The Chinese Calendar of

The Later Han Period

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We were very excited when we saw this project because we thought it would be very interesting and we would like to learn some basic knowledge in the Chinese calendar, which is an important part of the Chinese culture and Chinese history. However, by the time we decided to take this project, the deadline for taking it was over, so we went to Dr. Helmer's office and required joining the project. He accepted our request and helped us to contact Deans' office so that we could take this module. We are greatly appreciated all the things he did for us. We hope this report will not disappoint him; contrarily, we hope it can help him in some ways.

Abstract

The objectives of this report are to give some reasonable explanations to the Chinese calendar of the Later Han period, *Shi Fen Li,* and introduce some useful and effective formulas for the calculations of the calendar. The first part of this report, the Chinese calendar, will introduce some basic calendrical concepts for the Chinese calendar. In the second part, we give the details and some explanations for the *Si Fen Li.* Following that, some important calculation methods stated in *Hou Han Shu* (a classical Chinese text) will be give with explanations to each of those methods. In part 2, we explain the rules in the *hou han shu*. In the last part, we reformulate the rules in a form that suitable for computer implementation. Discussions on the pattern and distribution of leap months, leap years and consecutive big months will be given in the last section. In Appendix C, we also include source code for SiFenLi (executable file for SiFenLi computation).

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1. The Chinese Calendar

1.1 Brief Introduction to the History of the Chinese Calendar

The Chinese calendar has been in used for more than four millenniums. In ancient China, calendar was a sacred document sponsored and announced by the ruler. Every imperial court had a Bureau of Astronomy to make astronomical observations, calculated astronomical events such as eclipses, prepared astrological predictions, and maintained the calendar. This was because they believed that the accuracy of astrological predictions was related to the fate of their country. After all, an accurate calendar not only served practical needs, but also confirmed the consonance between Heaven and the imperial court.

From the earliest records, the beginning of the year was the new moon before the winter solstice (*dong zhi*). However, the choice of month for beginning the civil year varied with time. In 104B.C., a new Chinese calendar, *Tai Chu Li* ( ), was formed. It established the practice of requiring the winter solstice to occur in 11th month. Therefore the beginning of a year is the second or third month after winter solstice (if there was a leap month after 11th month, then it was the third month). This calendarical reformation also introduced a new rule for allocating leap month that was "a leap month without a solar term (we will talk about this in the next section)" (the Chinese calendar before *Tai Chu Li* allocated the leap month at the end of the year.

Years were counted from a succession of eras established by reigning emperors. Although the accession of an emperor would mark a new era, an emperor might also declare a new era at various times within his reign. The introduction of a new era was an attempt to reestablish a broken connection between Heaven and Earth, as personified by the emperor. The break might be revealed by the death of an emperor, the occurrence of a natural disaster, or the failure of astronomers to predict a celestial event such as an eclipse. In the latter case, a new era might mark the introduction of new astronomical or calendrical models.

Western (pre-Copernican) astronomical theories were introduced to China by Jesuit missionaries in the seventeenth century. Gradually, more modern Western concepts became known in China and were applied in the Chinese calendar. The modern Chinese calendar was formed in 1645, it is much more accurate than all others. In 1911, the Chinese Government abolished the traditional practice of counting years from the accession of an emperor and started using Gregorian calendar as Chinese general calendar.

1.2 Brief Introduction to the Chinese Calendar

The Chinese calendar is a combination of two calendars, a solar calendar and a lunisolar calendar (*yin yang li*) [2], and it is based on calculations of the positions of the sun and moon. Each month begins with the day of astronomical new moon, with an intercalary month being added every two or three years. Basically, the calculations of the Chinese calendar are based on the observations of the positions of the sun and moon, therefore, the accuracy of the calendar depends on the accuracy of the astronomical theories and calculations.

There are two kinds of year in the Chinese calendar, the *sui* ( ) and the *nian* ( ). A *sui* is the solstice year from one winter solstice (*dong zhi*) to the next; the length of a *sui* is close to the length of a tropical year. A *nian* is the Chinese year from one Chinese New Year to the next and it can contain 12 or 13 lunar months.

A month in the Chinese calendar is the period from one new moon (*shuo*) to the next. Every new moon appears in the first day of a month, as compared different with the present Gregorian calendar. Since the extreme maximum and minimum value of the length of a month is about 29.84 days and 29.27 days, a month in the Chinese calendar has 29 or 30 days.

The solar terms (*Jie qi*) are very important elements in the Chinese calendar. They are a generalization of the solstices and equinoxes, for example, the seasonal markers that cut the ecliptic into 4 sections of 90^{o} (figure 1) are some of the *jei qi*. There are 24 *jie qi* in a *sui*, and the 24 *jie qi* include 12 odd *jie qi* and 12 even *jie qi* which are named *zhong qi* ( ) (refer to Table 3 of Appendix B for the names of the 24 *jie qi*). Hence *jie qi* in the Chinese calendar has two meanings, one is the 24 *jie qi* and another is the 12 odd *jie qi* of the 24 *jie qi*. The 24 *jie qi* cut the orbit of the earth into 24 sections of 15^{o}. 24 *Jie qi* were especially useful for agriculture in the ancient China because they told the dates for farming.

One of the important concepts in the Chinese calendar is that a Chinese astronomical phenomenon such as new moon, full moon or *jie qi* can take the whole day to occur. For example, if an astronomical phenomenon takes place at 11:55pm, then the entire day is considered as an astronomical event, even though it started at 11:55pm.

In Chinese calendar, for every two or three years, there will be one leap month. This is because the length of one Chinese year with 12 months can only be 353, 354 or 355, therefore, after three years there will have about 33 days behind the solar calendar, a leap month has to be added so as to make it close to the solar calendar. There are two ways to decide where to insert the leap month. The earlier calendar before *Tai Chu Li* always inserts the leap month after a certain month, for example, in the calendar during the *Qin* dynasty and the early Han, the leap month was inserted after the 9th month (which was the end of the year at that time). After that, the *Tai Chu Li* was announced, and it introduced "*no zhong qi rule*" to determine leap month (this method to determine leap month is used till today). The "*no zhong qi rule*" for that period (and also for the *Si Fen Li* which is our main discussion of this report) has the definition of "A month without any *zhong qi* is a leap month", where the modern definition is "In a leap *sui*, the first month that doesn't contain a *zhong qi* is the leap month". There are different definitions to the rule is because the earlier calendars, which used *ping qi* and *ping shuo* (which we will discuss in the next paragraph), a *sui* could not have two adjacent months without *zhong qi*, but this could happen in the later Chinese calendars, including the modern Chinese calendar.

Both 24 *jie qi* and month are the basic units of Chinese calendar. The average value of the length of these two terms is 15.218 days and 29.53 days. For the earlier Chinese calendar, the scholars used the average value for the length of *jie qi* and month, this method is call *ping qi* (the interval between any two adjacent *jie qi* is a constant) and *ping shuo* (the interval between any two adjacent new moon is a constant).

Because of the inconsistent speed of motion of the sun and moon relative to the earth, the actual lengths of the 24 *jie qi* and month should be changing throughout the year. The calendrical scholars of the later Chinese calendar discovered this fact and then started to use the true values determined by observations as the lengths of *jie qi* and month. This method is called *ding qi* (the interval between any two adjacent *jie qi* depends on the speed of the earth relative to the sun) and *ding shuo* (the interval between any two adjacent new moon depends on the speed of the moon relative to the earth).

The Chinese calendars before early *Tang* dynasty (before middle of sixth century A.D.) used *ping qi* and *ping shuo* methods; in the period between *Tang* dynasty and early *Qing* dynasty (before middle of sixteenth century A.D.) the Chinese calendars used the *ping qi* and *ding shuo* methods; the modern Chinese calendar, which started from *Qing* dynasty, used the *ding qi* and *ding shuo* methods.

In order to clarify on the "*ping*" and "*ding*" methods, please refer to Figure 1 on next page, which show the difference between *ping qi* and *ding qi* methods.

