Polyhedra

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ABSTRACT

This report focuses on inclusions, compounds and stellations of regular polyhedra and deltahedra and zonohedra. Firstly, some terms used to describe a polyhedron are introduced. Then, we go on to talk about Platonic polyhedra and the duality of polyhedra. Using this duality property of polyhedra, we find some inclusions of polyhedra and we go on to find some other ways of including one polyhedron in another. Next, we also found three ways of generating compounds of polyhedra, namely using duality, inclusions and symmetries. Under the section of compounds of polyhedra, the transitivity property of the compounds is discussed. We carry on to discuss the stellations of polyhedra. There are two-types of stellations, namely the edge and face stellations and from there, we use the stellation of dodecahedron to illustrates the two type of stellations. Then, there is the deltahedra. Pyramids are introduced first since they are closely related to the deltahedra. Then, we start to prove that there are only 8 deltahedra possible. A geometric process is also used to prove the non-existence of some deltahedra. Finally, we discussed rhombic polyhedra which form a subset of the zonohedra. We talked about the 4 rhombohedra; the 2 rhombic dodecahedra and the 20 and 30 faced rhombic polyhedra.

PLATONIC POLYHEDRA

There are 5 Platonic polyhedra, namely the tetrahedron, the cube (hexahedron), the octahedron, the dodecahedron and the icosahedron. A Platonic polyhedron has regular faces where the faces are made up of one type of polygon and all the vertices have the same valency.

Duality of Polyhedra

The dual polyhedron of a polyhedron can be constructed by using the midpoints of the faces of the polyhedron as the vertices of its dual polyhedron. Hence, the number of vertices of a polyhedron is the same as the number of faces of its dual polyhedron. It is found that a tetrahedron is dual of itself, a cube is dual of an octahedron and a dodecahedron is dual of an icosahedron. For any polyhedron, there exist a dual polyhedron.

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INCLUSIONS OF POLYHEDRA

A polyhedron is included in another polyhedron if their centers coincide and the vertices of the included polyhedron lie on the outer polyhedron. There are many possible ways of including one polyhedron in another. We can make use of the duality property of the polyhedron. Other inclusions include the vertices on vertices inclusions, the vertices on edges inclusions, the vertices on faces inclusions, the edges on faces inclusions and the faces on faces inclusions. Mainly, for a polyhedron can be included in another polyhedron by the vertices on vertices inclusion, the number of vertices of the included polyhedron has to be smaller than the number of vertices of the outer polyhedron. For the vertices on edges or faces inclusions, the number of vertices of the included polyhedron must divide the number of edges or faces of the outer polyhedron. However, an integer answer from the division does not necessary imply that the inclusion is possible.

Nested Platonic Solid

A nested Platonic solid is a framework of all the Platonic polyhedra, with a series of inclusion of one polyhedron in another. It is found that we can construct a nested Platonic solid starting with any polyhedron. However, they are not preferred since they generate large models which are usually hard to build. The most compact and most easily built nesting is found to be the icosahedron included in an octahedron, which are subsequently being included in a tetrahedron, a cube and the dodecahedron.

COMPOUNDS OF POLYHEDRA

A compound is a collection of two or more interpenetrating polyhedra, whose centers coincide. We are more interested in dual compounds and compounds made up of only one type of polyhedron. Hence, we shall look into how we can construct those compounds. We can form compound by observing the duality of a polyhedron, by observing the possible inclusions of polyhedron and by observing the symmetries of the polyhedron.

Forming Compound By Observing Duality

We can obtain dual compounds by combining dual pairs such that their centers coincide. Here, we get the two tetrahedra compound (stella octangula), the cube and octahedra compound and the dodecahedron and icosahedron compound.

Forming Compound By Observing Inclusions

Another method of generating compound is to observe the possible inclusions. We need to find the number of ways in which a polyhedron can be included in another polyhedron. If one polyhedron can be included in another polyhedron in 3 different ways, where when the three polyhedra are included together in the outer polyhedron, they form a compound of three polyhedra, with each being orientated in different direction. From this method, we can generate the two tetrahedra compound, the three and five tetrahedra compounds and the two dodecahedra compound.

Of course there are other compounds that can be generated by this method, however, in this report, I only mention some compound which are more common.

