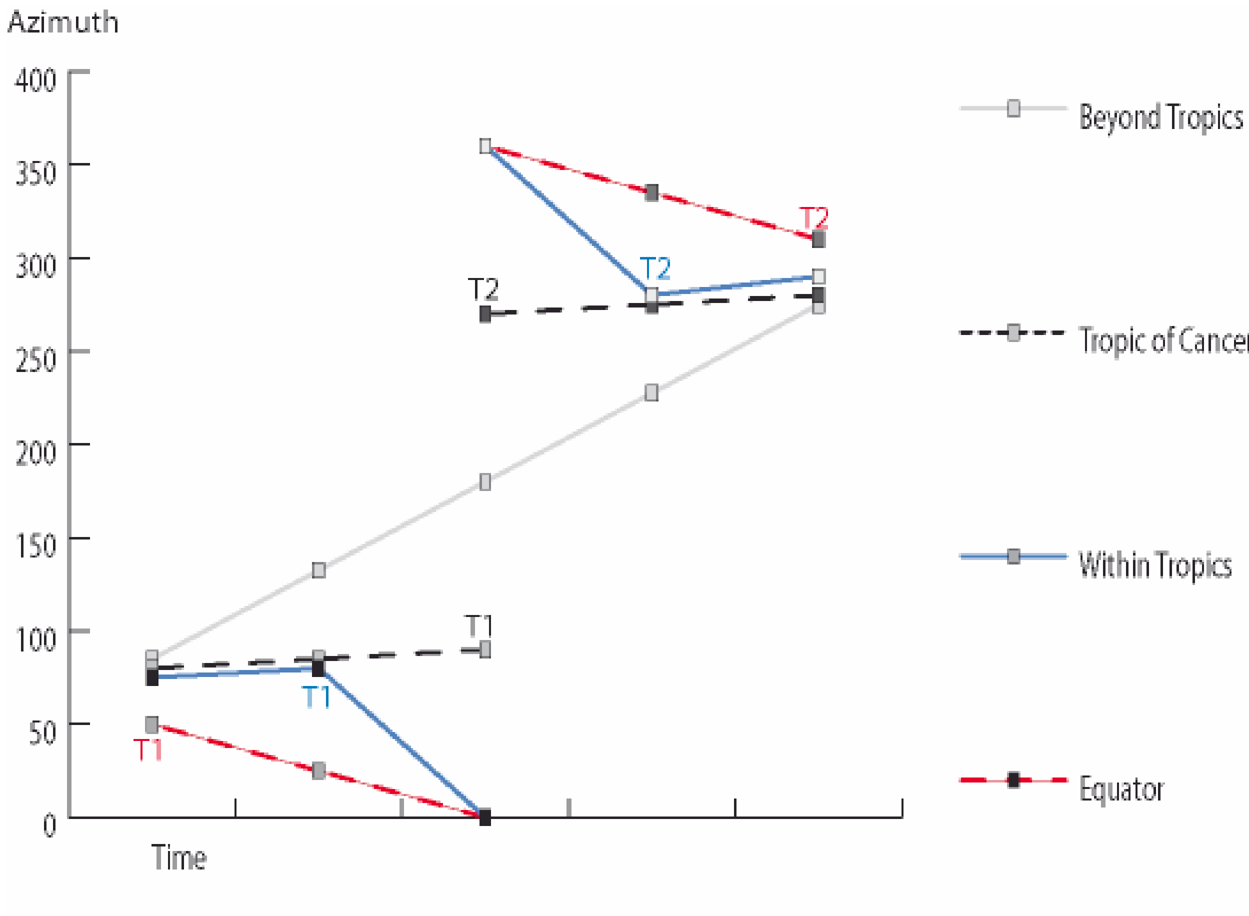

Figure 5.12: Graph showing Azimuth at Different Latitudes

Although the phenomenon is easily observable at higher latitudes, the trade off is that the number of days in which the phenomenon can occur, the critical period, will be reduced. At higher latitude there will be less number of days between the summer solstice and the zenith passage. At the Tropic of Cancer, the zenith passage and the summer solstice coincides, hence the critical period is only one day, thus the phenomenon is not visible there.

Glossary

Almucantar: a circle on the celestial sphere parallel to the horizon. Two stars that lie on the same almucantar have the same altitude.