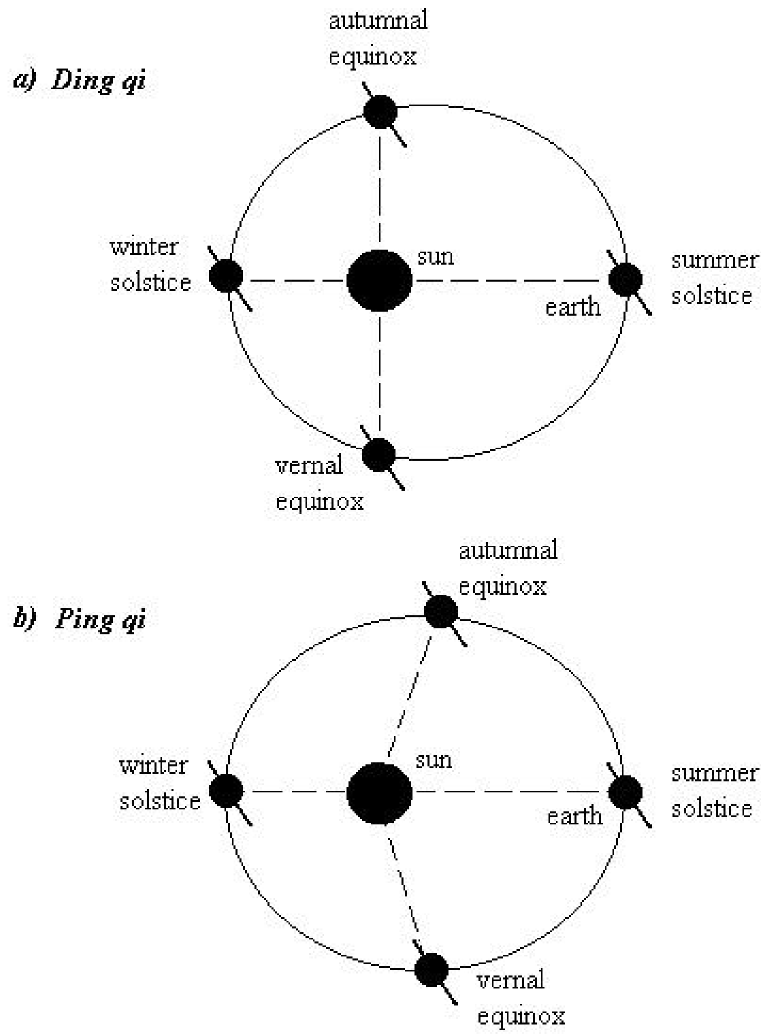


Figure 1: Positions of the earth at solstices and equinoxes based on *ding qi* and *ping qi* rules.

Figure 1 shows the seasonal markers cutting the ecliptic into 4 sections. For figure 1 a), the line between the solstices and the line between the equinoxes are perpendicular to each other. The modern Chinese calendar uses the four points to decide the four seasonal markers: winter solstice, vernal equinox, summer solstice and autumnal equinox. As we know, Kepler's second law states that "the area of the ecliptic swept by the line between the sun and the Earth is always the same for any same period of time interval". Therefore, we can easily see that the interval between the summer solstice and its following *jie qi* is longer than the interval between the winter solstice and its following *jie qi*. For figure 1 b), the period for the four seasonal markers are the same, but the angle between any two adjacent seasonal markers with respect to the sun is not the same.

There is no specific beginning for counting years in ancient Chinese calendar. In historical records, dates were specified by number of days and years in sexagenary cycles, which are called *gan zhi ji fa* ( ).

The sixty-year cycle or sixty-day cycle consists of a set of names that are created by pairing a symbol from a list of ten *tian gan* ( ) with a symbol from a list of twelve *di zhi* ( ), following the order specified in Table 1 (Appendix A). The ten *tian gan* are Chinese characters with no English translation and so are the twelve *di zhi*. However, the *di zhi* are matched by twelve animals. After six repetitions of the set of *tian gan* and five repetitions of the *di zhi*, a cycle (the table of *jia zhi* cycle, table 2 in Appendix A) is completed and a new cycle begins. This cycle is used to count years, days, and fractions of a day (time).

For earlier Chinese calendars, the scholars used to look for some ideal starting points for the calculations of the calendar. First they made astronomical observation to gather required information, such as the length of a year, day or the time of occurrence for *jie qi* or lunar eclipse and subsequently they used these information to determine repeating patterns for the position of the heavenly bodies. *Li yuan* ( ) is one of the ideal points in

Si Fen Li, it occurs when new moon and winter solstice take place at the midnight of the first day in the sexagenary day cycle (we called that day as *jia zi* (   ) day). A *li yuan* is called *shang yuan* (   ) provided that the sun, moon, and five planets (Jupiter, Venus, Saturn, Mars and Mercury) are lined up in conjunction at the same moment.

2. The Si Fen Li ( ) of Later Han Dynasty

The Chinese calendar used before the Later-Han *Si Fen Li* is *Tai Chu Li* ( ), which had been in used from 104B.C. Owing to the failure of predicting the motions of the moon and the sun in the later period (the development of basic constants for *Tai Chu Li* is based on the belief of the length of a month is $29\frac{43}{81}$ days which is larger than the actual value, therefore it caused a serious accumulated error over time), calendrical scholars decided to form a new calendar. Finally, the *Si Fen Li* was successfully created and announced in 85A.D. ( ) and it was used until 263A.D. ( ), lasting a total of 179 years. The *Si Fen Li* developed it's basic constants from the length of one year, which is equal to $365^1/_4$ days, so it is more accurate comparing to the *Tai Chu Li*.

The *Si Fen Li* has the basic arguments shown below:

1. Length of a year = $365\ ^1/_4$ days (in average).
2. 7 leap months in 19 years. (19 years with total 235 months)
3. The *Li Yuan* ( ) is in 161B.C.

   (Through calculations on the motion of planets, the scholars of *Si Fen Li* found that the *shang yuan ji nian* appeared at 9120 years before 161B.C., so the *ji nian* for 161B.C. is 9121.)
4. *Si Fen Li* used *ping qi* and *ping shuo* to determine the length of *jie qi* and month.
5. The year name for the year of *li yuan* is *geng chen* (17).

In the *Si Fen Li*, there are four important cycles:

1. One *yuan* is 4560 years.
2. One *ji* is 1520 years. (One third of one *yuan*)
3. One *bu* is 76 years. (One twentieth of one *ji*)
4. One *zhang* is 19 years. (One forth of one *bu*)

For the first day of every *zhang* (19 years), the winter solstice and new moon of the 11th month will fall together, and *zhang* is the leap period of 19 years (19 years with 7 leap months); For the first day of every *bu* (4 *zhang*), the winter solstice and new moon of the 11th month will fall together at midnight (76 years is the concordance of the lunation and the solar year). (Refer to part 4 for further details of these two statements)

For the first day of every *ji* (20 *bu*), the winter solstice and new moon of the 11th month fall together at midnight and the day name is *jia zi* (*ji* is the concordance of the sexagesimal cycle of day name, the lunation, the solar year, and the eclipse period); For every *yuan* (3 *ji*) the year name will repeat itself (*yuan* is the concordance period for the sexagesimal cycle of day name, the sexagesimal cycle of year name, the lunation, the solar year and the eclipse period). (Refer to the next section for further details.)

In order to make the calculations simpler, scholars of the early Chinese calendar had constructed a table of *bu-shou* to determine the name for the first day and first year of *bu*. By comparing names of different days or years, they can gather the information needed. Table A is the table of *bu-shuo* for the *Si Fen Li*.

Table of *bu-shou* ( )				
		Year Name		
	Day Name	*Heaven-Ji* ( )	*Earth-Ji*( )	*Man-Ji*( )
1	Jia zi (1)	Geng chen (17)	Geng zi (37)	Geng shen (57)
2	Gui mao (40)	Bing shen (33)	Bing chen (53)	Bing zi (13)
3	Ren wu (19)	Ren zi (49)	Ren shen (9)	Ren chen (29)
4	Xin you (58)	Wu chen (5)	Wu zi (25)	Wu shen (45)
5	Geng zi (37)	Jia shen (21)	Jia chen (41)	Jia zi (1)
6	Ji mao (16)	Geng zi (37)	Geng shen (57)	Geng chen (17)
7	Wu wu (55)	Bing chen (53)	Bing zi (13)	Bing shen (33)
8	Ding you (34)	Ren shen (9)	Ren chen (29)	Ren zi (49)
9	Bing zi (13)	Wu zi (25)	Wu shen (45)	Wu chen (5)
10	Yi mao (52)	Jia chen (41)	Jia zi (1)	Jia shen (21)
11	Jia wu (31)	Geng shen (57)	Geng chen (17)	Geng zi (37)
12	Gui you (10)	Bing zi (13)	Bing shen (33)	Bing chen (53)
13	Ren zi (49)	Ren chen (29)	Ren zi (49)	Ren shen (9)
14	Xin mao (28)	Wu shen (45)	Wu chen (5)	Wu zi (25)
15	Geng wu (7)	Jia zi (1)	Jia shen (21)	Jia chen (41)
16	Ji you (46)	Geng chen (17)	Geng zi (37)	Geng shen (57)
17	Wu zi (25)	Bing shen (33)	Bing chen (53)	Bing zi (13)
18	Ding mao (4)	Ren zi (49)	Ren shen (9)	Ren chen (29)
19	Bing wu (43)	Wu chen (5)	Wu zi (25)	Wu shen (45)
20	Yi you (22)	Jia shen (21)	Jia chen (41)	Jia zi (1)

Table A

The column 'Day Name" is used to determine the name for the first day of every *bu*, and column 'Year Name' is used to determine the name for the first year of every *bu. Heaven-ji, Earth-ji, Man-ji* are the names of the three *ji* in a *yuan*.