Forming Compound By Observing Symmetry

There is still another method which we can use to generate compound. For this method, we need to consider the rotational symmetries of the included polyhedron (kernel), the circumscribed polyhedron (shell) and the compound of the kernel and the shell (amalgam). Here the kernel refers to the polyhedron that is included in another polyhedron which is the shell. The symmetry of the amalgam is usually smaller than the symmetry of the kernel or the shell alone. The symmetries have been lost. The aim of this method is to reinstate the lost symmetries in the amalgam. The formulae (Cromwell (1996)) are used:

$$\text{Number of components in the compound of shell polyhedra} = \frac{\text{number of symmetries of the kernel}}{\text{number of symmetries of the amalgam}}$$

$$\text{Number of components in the compound of kernel polyhedra} = \frac{\text{number of symmetries of the shell}}{\text{number of symmetries of the amalgam}}$$

It is noted that by forming compound of the shell polyhedra, we are reinstating the symmetry of the kernel and vice versa. Using this method, we can generate the three octahedra compound, the three cubes compound, the two dodecahedra compound and the five cubes compound. Other compounds that can be generated are not mentioned here.

Transitivity

A polyhedron is said to be face-transitive if for any pair of faces, there is a symmetry of the polyhedron which carries the first face onto the other; vertex-transitive if any vertex can be carried onto any other by a symmetry operation; edge-transitive if any edge can be carried to any other by a symmetry operation; and flag-transitive if any one flag triples can be carried onto any other flag by a symmetry operation. Note that a flag triple is a set of face-edge-vertex where the edge is a side of a face and the vertex is an end of the chosen edge. A polyhedron that is face-transitive, edge transitive and vertex transitive is said to be totally transitive. Note that we are more interested in compounds that are totally transitive.

Five tetrahedra compound There are two five tetrahedra compound, namely the right five tetrahedra compound and the left five tetrahedra compound. They are actually mirror images of each other and when they are combined together vertices to vertices, they form the ten tetrahedra.

STELLATIONS OF POLYHEDRA

Stellation of polyhedra is a way of creating new polyhedron from existing ones. There is the edge stellation and the face stellation. The process of stellation is to extend the edges of faces of the polyhedron infinitely such that they intersect to form finite regions. The resulting polyhedron of edge stellation is called an echinus while that of a face stellation is called an ostera. However, stellations of a polyhedron does not always produce new polyhedron. But it does affect the dodecahedron. Applying the stellation process on a dodecahedron, we obtain the small stellated dodecahedron, the great dodecahedron and the great stellated dodecahedron.

DELTAHEDRA

Deltahedra is named after the Greek capital letter delta, which is a triangle. We denote deltahedra of n faces by D_n. A deltahedron can also be denoted by (V_3, V_4, V_5) where V_i is the number of vertices having valency i, where i = 3, 4, 5. From some calculations, we note that there are only 8 convex deltahedra, namely the D_4 tetrahedra, D_6 - triangular dipyramid, D_8 – octahedra, D_{10} - pentagonal dipyramid, D_{12} - snub disphenoid, D_{14} - tri-augmented triangular prism, D_{16} - gyro elongated square dipyramid and the D_{20} – icosahedra. Note that there are no D_{18}. A geometrical process can be used to prove this result. The process goes like this: starting from the tetrahedra, detach a 'line' consisting of 2 edges which joins 2 vertices of the minimum degree. Then add in two more faces, one to each detached edge. We get the next deltahedron which 2 additional faces. We can carry on the process and generate a series of deltahedra. However, we can only detach 3 edges that joins the last two vertices of the minimum degree of the D_{16} in order to form a convex deltahedron, in doing so, we get the icosahedron instead of the D_{18}.

ZONOHEDRA

A zonohedron is a convex polyhedron bounded by zonogons. The rhombic polyhedra form a special set of the zonohedra where all the faces are regular rhombi. We can generate different rhombic polyhedra by changing the combination of obtuse and acute angles of the rhombi at a vertex. The set of rhombic polyhedra are 4 rhombohedra (prolate and oblate), 2 rhombic dodecahedra, a 20 faced rhombic polyhedron and the triacontahedron. We can generate the rhombic dodecahedron and the 20 faced rhombic polyhedron from the rhombic triacontahedron by collapsing 'belts' of rhombi.

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