Altitude: angular distance measured positive upward from the horizon to the star along the star's vertical circle. (DC in Figure 1.1)

Angular aperture: an angle which is half of the opening angle of a right cone. (Angle *v* in Figure 4.1)

Azimuth: angular distance measured from the north point to the base of the star's *vertical circle* along the horizon in an easterly direction. (NOD in Figure 1.1)

Celestial equator: the projection of the earth's equator onto the celestial sphere. (Blue circle in Figure 1.2) This great circle will be 90^{o} distant from the celestial poles at all points. Just as the horizon serves as the fundamental reference circle in the horizon system, so, too, is the celestial equator the fundamental reference circle in the equatorial system.

Celestial poles: extension of the poles of rotation of the earth onto the celestial sphere. P is the north celestial pole and P' is the south celestial pole in Figure 3.5.

Celestial sphere: an imaginary sphere of arbitrary extent centered about an observer located at some position on earth where all the objects in sky can be thought of as lying upon the sphere. Projected, from their corresponding geographic equivalents, are the celestial equator and the celestial poles.

Declination: angular distance measured from the celestial equator to a star along the star's hour circle. It is designated as positive to the north of the celestial equator, negative to the south, and is measured in degrees.

Directrix: The line which, together with the point known as the focal point or focus, serves to define a conic section as the locus of points whose distance from the focus is proportional to the horizontal distance from the directrix. If the ratio $r = 1$, the conic is a parabola, if $r < 1$, it is an ellipse, and if $r > 1$, it is a hyperbola.

Eccentricity: A quantity defined for a conic section which can be given in terms of semimajor a and semiminor axes b. The eccentricity e can also be interpreted as the fraction of the distance along the semimajor axis at which the focus lies, $e = c/a$, where c is the distance from the center of the conic section to the focus.

Ecliptic: extension onto the celestial sphere of the sun's path in a geocentric model. It is also shown in Figure 1.2 as a great circle making an angle of 23.5° with the celestial equator. As far as terrestrial observers are concerned, this circle traces out the annual motion of the sun on the sky relative to the background of distant stars.

Equinoxes: the two points on the celestial sphere at which the sun crosses the celestial equator. That is, the two intersection points of the two great circles, the ecliptic and the celestial equator. The spring equinox (Labeled in Figure 1.2) is one of the points of intersections where the sun passes from the southern to the northern hemisphere. The fall equinox is the other point of intersection opposite to spring equinox, where the sun passes from north to south. The equinox dates are March 21 and September 22 approximately.

Fall equinox: See *Equinoxes*.

Focal point or focus: a point related to the construction and properties of conic sections. For example, an ellipse is a curve that is the locus of all points in the plane the sum of whose distances r_1 and r_2 from two fixed points F_1 and F_2 (the foci) separated by a distance of 2c is a given positive constant 2a. a is the semimajor axis and the origin of the coordinate system is at one of the foci.

Geocentric: a viewpoint that the Earth is at the center of the Universe and/or the Solar System.

Gnomon: a scientific instrument that can be used for finding the declination of the sun through the year and to study the path of the sun among other things. It is one of the first scientific instruments ever made, originating with the Chaldean astronomers of Babylon and from there brought to the Greek world. It is still very instructive, an excellent example of random and systematic errors of measurement, vividly demonstrating astronomical principles.

Heliocentric: a viewpoint that the Sun is at the center of the Universe and/or the Solar System.

Horizon (of an observer): the horizontal plane through the eyes of the observer and is perpendicular to the observer.

Hour circles: great circles passing through the celestial poles. They bear the same relation to the celestial equator as do vertical circles to the horizon.

Inclination: the angular distance of the plane of the orbit of the planet and the ecliptic, normally stated in degrees.

Latitude: the position north/south of the equator measured from 0^o to 90^o.

Mean anomaly: the angular distance of the mean sun from the perigee.

Mean longitude: the longitude at which an orbiting body could be found if its orbit was circular and its inclination was zero. For the Earth orbiting about the Sun, 1day = 1^{o}.

Mean sun: the sun which is assumed to move along the ecliptic at a constant speed.

Meridian: great circle on the celestial sphere that passes through the zenith and north point of horizon, through the celestial pole, up to the zenith, and through the south point on the horizon and is perpendicular to the local horizon. Thus the azimuth at meridian in the North is 0^{o}.

Midnight sun: a phenomenon occurring in latitudes north of the Arctic Circle and south of the Antarctic Circle where the sun is continuously visible for 24 hours.