Characteristics of the Table of *bu-shou*: the difference between two adjacent rows in the column 'Day Name' and the column 'Year Name' are 39 and 16 relatively (in sexagenary cycle). Every 20 *bu*, the day name will be repeated and every 15 *bu*, the year name will be repeated. The following paragraphs are the explanations of these characteristics.

The first day of the first *bu* was named *jia zi* (1). There are 27759 days in a *bu*, and dates in Chinese calendar are specified by counts in sexagenary cycles. Since the remainder of 27759 divided by 60 is 39, the name for the last day of the first *bu* is *ren yin* (39) (refer to Table 2 of Appendix B). Hence the name for the first day of the second *bu* is *gui mao* (40) and the name for the first day of the third *bu* is *ren wu* (19) (40+39   19(mod 60)). We conclude that the position (number in Table 2) of the first-day name of a *bu* is the addition of the position of the first-day name of the previous *bu* and 39 (mod 60). If we divide 27759 by 60, we will obtain $^{27759}/_{60} = 462\,^{13}/_{20}$, hence for every 20 *bu* (20 *bu* is 1 *ji*), the name for the first day of the *bu* will be repeated.

The following 'Year Name' column of Table A shows the names for the first year of every *bu*. The first year of the first *bu* in a *yuan* was named *geng chen* (17). The positions of names for the first years of any two consecutive *bu* are 16 apart in table of *jia zi* cycle (table 2) because 1 bu have 76 years and the recording method is a sexagenary cycles, so 76   16 (mod 60). By dividing 76 by 60, we will obtain $1\,^{4}/_{15}$, hence for every 15 *bu*, the name for the first year of the *bu* will be repeated. For one *ji* (20 bu), the total number of years is 76 x 20 = 1520. Since 1520   20 (mod 60), the position of first-year name of the *bu* is the addition of the position of the first-year name of the previous *bu* and 20 (mod 60). After one *ji*, we may see this by comparing the different year names between two adjacent columns.

The order of the year names is from top to bottom and then left to right, which means that for the first 20 *bu*, we use *Heaven-ji* to determine the year name for the first year of the *bu*, then the following 20 *bu* use *Earth-ji*, and the last 20 *bu* use *Man-ji*.

3. The Calculation Methods of the Si Fen Li

The calculation methods of the *Si Fen Li* are stated in *Hou Han Shu.* Due to the lack of translation, in this part, we are only able to show some original paragraphs stated in *Hou Han Shu* and some formulas described in [1]. For each of the method, we will do some explanations on the purpose of the methods and use year 146A.D. as an example to illustrate it.

There are some constants stated in *hou han shu* for calculations.

1. *yuan*-rule = 4560.
2. *ji*-rule = 1520. (One third of *yuan*-rule)
3. *bu*-rule = 76. (One twentieth of *ji*-rule)
4. *bu*-month = 940. (940 is the number of months in 76 years)
5. *zhang*-rule = 19. (There are seven leap month in 19 years)
6. *zhang*-month = 235. (235 is the number of months in 19 years)
7. *bu*-day = 27759. (27759 is the number of days in 76 years)
8. *ri yu* = 168. (Each *jie qi* has a surplus of 7/32 days. 24 *jie qi* have a surplus of 168/32 days. Or, in 32 years, the surplus is 168 days.)
9. *zhong*-rule = 32. (The *jie qi* have a surplus of 168 days in 32 years)

In the following calculation methods, Q denotes the quotient and R the remainder. The year we use to do computation for the Chinese calendar is a kind of astronomical year, which is the period between the new moon before winter solstice and the new moon before the next winter solstice, i.e. period between the first day of month 11 and the day before month 11 of the next year.

In each of the following calculation methods, we start with the original sentences copied from *hou han shu*, and then state the purpose of the calculation. After that, we will write down the formulas given in [1], and give explanations to them.

1. Calculation of ru bu ( )

The purpose of this method is to determine which *yuan*, *ji* and *bu* for a given year is belonging to and to find the day name for the first day of the *bu* that the year is belong to. In this method, we use *ji nian* to do the computations; therefore we have to convert the given year into *ji nian* before doing computations.

Step 1: *Ji nian* = *yuan*-rule x Q_1 + R_1

(Q_1 is the number of completed *yuan* since the beginning of *shang yuan*. The current *yuan* is Q_1+1 because we count the *yuan* starting at 1 but **Q_1 starts at 0**. R_1 is the remainder of *ji nian* divided by 4560.)

Step 2: R_1 = *ji*-rule x Q_2 + R_2

(Q_2 is the number of completed *ji* since the beginning of the (Q_1+1)-th *yuan*. The current *ji* is Q_2+1 because we count the *ji* starting at 1 but Q_2 starts at 0. R_2 is the remainder of R_1 divided by 1520. Since there are three *ji* in a *yuan*, 0 Q_2 2; we call Q_2 = 0 is Heaven-*ji*, Q_2 = 1 is Earth-*ji* and Q_2 = 2 is Man-*ji*)

Step 3: R_2 = *bu*-rule x Q_3 + R_3

(Q_3 is the number of completed *bu* since the beginning of the (Q_2+1)-th *ji*. The current *bu* is then Q_3+1 because we count the *bu* starting at 1 but Q_3 starts at 0. R_3 is the remainder of R_2 divided by 76 and it is called *ru bu* year, which is the number of years passed the beginning of current *bu*, including the given year.)

After that, we look at the Table of *bu-shou* (Table A); the day name for the first day of current *bu* the (Q_3 + 1)-th symbol of the day-name column. The (Q_3 + 1)-th symbol of the corresponding *ji* column is the year name for the first year of current *bu*.

To find the year name of any given year, we use the Table of *jia zi* cycle (Table 2) in Appendix A. (R_3–1) + the number of the name (in table 2) for the first year of current *bu* is the number of the name for the year. the reason for the "–1" is the R starts at 1.

E.g. 146 A.D. = 9427 year (*ji nian*)

9427 = 2 x 4560+ 307 (Q_1 = 2, R_1 = 307)

307 = 0 x 1520+ 307 (Q_2 = 0, R_2 = 307)

$307 = 4 \times 76 + 3$ ($Q_3 = 4$, $R_3 = 3$)

Since $Q_2 = 0$, this year is in Heaven-*ji*. The day name for the first day of current *bu*, which is the 5th symbol of the day-name column, is *geng zi* (37 in table 2). The 5th symbol of the Heaven-*ji* column is the year name of the first year of current *bu*, which is *jia shen* (21). The year name of the year is *bing xu* (23).

2. Calculation of Tian Zheng ( )

This method is used to determine whether there is a leap month between the winter solstice of the previous year to the winter solstice of a given year. In order to do this, we need to know the number of months from the beginning of current *bu* to the winter solstice of the previous year. So the number of years, which we are using to do the computation for R_3th year, count from the beginning of current *bu*, is $R_3 - 1$.

Step: $(R_3 - 1) \times$ *zhang*-month $=$ *zhang*-rule $\times Q_4 + R_4$

(Q_4 is *ji yue* , the number accumulated of months since the beginning of the current *bu* to the winter solstice of the previous year. R_4 is remainder of $(R_3 - 1)\times 235$ divided by 19, called *run yu*)

We have two ways to explain the above method:

(1) In the *Si Fen Li*, one year has $12\ ^{7}/_{19}$ months. When the fraction of a given year reaches 1, we add a leap month. For instance, the third year from the beginning of a *bu* has fraction $^{7}/_{19}+^{14}/_{19}=1\ ^{2}/_{19}$ (the second year has the fraction $^{14}/_{19}$), so it has a leap month. R_4 in the formula is the numerator in the fraction at the winter solstice of the previous year. If R_4 12, the fraction will pass or equal to 1 by the next winter solstice, then there will be a leap month in this year.

(2) Now, we are showing the argument mathematically.

Let the *ji yue* of the next year and this year be Q' and Q respectively and the *run yu* of the next year and this year be R' and R respectively.

Now, we suppose that $Q' - Q = 12$.

Then $R' - R = 7$ or $R' = R + 7$

Since $R' < 19$, we get $R < 12$.

