Quasi-Periodicity in Medieval and Islamic architecture and ornament

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Abstract

A recent article in Science by Lu and Steinhardt has caused a controversy over whether Medieval Islamic tilings are examples of aperiodic and quasiperiodic tilings. The tilings have five- or ten-fold symmetry, which cannot occur in traditional crystallography. Such tilings were only recently discovered in the West. The goal of this project is to read several articles and write a summary of the various claims regarding quasi-periodicity in Medieval Islamic tilings. We will focus on three properties: the method of construction, its relation to Penrose tilings and whether the tiling is quasi-periodic. Such tilings appear on buildings or on ornaments as decoration. Our main examples will be taken from the Gunbad-i Kabud tower in Maragha, Iran (1197 C.E.) and the Darb-i-Imam shrine in Isfahan, Iran (1453 C.E.).

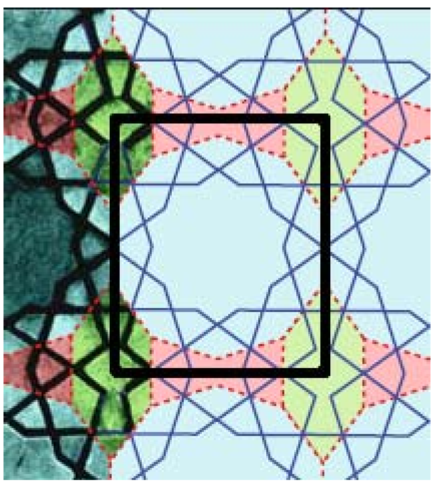


Figure 1
An Islamic tiling with its unit cell highlighted modified from (Lu & Steinhardt, 2007)

Introduction

Most Medieval Islamic tilings are periodic. A periodic tiling is one which possesses translational symmetry. It can be proven that within a periodic tiling, only two-fold, three-fold, four-fold and six-fold rotational symmetries are allowed. Five-fold and ten-fold rotational symmetries are forbidden in a periodic tiling. However, most Medieval Islamic tilings contain pentagons and decagons inside a periodic tiling. Pentagons and decagons have five and ten fold rotational symmetries respectively.

Actually, the use of pentagons and decagons within a periodic tiling can be explained through the use of unit cells. A unit cell tiles a plain with two different translations. A rectangular unit cell would have translations in the horizontal and vertical plain. Medieval Islamic tilings decorate these unit cells with pentagons and decagons, thus explaining how five and ten-fold rotational symmetries are present in a periodic tiling.

Discussion

Method of Construction

Early research into Islamic tilings proposed that the tilings were constructed using the direct strapwork method. In the direct strapwork method, an artist or mathematician constructs the entire tiling using a straightedge and compass. It is a tedious and long process that is likely to induce errors in the tiling, for example an accumulation of angular distortions when manually drafting the tiling.

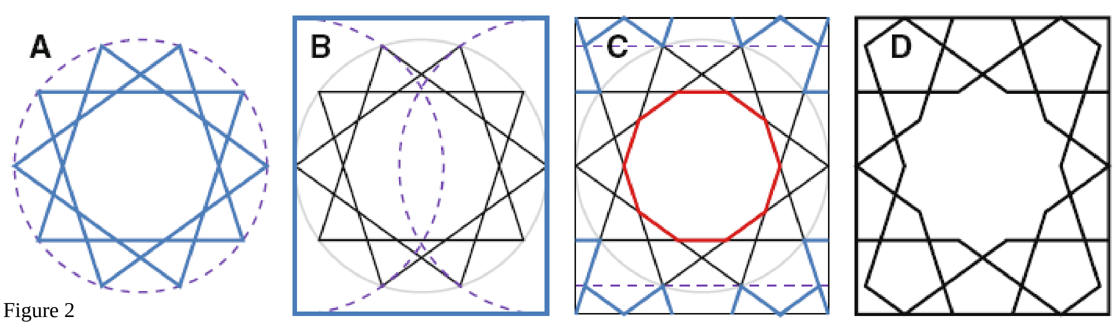


Figure 2
The steps in constructing a 10/3 star through direct strapwork, taken from (Lu & Steinhardt, 2007)

(Lu & Steinhardt, 2007) found some inconsistencies with the proposed direct strapwork method. Firstly, the direct strapwork method requires the construction of the 10/3 star to establish decagonal angles needed for direct drafting with straightedge and compass. However, there are Medieval and Islamic tiling which lack the star, such as the Mama Hatun Mausoleum in Tercan, Turkey. These tilings could not have been constructed using the direct strapwork method. Also, Lu and Steinhardt pointed out that errors which would result from the direct-strapwork method were absent from the tilings. Instead, the errors which were spotted were probably of the kind which was induced when repairing damaged sections of the tiling. Thus, they proposed that the tilings were constructed through the use of basic tiles which they termed girih tiles.