Nadir: the point directly opposite the zenith.

North celestial pole: See *Celestial poles*.

Obliquity of the ecliptic: the Earth's axial tilt which is the angle between the two planes, the Earth's orbital plane in which the ecliptic lies and the plane that contains celestial equator. The Earth has an axial tilt of about 23.5°.

Opening angle of a right cone: the angle made by a cross section through the vertex and center of the base of the cone.

Perihelion (perigee): the point of least distance of the elliptical orbit of earth from the sun.

Right ascension: angular distance measured from the spring equinox to the star's hour circle along the celestial equator in an easterly direction (spring equinox to D in Figure 1.2) Because this direction lies along the equator and we measure the passage of time by rotation of the earth, the right ascension coordinate is usually expressed in hours and minutes of time instead of angular measure. The celestial equator is divided into twenty-four hours instead of 360^o; therefore, one hour of time measure is equivalent to 15^o of angle measure.

Right cone: a cone with its vertex above the center of its base.

Semimajor axis: half the distance across an ellipse along its long principal axis.

Semiminor axis: half the distance across an ellipse along its short principal axis.

Solstices: See *Summer solstice* and *Winter solstice*.

Spring equinox: See *Equinoxes*.

Summer solstice: point on the celestial sphere where the sun reaches its greatest distance north of the celestial equator, about June 21.

True anomaly: the angle between the vectors Sun-Planet and Sun-Perihelion.

True longitude: is the longitude at which an orbiting body could actually be found if its inclination were zero.

Vertical circles: great circles passing through the zenith and nadir, perpendicular to the horizon. ZS, ZD, and ZW each depict quarters of vertical circles in Figure 1.1.

Winter solstice: point on the celestial sphere where the sun reaches its greatest distance south of the celestial equator, about December 21.

Zenith: Z, of the observer, O, the point directly overhead (opposite the direction of a plumb line) in Figure 1.1.

Zenith passage: The day where the Sun passes the zenith.

Appendix 1

Day		SUN SD 16' Dec	d
15	FRI	S23 16	3
16	SAT	S23 19	2
17	SUN	S23 21	2
18	MON	S23 23	2
19	TUE	S23 25	1
20	WED	S23 26	0
21	THU	S23 26	0
22	FRI	S23 26	0
23	SAT	S23 26	1
24	SUN	S23 25	1
25	MON	S23 24	2
26	TUE	S23 22	3
27	WED	S23 19	2
28	THU	S23 17	4
29	FRI	S23 13	3

Extract from the table of SUN & PLANETS AT GMT 0hrs DECEMBER 2000

Observing the declinations of the sun, we realize that there are a few days in a row that has a minimum declination of $S\,23°\,26'$. We will assume 21 December as the day of winter solstice in year 2000.

A $5'$ difference from this minimum declination leads us to two dates: 17 December and 26 December which are 4 and 5 days from 21 December respectively.

Appendix 2

Day		SUN SD 16' Dec	d
10	SAT	N23 01	4
11	SUN	N23 05	4
12	MON	N23 09	4
13	TUE	N23 13	3
14	WED	N23 16	3
15	THU	N23 19	2
16	FRI	N23 21	2
17	SAT	N23 23	1
18	SUN	N23 24	1
19	MON	N23 25	1
20	TUE	N23 26	0
21	WED	N23 26	0
22	THU	N23 26	0
23	FRI	N23 26	1
24	SAT	N23 25	2
25	SUN	N23 23	2
26	MON	N23 21	2
27	TUE	N23 19	3
28	WED	N23 16	3
29	THU	N23 13	3
30	FRI	N23 10	4
31	SAT	N23 06	

Extract from the table of SUN & PLANETS AT GMT 0hrs JUNE 2000

Observing the declinations of the sun, we realize that there are a few days in a row that has a maximum declination of $N\,23°\,26'$. We will assume 21 June as the day of winter solstice in year 2000.

A $17'$ difference from this maximum declination leads us to two dates: 12 June and 30 June which are both 9 days from 21 June.