However, if $Q' - Q = 13$, then $R' - R =$ 12 or $R' + 12 = R$.

Since R' 0, we get R 12. (Shown)

E.g. Year 146 A.D. is 3rd year of *bu* ($R_3 = 3$)

$(3 - 1) \times 235 = 19 \times 24 + 14$ ($Q_4 = 24$, $R_4 = 14$)

Since the *run yu* = 14 (14 > 12), hence there is a leap month between the winter solstice of 145A.D. and the winter solstice of 146 A.D.

3. Calculation of the position of the leap month

This method is used to determine the approximate position of the leap month in the year, if any. This approximate position may be one month earlier or later than the actual position, which is based on the "*no zhong qi rule*".

Step: (*chang*-rule – R_4) x 12 = 7 x Q_5 + R_5

(R_5 is the remainder of (19– R_4) x 12 divided by 7.)

If $R_5 < 4$, then the (Q_5 + 1)-th month, count from the month 11 of the previous year, is a leap month. If R_5 4, then the (Q_5 + 2)-th month, count from the month 11 of the previous year, is a leap year. However, we have to ensure that there has no *zhong qi* in the month, or otherwise, we cannot say that it is a leap month.

Explanation:

Recall the last formula, *run yu* (R_4) is the numerator of the fraction at winter solstice, so the fraction of the previous year is $R_4/19$. Therefore, the interval between the new moon before the winter solstice and the winter solstice is ($R_4/19$) M, where M = 29 $^{499}/_{940}$. So the interval between the winter solstice and the new moon after it is equal to (1–$R_4/19$) M.

From the figure below, we see that the interval between a *zhong qi* and the new moon after it will be $^{3409}/_{3760}$ days longer than the interval between the next *zhong qi* and the next new moon.

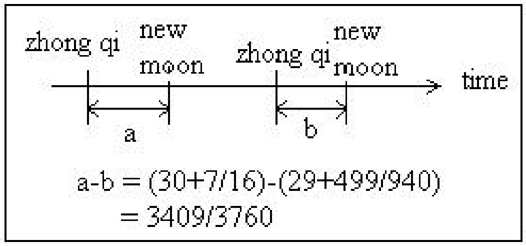


Figure 2

If a = (1–R_4/19) M, then we suppose that after *P* month(s), the interval between the *zhong qi* of the month and the first new moon after the *zhong qi* is C days (let C be negative if *zhong qi* is over the new moon). Then we have (1– R_4/19)M = *P*(3409/3760) + C or (19–R_4) x 12 = 7*P* + 228C/M.

Since we do not know whether the midnight will fall in between the *zhong qi* and the new moon, we cannot say that the month in the figure below is leap month even if the two new moons are inside the two *zhong qi*. This is because if the midnight starts before the first *zhong qi*, then the *zhong qi* is in the month of the new moon after it, and hence the month is not a leap month). Instead of looking at the midnight position, we assume the leap month appears as in the following figure so as to find an approximate position for leap month.

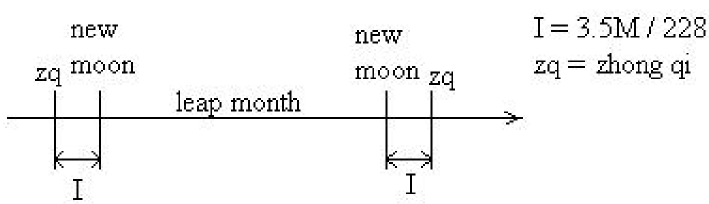


Figure 3

We first consider the first *zhong qi* (1) and then the second *zhong qi* (2).

(1) For 0 C I => 0 228C / M 3.5, we take R_5=228C/M, Q_5=*P*.

So the leap month is at the $P+1=Q_5+1$ th month for $0 \quad R_5<4$.

(2) For –I C 0 => 3.5 228C/M + 7 7, we take $R_5=228C/M+7$, $Q_5=P-1$.

So the leap month is at the $P+1=Q_5+2$ th month for $4 \quad R_5<7$.

The leap month found by this method may not be the true leap month since the method is just an approximation for the position of leap month. We are not sure that whether the formula of the *Si Fen Li* was found in this way, but it is a reasonable way for finding the approximate position of a leap month in the calendar.

E.g. Year 146 A.D. has a *run yu* of 14 ($R_4 = 14$)

$(19 - 14) \times 12 = 7 \times 8 + 4$ ($Q_5 = 8$, $R_5 = 4$)

$R_5 = 4$, $\therefore$ the (8+2)-th month, count from month 11 of year 145 A.D., is a leap month (this will be the 7th month of year 146 A.D.). To find the exact leap month, we have to make use of the '*no zhong qi rule*'.

4. Calculation of the length of a month

This method is used to determine the length of a month.

A month has only 29 days is called small month. Otherwise, it is called big month, which has 30 days.

Step 1: Q_4 x *bu*-day = *bu*-month x $Q_6 + R_6$

(R_6 is the remainder of Q_4 x *bu*-day divided by *bu*-month.)

If R_6 441, then the month 11 of the previous year is a big month.

To determine the length of the next month, we use the following formula:

Step 2: $R_6 + 499$ $940 \times Q_7 = R_7$

(R_7 is the remainder of $R_6 + 499$ divided by 940.)

If R_7 441, then the next month is a big month. The length of the following months can be determined by using the same formula.

Explanation:

In the *Si Fen Li*, one month has 29 $^{499}/_{940}$ days. When the fraction of a given month reaches 1, the month is a big month. E.g. the second month from the beginning of a *bu* has the fraction $^{499}/_{940}+^{499}/_{940}=1\ ^{58}/_{940}$ (the fraction of the first month is $^{499}/_{940}$), so it has a big month. R_6 in the formula is the numerator in the fraction of this month. If R_6 441, the fraction of the next month will pass or equal to 1, then the next month is a big month.

E.g. the *ji yue* (Q_4) of 146 A.D. is 24

$24 \times 27759 = 940 \times 70 + 416$ ($Q_6 = 70$, $R_6 = 416$)

$416 + 499$ $940 \times 0 = 915$ ($Q_7 = 0$, $R_7 = 915$)

R6 = 416 < 441, R7 = 915 >441, hence the month 11 of 145A.D. is 29 days and the month 12 of 145A.D. is 30 days. The first month of 146A.D. has *run yu* = 915 + 499 – 940 = 474 > 441, so it is a big month.

5. Calculation of the 24 jie qi of the year

This method is used to calculate the days of the 24 *jie qi*.

Step 1: $(R_3 - 1)$ x *ri yu* = *zhong*-rule x $Q_8 + R_8$

(R_8 is the remainder of $(R_3 - 1)$ x 168 divided by 32.)

Step 2: $Q_8 = 60 \times Q_9 + R_9$

(R_9 is the remainder of Q_8 divided by 60.)

Using the table of *jia zi* cycle (Table 2), the day-name number for the winter solstice of the previous year is R_9 + the number of the first day of current *bu*.

Explanation:

The interval between two consecutive winter solstice is 365.25, therefore, the number of accumulated days from the beginning of current *bu* to the beginning of the winter solstice in the previous year is $\lfloor 365.25 \times (R_3 - 1) \rfloor = \lfloor (360 + 168/32) \times (R_3 - 1) \rfloor \quad \lfloor 168(R_3 - 1) / 32 \rfloor$ (mod 60). The integral part of $168(R_3 - 1) / 32$ is the Q_8 in the above formulas. Therefore, the day-name number for the winter solstice of the previous year is the remainder of (Q_8 + the day-name number for the first day of current *bu*) divided by 60.

To find the next *jie qi*, we use the following formulas:

Step 3: $R_8 + 7 =$ *zhong*-rule x $Q_{10} + R_{10}$

(R_{10} is the remainder of $R_8 + 7$ divided by 32)

Step 4: $R_9 + 15 + Q_{10} - 60 \times Q_{11} = R_{11}$

(R_{11} is the remainder of $R_9 + 15 + Q_{10}$ divided by 60.)

The way we name the day of this *jie qi* is similar to the way we name the day of winter solstice (i.e. R_{11} + the day-name number for the winter solstice we just found).

Explanation:

The interval of two *jie qi* is 15 $^{7}/_{32}$, so its fraction is 7/32. The R_8 in step 1 is the numerator of the fraction for the winter solstice and the R_{10} in step 3 is the numerator of the fraction for the next *jie qi*. If the numerator reaches 32 then Q_{10} is 1 and the day-name number of the next *jie qi* is 15+1+day-name number of the winter solstice (mod 60).