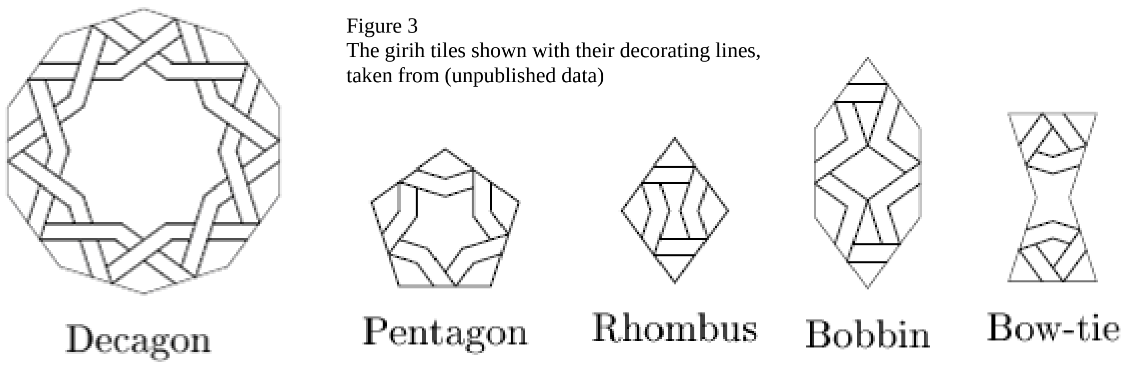


Figure 3
The girih tiles shown with their decorating lines, taken from (unpublished data)

The girih tiles consist of 5 tiles, a pentagon, a bow-tie, a rhombus, a decagon and a bobbin. The pentagon has 5-fold symmetry, the decagon 10-fold symmetry while the bow-tie, rhombus and bobbin have two-fold symmetry. Each of their edges has the same length. The girih tiles also have decorating lines printed on them. These decorating lines intersect the mid-point of every edge at 72^o and 108^o angles, which allow the lines to form a continuous network across an entire tiling. The girih tiles allow rapid, exact pattern generation. Using the girih tiles, one could construct an entire tiling with minimal errors.

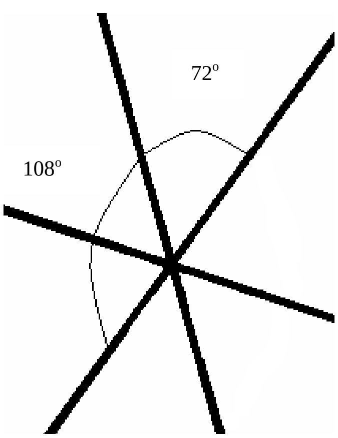


Figure 4
Illustration of how decorating lines intersect

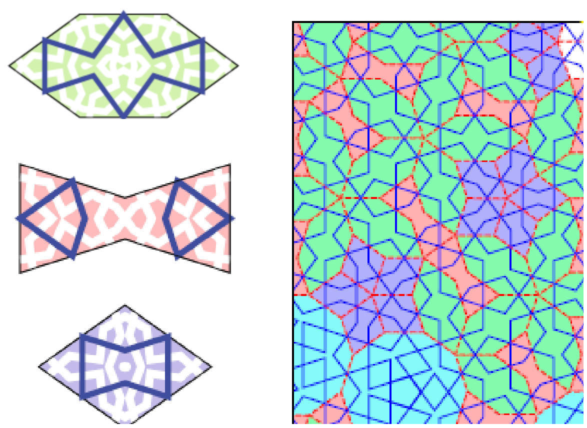


Figure 5
A section of the Gunbad-I Kabud shown together with the bobbin, bow-tie and rhombus. The tiles are shown with a second set of decorating lines as they are found on the Maragha, , taken from (Lu & Steinhardt, 2007)

Lu and Steinhardt referred to the tiling on the Gunbad-I Kabud in Maragha, Iran as compelling evidence for the use of girih tiles over the direct-strapwork method. They found that the bobbin, bow-tie and rhombus tiles on the Maragha had a smaller-line decoration with a two-fold symmetry, as shown by the white lines in Figure 5. This corresponds to the bobbin, bow-tie and rhombus' own rotational symmetries. Thus, they argue that the tiling is more likely to have been constructed using girih tiles.