Appendix 3

o , o ,
E103 55, N 1 14 Singapore

Altitude and Azimuth of the Sun
Mar 27, 2006
Zone: 8h East of Greenwich

Azimuth
(E of N)

h m	o
06:30	87.3
06:40	87.4
06:50	87.4
07:00	87.5
07:10	87.6
07:20	87.6
07:30	87.7
07:40	87.7
07:50	87.7
08:00	87.8
08:10	87.8
08:20	87.8
08:30	87.8

h m	o
08:40	87.8
08:50	87.8
09:00	87.9
09:10	87.9
09:20	87.8
09:30	87.8
09:40	87.8
09:50	87.8
10:00	87.8
10:10	87.7
10:20	87.7
10:30	87.6
10:40	87.5
10:50	87.4
11:00	87.3
11:10	87.1
11:20	86.9
11:30	86.7
11:40	86.4
11:50	86.0
12:00	85.5
12:10	84.8
12:20	83.9

h m	o
12.30	82.4
12:40	80.0
12:50	75.2
13:00	61.9
13:10	357.9
13:20	297.3
13:30	284.7
13:40	280.0
13:50	277.6
14:00	276.2
14:10	275.3
14:20	274.6
14:30	274.1
14:40	273.7
14:50	273.4
15:00	273.2
15:10	273.0
15:20	272.9
15:30	272.7
15:40	272.6
15:50	272.6
16:00	272.5
16:10	272.4

h m	o
16:20	272.4
16:30	272.4
16:40	272.3
16:50	272.3
17:00	272.3
17:10	272.3
17:20	272.3
17:30	272.3
17:40	272.3
17:50	272.3
18:00	272.4
18:10	272.4
18:20	272.4
18:30	272.5
18:40	272.5
18:50	272.5
19:00	272.6
19:10	272.6
19:20	272.7
19:30	272.8
19:40	272.8
19:50	272.9

Data extracted from U.S. Naval Observatory, Sun or Moon Altitude/Azimuth Table for One Day

The above table shows the Maya Phenomenon taking place in Singapore. Data is taken 7 days after the spring equinox. The first set of data highlighted in red indicates the time and the azimuth at T1 and the next set of data highlighted in red shows the time and azimuth of T2.

o , o ,
W 88 32, N20 40 Chichen Itza

Altitude and Azimuth of the Sun
Jun 20, 2006
Zone: 6h West of Greenwich

Azimuth
(E of N)

h m	o
04:20	59.0
04:30	60.1
04:40	61.2
04:50	62.2
05:00	63.2
05:10	64.1
05:20	65.0
05:30	65.9
05:40	66.7
05:50	67.5
06:00	68.2
06:10	69.0
06:20	69.7
06:30	70.4

h m	o	h m	o	h m	o
06:40	71.0	10:40	77.5	14:40	282.0
06:50	71.6	10:50	76.7	14:50	282.3
07:00	72.2	11:00	75.3	15:00	282.6
07:10	72.8	11:10	73.2	15:10	282.9
07:20	73.4	11:20	69.8	15:20	283.3
07:30	73.9	11:30	63.9	15:30	283.7
07:40	74.4	11:40	52.0	15:40	284.1
07:50	74.9	11:50	25.2	15:50	284.6
08:00	75.4	12:00	340.4	16:00	285.0
08:10	75.8	12:10	310.5	16:10	285.5
08:20	76.3	12:20	297.3	16:20	286.0
08:30	76.7	12:30	290.8	16:30	286.6
08:40	77.0	12:40	287.1	16:40	287.1
08:50	77.4	12:50	284.9	16:50	287.7
09:00	77.7	13:00	283.5	17:00	288.3
09:10	78.0	13:10	282.6	17:10	288.9
09:20	78.2	13:20	282.0	17:20	289.6
09:30	78.4	13:30	281.6	17:30	290.2
09:40	78.6	13:40	281.4	17:40	290.9
09:50	78.7	13:50	281.3	17:50	291.6
10:00	78.7	14:00	281.3	18:00	292.4
10:10	78.6	14:10	281.4	18:10	293.2
10:20	78.4	14:20	281.6	18:20	294.0
10:30	78.1	14:30	281.7	18:30	294.9

Data extracted from U.S. Naval Observatory, Sun or Moon Altitude/Azimuth Table for One Day

This table shows the Maya Phenomenon taking place in Chichen Itza. Data shown is the expected azimuth on the day before the summer solstice. Note that we are unable to pick the 27th March 2006 as before since this day lies outside the critical period for an observer at Chichen Itza. The first set of data highlighted in red indicates the time and the azimuth at T1 and the next set of data highlighted in red shows the time and azimuth of T2.

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