E.g. $R_3 = 3$, $(3 - 1) \times 168 = 32 \times 10 + 16$ ($Q_8 = 10$, $R_8 = 16$)

$10 = 60 \times 0 + 10$ ($R_9 = 10$)

Hence the name for the day of winter solstice of 145A.D. is *geng xu* (47), which is 37+10.

$16 + 7 = 32 \times 0 + 23$ ($Q_{10} = 0$, $R_{10} = 23$)

$10 + 15 + 0 - 60 \times 0 = 25$ ($Q_{11} = 0$, $R_{11} = 25$)

The name of 2^{nd} *jie qi* (*xiao han*) is *yi chou* (2), which is $47+15 \equiv 2 (\mathrm{mod} 60)$.

4. Reformulation and Analysis

In this part, we will introduce some reformulated methods to determine position of leap year, leap month, *jie qi*, *zhong qi*, etc. These methods are easier for us to do implementation through computer. At the later section of this part, we will make some discussions on the pattern of big/small month and leap year.

4.1 Methods to Determine Date of Astronomical Event

By using the basic arguments in the *Si Fen Li* (refer to part 2), we can develop the following three important constants:

1. 1 year = $12\,{}^{7}\!/_{19}$ months (in average)
2. 1 months = $29\,{}^{499}\!/_{940}$ days (in average)
3. 1 *jie qi* = $15\,{}^{7}\!/_{32}$ days (in average)

We mention these constants because the majority of the calculations in the *Si Fen Li* (excluding the planets' motion part which is not included in our project) depend on them.

Since the least common multiple of 4 and 19 is 76, we can know that the numbers of years, months, days and *jie qi* will be an integer at the beginning of each *bu* (1 *bu* = 76 years). This cycle is important because it is the cycle of the distribution for leap year/month, big/small month and *jie qi*.

The following figure illustrates the significance of a *bu* cycle.

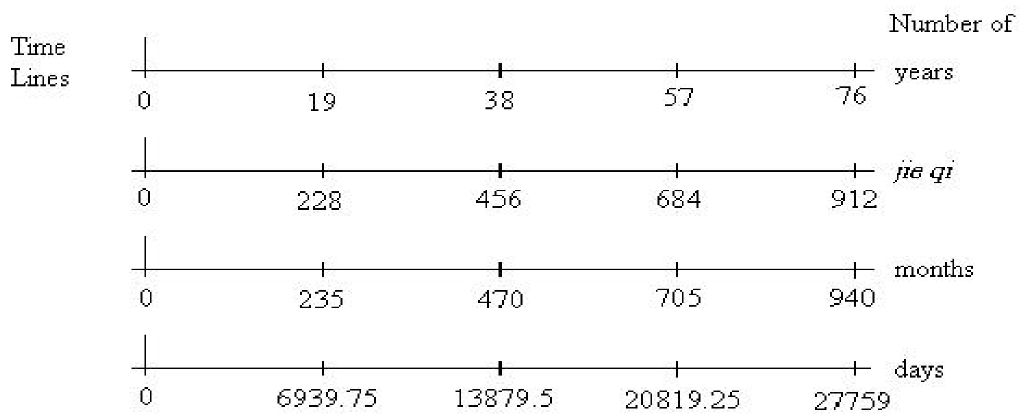


Figure 4

Figure 4 contains four time lines; they represent the numbers of four basic units (years, *jie qi,* months and days) in the *Si Fen Li*. The starting point of these time lines having value of zero represent the beginning of a *bu*.

From the graph, we can easily see that for every 19 years (1 *zhang*), number of *jie qi* and number of months are both integers, and for every 76 years (1 *bu*), numbers of *jie qi*, months and days are integers. These phenomena tell us that for every 19 years, the date of the first *jie qi* (which is also the first *zhong qi,* winter solstice) will be repeated. For every 76 years, the first *jie qi* will not only repeat its date but also will start at midnight. And hence, the distribution cycle for date of each *jie qi* is 76 years.

As the positions of various Chinese astronomical events such as *jie qi*, new moon, etc, have a cycle of 76 years, we may do all of the computations (again, excluding motion of planet and the sexagenerary cycle recording method) for a given year, base on the *ru bu* year (   ) of that given year.

The way to convert a particular year into a number in the current *bu* cycle (*ru bu year*) is shown below (you may also refer to the part 3):

For any given year *X*, we convert it into *ji nian* by using the fact that 161B.C. is year 9121 in *ji nian*, and then determine the remainder of *X* (in *ji nian*) divided by 76.

E.g. 130 A.D. = 9411 year (in *ji nian*) (note that there is no 0A.D or 0B.C.)

63 (mod 76)

∴130 A.D. is the 63th year in the current *bu* cycle.

(63 is said to be the *ru bu* year of the 130A.D.)

Note that *ru bu* year is an astronomical year in the Chinese calendar. It starts at the new moon just before winter solstice and ends at the new moon just before the next winter solstice. From the view of the civil year in the Chinese calendar, we may say that all *ru bu* years are started at 11^{th} month and end at 10^{th} month. Therefore, the first month of a *ru bu* year is the month 11 of previous year. This is very important for our calculations because whenever we want to do a calculation, for example, on A year B month, we convert A year into *ru bu* year and B month into month in *ru bu* year (if B < 11, we just add two into B, i.e. month in *ru bu* is B+2; but if B = 11 or 12, B will be to the 1^{st} or 2^{nd} new moon of next *ru bu* year)

Before going into our topics, we introduce a new term "accumulated days" (which is similar to *ji ri* in part 3) to you as we are often using it to do our calculations. **We define**

the "Accumulated days" as the fractional number of days, which is counted from the first day of the current *bu* cycle to a particular event that we want to discuss. The main usage of the 'accumulated days' is to find the date of an astronomical event by subtracting the accumulated days of that event with the accumulated days of the new moon before the event. (From now on, we will use 'AcD' to represent the accumulated days)

From the three constants stated in part 4.1, we know that the lengths of a month, year and *jie qi* are not integers. However, when we talk about date of an astronomical event, we are referring to the whole day of the day where the event occurs. E.g. if we found that the coming winter solstice will appear at 11a.m. of Dec 22, then we say that the *dong zhi* starts at 0a.m. of Dec 22 and ends at 11:59:59p.m. of that day.

4.1.1 Date of the Full Moon and the Length of a Month

To determine date of a given full moon, we need two values, the AcD for the given full moon and the AcD for the new moon just before the given full moon. To determine length of a given month, we need the AcD for the new moon of the given month, and also the AcD for the new moon of the next month. (Note that there is not need to determine date of new moon, because by definition, a month start at the day of the new moon and end before the day of the next moon, a new moon has to be the first day of the month.)

Method to determine AcD for new moon is: first we determine the total number of new moons passed by since the beginning of the current *bu* cycle, then we multiply the total number of new moon by the average length of a month.

Let's use *X ru bu* year *Y* month to illustrate,

Total month since the beginning of *bu* cycle = $\lfloor (X-1)\times 12\tfrac{7}{19} \rfloor + (Y - 1)$

AcD for the new moon of *Y* month in *X ru bu* year

$$= \left(\lfloor (X-1)\times 12\tfrac{7}{19} \rfloor + (Y-1) \right) \times 29\tfrac{499}{940}$$

$(X - 1)$ is the completed year since the beginning of the current *bu* cycle; $\lfloor (X-1)\times 12\tfrac{7}{19} \rfloor$ is the number of completed month since the beginning of the current *bu* cycle until *X ru bu* year, $12\tfrac{7}{19}$ is the average number of new moon in one year; $(Y-1)$ is the completed month since the beginning of *X ru bu* year.

(Take note that, while we are counting the AcD for the new moon, it is wrong if we just multiply $(X - 1)$ by 365.25 and $(Y - 1)$ by length of a month, then do the addition. This is because 365.25 days is just the average value of a year, but not the exact value, a year in Chinese calendar may contain 12 or 13 months and each month may contain 29 or 30 days.)

Method to determine the AcD for full moon is same to method for new moon, but we just need to add $\tfrac{1}{2}$ to the total number of completed month,

E.g. Find AcD for full moon of *X ru bu* year *Y* month,

AcD for full moon of *Y* month in *X ru bu* year

$$= \left(\lfloor (X-1)\times 12\tfrac{7}{19} \rfloor + (Y-1) + \tfrac{1}{2} \right) \times 29\tfrac{499}{940}$$

(The reason why we can complete our calculation by just adding “$^{1}/_{2}$ ” to the total completed month is by making use of the properties *ping shou* (refer to part 1) which make the full moon always occurs at the middle point of two adjacent new moons.)

To determine the date of full moon, we take the integral part of the AcD for the full moon and new moon and then make subtraction. For the time where the full moon fall, it is represent by the decimal part of the AcD for the full moon.