However, Cromwell (unpublished data) has proposed that it is likely the Islamic and Medieval mathematicians used a simpler method to construct their fancy tiling. The technique is known as "polygons in contact" (PIC), first described in the West by Hankin. Polygons are drawn over an entire plain, forming an underlying polygonal network. Using the decagons in the underlying polygonal network, stars can be drawn within the decagon. These stars act as a base from which the rest of the pattern is extended upon. The lines comprising the start are extended till they meet. In tilings without the regular star, patterns can be generated by extending and joining lines arising from the midpoint of each edge. More complex patterns and a larger variety of designs could be constructed using the PIC method compared to using girih tiles.

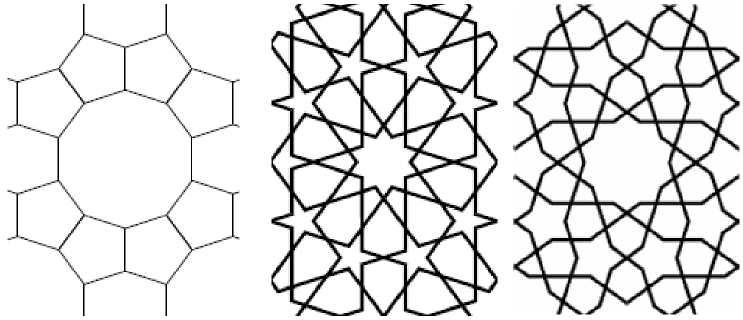


Figure 6
Underlying polygonal network (left) with two different Medieval/Islamic tilings derived from the network, taken from (unpublished data)

Relations to Penrose Tiling

A non-periodic tiling does not have translational symmetry. Some set of tiles can form either periodic or non-periodic tilings. An aperiodic set of tiles are a set of tiles which only tile in a non-periodic manner. The earliest form of non-periodic tilings that were discovered in the West was Wang Tiles, first found by Berger in 1966. (Grunbaum and Shephard, 1987) The most famous aperiodic tiles are the Penrose kite and dart. The kite and dart tile in a non-periodic manner using matching rules. By decorating the kite and dart using coloured lines, a set of matching rules can be described. The kite and dart tile in a non-periodic manner when continuity of the red and blue curves is maintained throughout the entire tessellation.

Lu and Steinhardt found that the Darb-I Imam pattern can be tiled using Penrose kites and darts. By placing kites and darts within the girih tiles, the Darb-I Imam pattern can be converted to a Penrose kite and dart tiling. There are several ways in which Penrose kites and darts can be tiled to form girih tiles. For the decagon, there are ten ways, while there are two for the bobbin and bow-tie. Lu and Steinhardt show how this can be for the decagon, bobbin and bow-tie. The same process can be applied to the pentagon and rhombus.

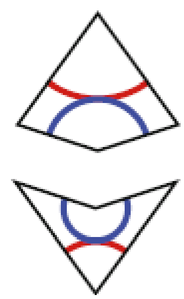
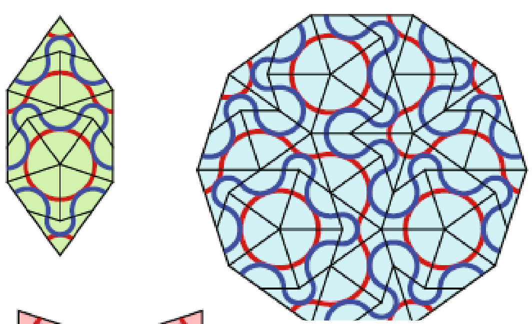


Figure 7
Penrose Kite (top) and Dart (bottom), taken from (Lu & Steinhardt, 2007)

Lu and Steinhardt replaced girih tiles with Penrose kites and darts for the Darb-I Imam pattern, minimizing Penrose tile edge mismatches as much as possible. They found 11 mismatches out of 3700 Penrose tiles. These mismatches are easily corrected by repositioning nearby tiles and the tile in question. They are thought to be an error which occurs when an artisan repairs or constructs a complex pattern.



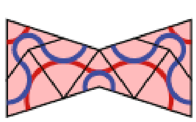


Figure 8
Mapping of Penrose kite and dart onto the bobbin, decagon and bow-tie, , taken from (Lu & Steinhardt, 2007)

Penrose proved that the ratio of kites to darts in a Penrose kites and darts tiling approaches the golden ratio, 1.618…. (Gardner, 1989) In a tiling which is expanded to infinity, the ratio is exactly the golden ratio, an irrational ratio. The ratio of hexagons to bowties in an arbitrarily large Darb-I Imam pattern approaches the golden ratio too. (Lu & Steinhardt, 2007) This indicates the the Darb-I Imam pattern is non-periodic, as the ratio of tiles in a periodic tiling must necessarily be rational.

