Polyhedra

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Abstract

Introduction

The report focuses on three main families of polyhedra-the Platonic solids, the Archimedean solids and the Kepler-Poinsot solids. Firstly we will introduce the basic terms used for polygons and polyhedra. Then there will be a brief discussion on polygons since they form the basis of polyhedra. Next, we will define regular polyhedra because two of the families that are focused in the report, the Platonic solids and the Kepler-Poinsot solids, are regular polyhedra. Subsequently, we will define polyhedra. Overall, when discussing each of the three families of polyhedra in the report, we will find out the following; the processes by which the polyhedra in the family are formed, the reasons for their respective names and the duals of the polyhedra in the family.

Polyhedra

There are five Platonic solids, namely the tetrahedron, cube, octahedron, dodecahedron and icosahedron. The name 'Platonic' solid was derived from the name of the great Greek philosopher Plato, who wrote about them in about 400B.C.

The convex deltahedra are the polyhedra that are made of equilateral triangles. They are called *deltahedra* because an equilateral triangle looks like the Greek capital letter *delta*, Δ.

Both the Platonic solids and the deltahedra can be considered as regular polyhedra if we define regular polyhedra using the following two conditions.
(i) The polygonal faces of the polyhedra must be congruent.

(ii) The polygonal faces must be regular polygons.

However, in the report we will modify the definition of regular polyhedra to exclude the family of deltahedra but include the five Platonic solids. This modified definition is taken from the book “Polyhedra” by Cromwell. Following this we will compare the number of Platonic solids and the number of deltahedra. In the report we will proof that there are only five possibilities Platonic solids and all the five Platonic solids can be constructed. For the family of deltahedra the proof will give nine possibilities of deltahedra but only eight of them can be constructed. The convex deltahedron with eighteen faces cannot be constructed.

The five Platonic solids are the convex regular polyhedra. There are also concave regular polyhedra that satisfy the modified definition of the regular polyhedra given by Cromwell. The concave regular polyhedra are the Kepler-Poinsot solids. There are only four Kepler-Poinsot solids. Thus, there are exactly nine regular polyhedra.

In his book “Polyhedra” Cromwell also gave a definition for polyhedra that will be discussed in the report.

Platonic solids

The names of the Platonic solids tell us the number of faces that each Platonic solid has. In Greek tetra, hexa, octa, dodeca and icosa refer to the numbers four, six, eight, 12 and 20 respectively. The term ‘hedron’ at the end of the name means “base” in Greek Thus, tetrahedron is a Platonic solid with four faces and each face can be used as a base to set the tetrahedron on a table.

What are duals and how are they formed? There are two methods to form duals. The first method involves replacing each edge of a polyhedron by a new edge such that the two edges are perpendicular to each other. The second method involves replacing each face of a polyhedron by a vertex and joining the new vertices. The new polyhedron and the original polyhedron are a dual pair. The number of vertices and faces are interchanged for a dual. For Platonic solids, the cube and the octahedron are a dual pair while the dodecahedron and the icosahedron are a dual pair.

Lastly, for Platonic solids we will find out the number of possible ways of proper colouring each Platonic solid. Proper colouring ensures that the faces that share a common edge have different colours.

Archimedean solids

There are 13 Archimedean solids. They are defined as semiregular polyhedra. A semiregular polyhedron has a wide variety of regular polygons as faces and all the vertices are congruent, meaning each vertex has the same arrangement of faces.

11 out of the 13 Archimedean solids are formed by a process called truncation. They are the truncated tetrahedron, truncated cube, truncated octahedron, truncated icosahedron, truncated dodecahedron, cuboctahedron and the icosidodecahedron, the great rhombicuboctahedron, the rhombicuboctahedron, the great rhombicosidodecahedron and the rhombicosidodecahedron. The two remaining Archimedeans, the snub cube and snub dodecahedron, are formed by snubbing the cube and the dodecahedron, respectively.

Truncation is the process of removing all the corners of a polyhedron in a symmetrical fashion. The measurements of truncation play a role in the names of the Archimedean solids obtained from truncation. The process of snubbing will also be discussed. The snub cube = snub octahedron and the snub dodecahedron = snub icosahedron. Snubbing of a polyhedron can give two different polyhedra, the left snub and the right snub, that are mirror images of each other. The two polyhedra are said to be enantiomorpic. The left snub cube = right snub octahedron and vice versa. Similarly, the left snub dodecahedron = right snub icosahedron and vice versa.

Isomers are polyhedra with different arrangement of the same faces. Miller's solid is an isomer of the rhombicuboctahedron. Miller's solid can be considered as an Archimedean solid because it has congruent vertices. But unlike Archimedean solids Miller's solid is not vertex-transitive, meaning any vertex can be carried to any other by a symmetry operation. Lastly, in the report we will focus on the dual of the cuboctahedron called rhombic dodecahedron and the dual of the icosidodecahedron called the rhombic triacontahedron.

Kepler-Poinsot solids

The Kepler-Poinsot solids are the small stellated dodecahedron, the great stellated dodecahedron, the great dodecahedron and the great icosahedron. Kepler described the first two and Poinsot described the last two.

The Kepler-Poinsot solids are obtained by stellation. There are two kinds of stellation, edge stellation and face stellation. Edge stellation is the extending of the edges of the polyhedra until they re-intersect to form a new polyhedron and face stellation is the extending of the faces. The first three Kepler-Poinsot solids mentioned above are obtained by stellating the dodecahedron and the last one is obtained by stellating the icosahedron. There can be alternative names for each of the four solids depending on the stellations.

The small stellated dodecahedron and the great dodecahedron are a dual pair while the great stellated dodecahedron and the great icosahedron are a dual pair.

References:

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