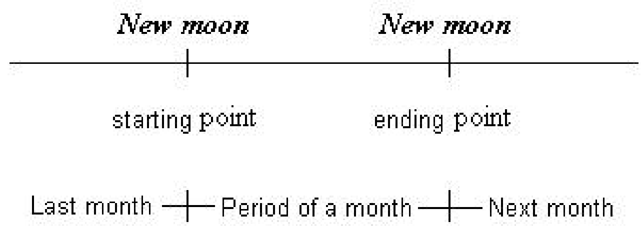
E.g. Let the AcD for the full moon be AcD_1 and the AcD for the new moon just before the full moon be AcD_2.

Date for the given full moon $= \lfloor AcD_1 \rfloor - \lfloor AcD_2 \rfloor + 1$

Occurring time for the given full moon = $(AcD_1 - \lfloor AcD_1 \rfloor) \times 24$ hours

(note that the value we obtained is only the value from computation, and it is base on midnight is the “starting of a day”. At later Han period, the definition of the “starting of a day” for the days with full moon is not at mid night, but is decided by the time of sun-rise and sun-set, and for different *jie qi* period, the “starting of a day” is different. After we obtain the computation value, we have to make comparison with the “starting of a day” to decide which day will the full moon belong to. Due to the lack of information for the “starting of a day” at later Han period, we are not able to provide the actual date of the full moon.

Length of a month is the subtraction of the integer part of the AcDs for two adjacent new moons. (The starting point of a month is the new moon of that particular month, and the ending point of a month is the new moon of next month of that particular month.)



Period of a month is a fractional number; length of a month, which is an integral number, will be the subtraction integral part of Acd for two adjacent new moons.

E.g. Length of month for *X ru bu* year *Y* month

$$\text{Accumulated days of the new moon} = (\lfloor X \times 12\tfrac{7}{19} \rfloor + Y) \times 29\tfrac{499}{940}$$

Accumulated days of next new moon

$$= (\lfloor X \times 12\tfrac{7}{19} \rfloor + Y + 1) \times 29\tfrac{499}{940}$$

$$\text{Length of a month} = \lfloor (\lfloor X \times 12\tfrac{7}{19} \rfloor + Y + 1) \times 29\tfrac{499}{940} \rfloor - \lfloor (\lfloor X \times 12\tfrac{7}{19} \rfloor + Y) \times 29\tfrac{499}{940} \rfloor$$

4.1.2 Date of Jie qi and Zhong qi

In *Si Fen Li*, time interval of two adjacent *zhong qi* or *jie qi* is always a constant (the property of *ping qi*), so we can simplify our calculation for position *zhong qi* and *jie qi* into calculation for date of *zhong qi* only (the 12 odd *jie qi* occurs at the middle of two *zhong qi*). In addition, by definition of leap month (month that without any *zhong qi* is a leap month), we know every non-leap month has one and only one specific *zhong qi* (note that there are 12 non-leap month and 12 *zhong qi* in a year). E.g. the *Dong zhi* (Winter solstice) always fall on the 11th month, *Da han* (Great Cold) on the 12th month, *Yu shui* (Rain water) on the 1st month, *Chun fen* (Spring equinox) on the 2nd month, etc. By these two properties, determination for date of *jie qi* or *zhong qi* will simplify into determination for the Acd for the *zhong qi*.

To determine date of a given *zhong qi*, we need two values, the AcD for the given *zhong qi* and the AcD for the new moon just before the given *zhong qi*.

To determine the AcD for a given *zhong qi*, we need to calculate the total number of the completed *zhong qi* since the beginning of the current *bu* cycle, then we multiply the total number of the completed *zhong qi* by the average length of a *zhong qi*. The value we get from the multiplying will be the AcD for the given *zhong qi*.

To determine the AcD for the new moon just before the given *zhong qi*, we have to make use of the AcD for the given *zhong qi* which we just determined. First we convert the AcD for the given *zhong qi* into number of months (just divide it by the average length of a month). Then, we take the integer part of the number of months and multiply it by the length of a month, this will give us the AcD for the new moon just before the given *zhong qi*.

By subtracting the two AcDs, we can know date of the *zhong qi*.

E.g. *X ru bu* year *Y zhong qi*. (*Y* = 1 for Dong zhi, 2 for Da han, etc. refer to Table 3 in Appendices B)

Total number of completed *zhong qi* since the beginning of the Current *bu* cycle = $(X - 1) \times 12 + (Y - 1)$

$(X - 1) \times 12$ is the total number of completed *zhong qi* since the beginning of the current *bu* cycle until *X ru bu* year. This calculation is valid only when the "*no zhong qi rule*" (month without *zhong qi* is a leap month) holds. For every year, there are exactly twelve non-leap month and one leap month (if the year is a leap year), every non-leap month will have one *zhong qi*, so we may conclude that every year will have an exact number of twelve *zhong qi*. (Take note that the same calculation idea does not apply to the twelve odd *jie qi*, because there are not guaranties for having exactly twelve odd *jie qi* in one year.)

AcD for the *Y zhong qi* in *X ru bu* year

$$= ((X - 1) \times 12 + (Y - 1)) \times 30\tfrac{7}{16}$$

AcD for the new moon just before the *Y zhong qi* in *X ru bu* year

$$= \left\lfloor ((X - 1) \times 12 + (Y - 1)) \times 30\tfrac{7}{16} \div 29\tfrac{499}{940} \right\rfloor \times 29\tfrac{499}{940}$$

Let AcD for the *Y zhong qi* in *X ru bu* year be AcD_1, and AcD for the new moon just before the *Y zhong qi* in *X ru bu* year be AcD_2,

Date for the given *zhong qi* = $\lfloor AcD_1 \rfloor - \lfloor AcD_2 \rfloor + 1$

Happening time for the given *zhong qi* = $(AcD_1 - \lfloor AcD_1 \rfloor) \times 24$ hours

(The reason behind the adding "1" method is, while we doing subtraction, the number start at zero, but when we talk about calendar terms such as date etc., the number start at one, e.g. We ask which day is the next day of "1st of July", the difference of "next day" and "1st of July" is one, but if referring to the date of "next day", we will say "2nd of July", while 2 = 1 + 1. In other words, the subtraction of the calculation we do is excluded the number (date) of the new moon, but while doing addition for calendar, the number (date) of the new moon must be include.

In order to make a clear understanding on the formula, let's do an example on the date for every *zhong qi* in *ru bu nian* 35.

Zhong Qi Z *[1]	AcD for Z AcD(Z) *[2]	AcD(Z) in months Z(m) *[3]	AcD for new moon AcD(m) *[4]		Date of *Zhong Qi* *[5]	Time of *Zhong Qi*
1	12418.50	420.53	12402.96		17	12.0
2	12448.94	421.56	12432.49		17	22.5
3	12479.38	422.59	12462.02		18	9.0
4	12509.81	423.62	12491.55		19	19.5
5	12540.25	424.65	12521.08		20	6.0
6	12570.69	425.68	12550.61		21	16.5
7	12601.13	426.71	12580.14		22	3.0
8	12631.56	427.74	12609.67		23	13.5
9	12662.00	428.77	12639.20		24	0.0
10	12692.44	429.80	12668.74		25	10.5
11	12722.88	430.83	12698.27		25	21.0
12	12753.31	431.86	12727.80		27	7.5

Table B

*[1] number 1-12 represent different *zhong qi*, eg 1 - Dong zhi, 2 - Da han, etc.

*[2] $AcD(Z) = ((35 - 1) \times 12 + (Z - 1)) \times 30\,{}^{14}\!/_{32}$, which means AcD(Z) = number of completed *zhong qi* (since the beginning *bu* cycle) x length between two *zhong qi*.

*[3] $Z(m) = AcD(Z) \div 29\,{}^{499}\!/_{940}$, which mean Z(m) = AcD(Z) / Length of one month.

*[4] $AcD(m) = \lfloor Z(m) \rfloor \times 29\,{}^{499}\!/_{940}$.

*[5] the values in this column is equal to $\lfloor AcD(Z) \rfloor - \lfloor AcD(m) \rfloor + 1$.

Caption: for the *ru bu* year 35, *Dong zhi* (Winter solstice) will fall on the 17th day of the 11th month of previous year; *Da han* (Great Cold) on the 17th day of the 12th month of the previous year; *Yu shui* (Rain water) on the 18th day of the 1st month; *Chun fen* (Spring equinox) on the 19th day of the 2nd month, etc. (year in this conclusion refer to civil year).