The similarities between the kite and darts and the Darb-I Imam hints at it being a quasi-periodic tiling. The Penrose kite and dart tile in a quasi-periodic manner.

Quasi-periodicity in Medieval Islamic tiling

A quasi-periodic tiling is one which is non-periodic and has the following local isomorphism property: when the pattern is extended to infinite proportions, copies of any finite portion can be found evenly distributed throughout the tiling. However, it can be shown that a non-periodic tiling with self-similarity is quasi-periodic. (Grunbaum and Shephard, 1987)

Self-similarity transformation occurs when over-lapping patterns at two different length scales are present in a tiling. One such example is the tiling of sphinxes as shown in Figure 9. The tiling is the only way to tile the sphinxes in a non-periodic manner. The basic tile is a small sphinx, shaded in black. Four of them make up a larger sphinx. Four of the larger sphinxes make up an even larger sphinx. As sphinxes of different length scales tile throughout the pattern, the tiling has self-similarity.

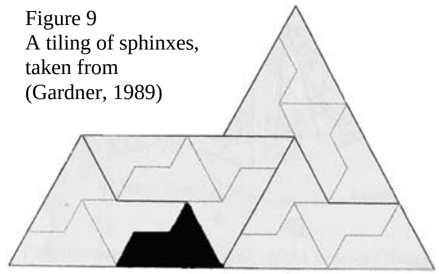


Figure 9
A tiling of sphinxes, taken from (Gardner, 1989)

Lu and Steinhardt have pointed out that the Darb-I Imam pattern is self-simlar. The Darb-I Imam pattern tiles at two different length scales. The first length scale is the tessellation of girih tiles while the second length scale is of a bow-tie surrounded by four decagons. However, the pattern of the larger length scale tiling, a bow-tie surrounded by four decagons, is not found in the smaller length scale tiling. Thus, strictly speaking, the Darb-I Imam does not have self-similarity and cannot be considered quasi-periodic. Also, the non-periodicity of the Darb-I Imam is also questionable. It is likely that the Darb-I Imam pattern is periodic, as can be seen from the larger length scale tiling. (Makovicky, 2007)

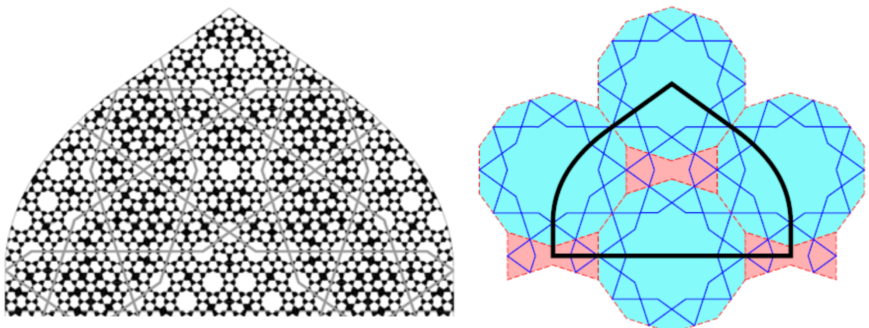


Figure 10
The Darb-I Imam tiling (left), taken from (unpublished data) and the larger length scale tiling of the Darb-I Imam (right), taken from (Lu & Steinhardt, 2007)

The Topkapi scroll also provides evidence that Medieval Islamic tiling do indeed allow a self-similar transformation. The Topkapi scroll is actually a pattern scroll kept in the Topkapi Palace Museum Library. The scroll contains several geometric drawings and is thought to have been around since the 15th century. Although its author is not known, it is a historical documentation of the methods used by the artisans and mathematicians of that time. In Figure 11, the tiling is outlined at two different length scales. The first length scale is outlined in blue while the second and larger length scale is bolded in black. It is clear that the tilings shown in the scroll are self-similar.