4.1.3 Position of Leap Year and Leap Month

Recall the definition of leap month, which states "month without a *zhong qi* is a leap month". By making use of this definition, we can easily reformulate a simple method to determine whether a given month is a leap month or not.

Again, this method needs two AcDs, the AcD for the starting point and the ending point of the given month. After we obtain the two AcDs, we convert them into total number of *zhong qi* since the beginning of current *bu* (dividing them by the interval of two adjacent *zhong qi*). If the integer parts of the two values (after converting) are equal, and the decimal part of the value for the starting point is not equal to zero, then the given month is a leap month. The integer parts of the two values (after converting) are equal implies that the starting point and the ending point are in the same number of accumulated *zhong qi*, and hence there has no *zhong qi* between the starting point and the ending point of the month. However, if the *zhong qi* occur at the starting point of the given month (number of *zhong qi* will be integer in this case, decimal part of the value for the starting point equals to zero), their integral parts will be the same but the month is not a leap month.

Method to determine the AcD for the starting point of a given month is same as the method to determine the AcD for a given new moon (refer to part 4.1.1), and the AcD for the ending point of a given month is equal to the AcD for the starting point of next month.

Suppose that the AcDs for the starting and ending points of a given month are already determined and equal to AcD_1 and AcD_2 respectively. Let AcD_1(zq) and AcD_2(zq) be the value after converting them into the number of *zhong qi*.

Then $AcD_1(zq) = AcD_1 / 30\,{}^{7}\!/_{16}$; $AcD_2(zq) = AcD_2 / 30\,{}^{7}\!/_{16}$

If $\lfloor AcD_1(zq) \rfloor = \lfloor AcD_2(zq) \rfloor$ and $AcD_1(zq) - \lfloor AcD_1(zq) \rfloor \neq 0$, then the given month is a leap month.

In order to make a clear understanding on the formula, let's do an example for every month in *ru bu nian 41*.

Month	AcD_1	AcD_2		AcD_1(zq)	AcD_2(zq)	Leap
1	14588.24	14617.77		479.28	480.23	FALSE
2	14617.77	14647.30		480.23	481.22	FALSE
3	14647.30	14676.83		481.22	482.17	FALSE
4	14676.83	14706.36		482.17	483.15	FALSE
5	14706.36	14735.89		483.15	484.11	FALSE
6	14735.89	14765.43		484.11	485.09	FALSE
7	14765.43	14794.96		485.09	486.05	FALSE
8	14794.96	14824.49		486.05	487.03	FALSE
9	14824.49	14854.02		487.03	488.02	FALSE
10	14854.02	14883.55		488.02	488.97	TRUE
11	14883.55	14913.08		488.97	489.95	FALSE
12	14913.08	14942.61		489.95	490.91	FALSE

Table C

Caption: for the *ru bu* year 41, the 10th new moon after the beginning of current *ru bu* year is a leap month (in civil calendar, we called that month as leap 7th month).

By using the calculations introduced in the previous sections, we have constructed a table for leap months in one *bu* and their corresponding *ru bu* years.

Ru bu Year	3	6	9	11	14	17	19	22	25	28	30	33	36	38
Leap month	6	3	12	9	5	1	10	7	3	11	8	5	1	9
Ru bu Year	41	44	47	49	52	55	57	60	63	66	68	71	74	76
Leap month	7	4	12	8	5	2	10	6	3	12	8	4	1	10

4.2 Pattern and Distribution

4.2.1 Pattern of Consecutive Big Months (Lian Da)

In *Si Fen Li*, it is easy to see that there exist consecutive big months, but there does not exist consecutive small months since the average length of month is longer than 29.5 days. Furthermore, there must not have more than two big months that are adjacent.

The other method of determining big/small of a month is deduced as the following:
The number of Acd since the beginning of the current *bu* to the end of N-th month (N>1) is $\lfloor (29\frac{499}{940})N \rfloor$ and since the beginning of the current *bu to* the end of (N-1)-th month is $\lfloor (29\frac{499}{940})(N-1) \rfloor$.

(Let $\lfloor n \rfloor$ denote the greatest integer which is less than or equal to n.)

Therefore, the number of days in N-th month

$$= \lfloor (29\frac{499}{940})N \rfloor - \lfloor (29\frac{499}{940})(N-1) \rfloor$$

$$= 29 + \lfloor (499/940)N \rfloor - \lfloor (499/940)(N-1) \rfloor$$

If N-th month is a big month, then $\lfloor (499/940)N \rfloor - \lfloor (499/940)(N-1) \rfloor = 1$

$$\Rightarrow \quad \lfloor (499/940)N \rfloor = \lfloor (499/940)N + (441/940) \rfloor$$

$$\Rightarrow \quad 499N \quad 0, 1, 2, \ldots, 498 \pmod{940} \qquad (1)$$

Now we are going to look for the pattern of consecutive big months.

Suppose N-th month from the beginning of a *bu* is a big month.

Then 499N 0, 1, 2, …, 498 (mod 940) (2)

If (N+1)-th month is a big month as well,

then 499(N+1) 0, 1, 2, … , 498 (mod 940)

or 499N 441, 442, … , 939 (mod 940) (3)

Now, we suppose that the N-th and the (N+1)-th months are big months.

Then, from (2) and (3), we get 499N 441, 442, … , 498 or N

or N 939 + 859K (mod 940), where K=0, 1, … , 57

or N 16, 33, 48, 65, 82, 97, 114, 129, 146, 163,

178, 195, 210, 227, 244, 259, 276, 291, 308, 325,

340, 357, 372, 389, 406, 421, 438, 453, 470, 487,

502, 519, 534, 551, 568, 583, 600, 615, 632, 649,

664, 681, 696, 713, 730, 745, 762, 777, 794, 811,

826, 843, 858, 875, 892, 907, 924, 939 (mod 940)

Now **we define the length of two adjacent *lian da* to be the number of months between the beginnings of the first months of the two *lian da*.** E.g. the 16th and 17th months in a *bu* form a *lian da* and the 33rd and 34th months form the next *lian da*, then the length between the two *lian da* is equal to 17 (=33-16).

The lengths of all *lian da* in one *bu* (940 month) are listed in the table below, the months after 940th month will repeat the same pattern of *lian da*. The numbers in *Lian Da* columns represent the number of the first month of a *lian da*.

Lian Da	Length		*Lian Da*	Length		*Lian Da*	Length
16	17		340	17		664	17
33	15		357	15		681	15
48	17		372	17		696	17
65	17		389	17		713	17
82	15		406	15		730	15
97	17		421	17		745	17
114	15		438	15		762	15
129	17		453	17		777	17
146	17		470	17		794	17
163	15		487	15		811	15
178	17		502	17		826	17
195	15		519	15		843	15
210	17		534	17		858	17
227	17		551	17		875	17
244	15		568	15		892	15
259	17		583	17		907	17
276	15		600	15		924	15
291	17		615	17		939	17
308	17		632	17		956	
325	15		649	15		Total:	940

Table D

From table D, we found that the length of *lian da* will be repeated for every 81 months, except in the last 49 years of a *bu*.

4.2.2 Distribution of Leap Year

In this section, we introduce a simple way to determine whether a given year is leap year.

Recall the formula of the calculation method 2 in the third part:

“(R_3 - 1) x *zhang*-month = *zhang*-rule x Q_4 + R_4 (*run yu*)

If R_4 12, then the year has a leap month”

Since *zhang*-month = 235 and *zhang*-rule = 19, we can restate the above method by “if (R_3 –1) x 235 12, 13, … , 18 (mod 19) or R_3 0, 3, 6, 9, 11, 14, 17 (mod19), then there is a leap month between the last winter solstice and the winter solstice of this year.” However, the leap months from the formulas are not the exact leap months because of the “*no zhong qi rule*”.

Note: R_3 a (mod 19) => *ji nian* Y = R_3 + 76k a (mod 19).

Combining with calculation method 3 in the third part, we have constructed a table for leap months in one *zhang* cycle:

R_3 (mod19)	0	3	6	9	11	14	17
Leap months (Using formulas 2 & 3)	10	7	3	12	8	5	1
Leap months (Using *no zhong qi rule)*	9 or 10	6 or 7	3 or 4	11 or 12	8 or 9	4 or 5	1 or 2

From the table, we found that when *ji nian* Y 9 (mod 19), the previous year (Chinese year, not *ru bu* year) has a leap month, which is either after month 11 or month 12, so any *ji nian* Y 8 (mod 19) is leap year. For the other cases, leap months fall in the year. So, if Y 0, 3, 6, 8, 11, 14, 17 (mod 19), then Y has a leap month.

In conclusion, when we want to know whether a year (say Y) is leap year, we just need to find the remainder of Y divided by 19; if the remainder is 0, 3, 6, 8, 11, 14 or 17, then the year Y is a leap year.

Appendix A

Table 1 Chinese Sexagenary Cycle of Days and Years			
	Celestial Stems		*Earth Branches*
1	Jia	1	Zi (rat)
2	Yi	2	Chou (ox)
3	Bing	3	Yin (tiger)
4	Ding	4	Mao (hare)
5	Wu	5	Chen (dragon)
6	Ji	6	Si (snake)
7	Geng	7	Wu (horse)
8	Xin	8	Wei (sheep)
9	Ren	9	Shen (monkey)
10	Gui	10	You (chicken)
		11	Xu (dog)
		12	Hai (pig)

Table 2 Table of *Jia Zi* Cycle					
1	Jia zi	21	Jia shen	41	Jia chen
2	Yi chou	22	Yi you	42	Yi si
3	Bing yin	23	Bing xu	43	Bing wu
4	Ding mao	24	Ding hai	44	Ding wei
5	Wu chen	25	Wu zi	45	Wu shen
6	Ji si	26	Ji chou	46	Ji you
7	Geng wu	27	Geng yin	47	Geng xu
8	Xin wei	28	Xin mao	48	Xin hai
9	Ren shen	29	Ren chen	49	Ren zi
10	Gui you	30	Gui si	50	Gui chou
11	Jia xu	31	Jia wu	51	Jia yin
12	Yi hai	32	Yi wei	52	Yi mao
13	Bing zi	33	Bing shen	53	Bing chen
14	Ding chou	34	Ding you	54	Ding si
15	Wu yin	35	Wu xu	55	Wu wu
16	Ji mao	36	Ji hai	56	Ji wei
17	Geng chen	37	Geng zi	57	Geng shen
18	Xin si	38	Xin chou	58	Xin you
19	Ren wu	39	Ren yin	59	Ren xu
20	Gui wei	40	Gui mao	60	Gui hai

Appendix B

Table 3 **Names of the 24 *jie qi***				
	Name of *Jie Qi*		Translation	Date
1	Da xue		Great snow	Dec 7
2	Dong zhi		Winter solstice	Dec 22
3	Xiao han		Slight cold	Jan 6
4	Da han		Great cold	Jan 20
5	Li chun		Beginning of spring	Feb 4
6	Yu shui		Rain water	Feb 19
7	Jing zhe		Waking of insects	Mar 6
8	Chun fen		Spring equinox	Mar 21
9	Qing ming		Bure brightness	Apr 5
10	Gu yu		Grain rain	Apr 20
11	Li xia		Beginning of summer	May 6
12	Xiao man		Grain full	May21
13	Mang zhong		Grain in ear	Jun 6
14	Xia zhi		Summer solstice	Jun 22
15	Xiao shu		Slight heat	Jul 7
16	Da shu		Great heat	Jul 23
17	Li qiu		Beginning of autumn	Aug 8
18	Chu shu		Limit of heat	Aug 23
19	Bai lu		White dew	Sept 8
20	Qiu fen		Autumnal equinox	Sept 23
21	Han lu		Cold dew	Oct 8
22	Shuang jiang		Descent of frost	Oct 24
23	Li dong		Beginning of winter	Nov 8
24	Xiao xue		Slight snow	Nov 22

Odd numbers are *jie qi*, Even numbers are *zhong qi*.

Date in the Table 3 are in Gregorian calendar form

Appendix C

```
/* SiFenLi.c, c-programming source code of exe. file for dos-prompt.
   Ask an input year, then print out the date, time of shuo, wang,
   zhong qi & length of month for that year on the screen. */


#include <stdio.h>


int Leap_y(int rbyear);
int Leap_m(int rbyear, int moon);

void print_s(int rbyear, int moon);

float acd_shuo(int rbyear, int moon);
float acd_wang(int rbyear, int moon);
float acd_qi(int rbyear, int qi);

int month, zq; // number of non-leap month & zhong qi.


main()
{
  int year=1, rb_year;
  int moon; // number of new moon since beginning of ru bu year.
  int tmonth;
  char continues;

  while(year != 0)
  {
 start:
    printf("\n\nPlease Enter a year to do computation (range:85~236
                    , 0 to exit):");
    scanf("%d", &year);

    if(year == 0) goto end;
    if(year<85 || year>236)  {
      printf("\nCalculation of Si Fen Li doesn't apply to your input value.");
      printf("\nPress any key to continue.");
      getch(continues);
      goto start;
    }

    rb_year = (year + 9281) % 76;
    tmonth = 14 + Leap_y(rb_year);
```

```
    printf("\n   MONTH      SHUO        WANG    LENGTH   ZHONG QI ");
    printf("\n          DATE  TIME   DATE  TIME         DATE  TIME");
    printf("\n__________________________________________________");

    month = 11;    zq = 0;
    for(moon = 1; moon <= tmonth; moon++)  {
      if(moon % 8 == 0)  printf("\n");
      print_s(rb_year, moon);
      month ++;
      if(month > 12)  month = 1;
    }

    printf("\n\n\tPress any key to continue.");
    getch(continues);
 end:
  }
  return 0;
}


int Leap_y(int rbyear)
{
  int isLeap=0;

  if( ( (rbyear - 1) * 235 ) % 19 >= 12)   isLeap = 1;
  return isLeap;
}

int Leap_m(int rbyear, int month)
{
  int isLeap = 0,  completed_month;
  float acd_first, acd_last, qi_first, qi_last;

  completed_month = (rbyear - 1) * (235.0/19.0) + (month - 1);

  acd_first = (completed_month * (29.0+499.0/940.0) );
  acd_last  = ((completed_month + 1.0) * (29.0+499.0/940.0) );
  qi_first  = (int)(acd_first)/(30.0+14.0/32.0);
  qi_last   = (int)(acd_last) / (30.0+14.0/32.0);

  if(qi_first-(int)(qi_first) != 0)
    if( ((int)(qi_first) == (int)(qi_last)) || (qi_last-(int)(qi_last) == 0) )
       isLeap = 1;

  return isLeap;
}
```

```
void print_s(int rbyear, int moon)
{
  int name_s = acd_shuo(rbyear, moon), name_w = acd_wang(rbyear, moon);
  int name_q;
  int time_s = ( acd_shuo(rbyear, moon) - (float)(name_s) ) * 24.0;
  int time_w = ( acd_wang(rbyear, moon) - (float)(name_w) ) * 24.0;
  int time_q;
  int length = (int)(acd_shuo(rbyear, moon + 1)) - name_s;

  if(Leap_m(rbyear, moon) == 0)  {    // if is not a leap month.
     zq ++;
     name_q = acd_qi(rbyear, zq);
     time_q = ( acd_qi(rbyear, zq) - (float)(name_q) ) * 24.0;
     printf("\n   %3d       1    %2d      %2d    %2d     %2d"
             , month, time_s, name_w - name_s + 1, time_w, length);
     printf("      %2d    %2d",name_q - name_s + 1, time_q);
  }  else  {
     month --;
     printf("\n   %3d(Leap)  1    %2d      %2d    %2d     %2d"
             , month, time_s, name_w - name_s + 1, time_w, length);
  }
}


float acd_shuo(int rbyear, int moon)
{
  int completed_month;

  completed_month = ((rbyear - 1) * 235 / 19) + moon - 1;
  return completed_month * (29.0+499.0/940.0);
}

float acd_wang(int rbyear, int moon)
{
  int completed_month;

  completed_month = ((rbyear - 1) * 235 / 19) + moon - 1;
  return (completed_month + 0.5) * (29.0+499.0/940.0);
}

float acd_qi(int rbyear, int qi)
{
  int completed_qi;

  completed_qi = ((rbyear - 1) * 12) + qi - 1;
  return completed_qi * (30.0+14.0/32.0);
}
```

Screen output of program SiFenLi.exe:

Please Enter a year to do computation (range:85~236, 0 to exit): 230

MONTH	SHUO		WANG		LENGTH	ZHONG QI	
	DATE	TIME	DATE	TIME		DATE	TIME
11	1	7	16	1	29	21	12
12	1	19	16	14	30	22	22
1	1	8	16	2	29	23	9
2	1	21	16	15	30	24	19
3	1	10	16	4	29	25	6
4	1	22	16	17	30	26	16
5	1	11	16	5	30	27	3
6	1	0	15	18	29	27	13
7	1	12	16	7	30	29	0
8	1	1	15	20	29	29	10
9	1	14	16	8	30	30	21
9(Leap)	1	3	15	21	29		
10	1	15	16	10	30	2	7
11	1	4	15	23	29	2	18
12	1	17	16	11	30	4	4

Press any key to continue.

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