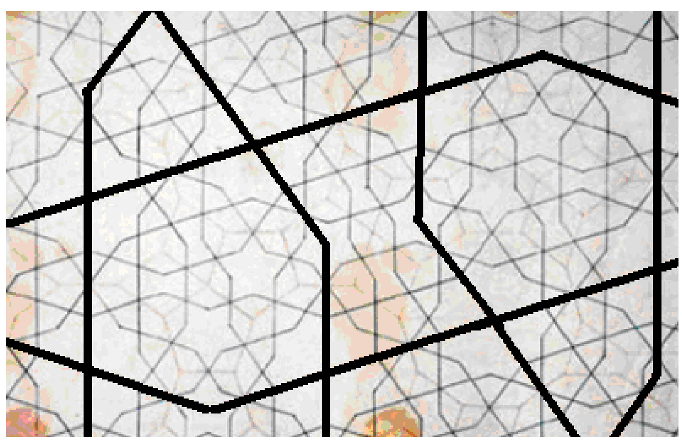


Figure 11
A segment of the Topkapi Scroll, with the larger length scale lines gone over in black, modified from (Lu & Steinhardt, 2007)

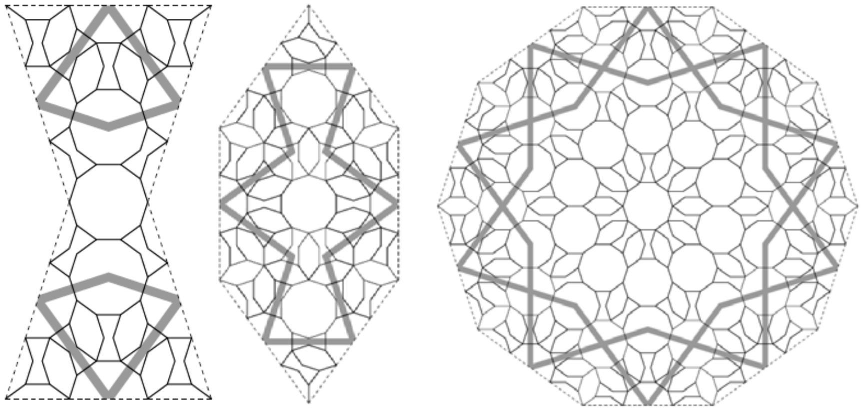


Figure 12
Large scale girih tiles formed from smaller girih tiles. The bow-tie (left), bobbin (middle) and decagon (right) are shown, taken from (unpublished data)

Shown in Figure 12 are the large scale versions of girih tiles formed from their smaller counterparts. The bow-tie and decagon were constructed by Lu and Steinhardt while Cromwell followed the same procedure in deriving the bobbin. As the scale difference between the two different length scales is quite big, it is difficult to spot whether a Medieval Islamic tiling is self-similar. However, many complications arise when attempting to extend a tiling. As the method of construction and the intention of the artist or mathematician are not known, there are various ways in which the tiling can be extended. This creates problems in interpreting whether a tiling is quasi-periodic.

Using Medieval Islamic tilings methods, non-periodicity and self-similarity in a tiling can be satisfied. Thus one can possibly construct a quasi-periodic pattern using the girih tiles. Cromwell demonstrates that this is possible, as can be seen in Figure 13. He starts out with a decagon and expands upon the tiling till it forms a larger scale decagon. Tiles are then placed till an even larger scale decagon is formed from the previous tiling. This process is repeated to get a quasi-periodic tiling with decagons present at different length scales.

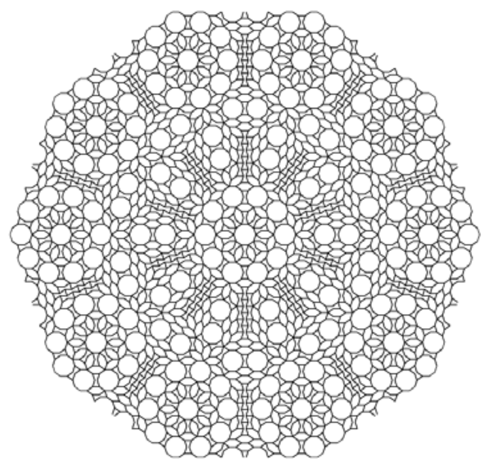


Figure 13
A quasi-periodic tiling formed from girih tiles, taken from (unpublished data)

Conclusion

Three different methods have been proposed as to how the tilings were constructed. They are the direct-strapwork method, use of girih tiles and the polygons in contact method. Since a finite tiling can often be extended to infinity in many ways (Grunbaum and Shephard, 1987), there can be multiple explanations for how a tiling was constructed. This can lead to controversy as to whether a finite tiling is intended as part of a larger quasi-periodic tiling or not, and whether the mathematicians at that time could have constructed a quasi-periodic tiling. Further research has to be done to ascertain the claims regarding the method of construction and the quasi-periodicity of the tilings. Also, there are still many more examples of Medieval Islamic which have not been looked into with such detail. These are possible avenues of further research.

References

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