HONOURS PROJECT

TILINGS AND PATTERNS

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Summary

This project serves as an introductory text on the concept and properties of tilings. Chapter 1 provides readers with background information on the use of tiling or patterns since the beginning of recorded history and followed by providing the mathematical definition of tiling and the various terminology used in tilings and tiles. As many important properties of tilings depend on the idea of symmetry, it gives a basic understanding of isometry, symmetry and transitivity classes.

Chapter 2 studies tilings made up of regular polygons. We consider tilings that are edge-to-edge. Chapter 2.1 looks at regular and uniform tilings and explains what are Archimedean tilings and illustrates the 11 types of vertices that can be extended to form Archimedean tilings. Chapter 2.2 studies k-uniform tilings and discuss Krotenheerdt's results. Chapter 2.3 goes a step further to look at demi-regular tilings. We study and analyse the demi-regular tilings given in the various references, compile a table of the pictures in the 5 references and end with a conclusion as given in the report.

Author's Contribution

My humble contribution is the following interesting discovery on demi-regular tilings.

Several authors, including Ghyka, Critchlow, Williams, Lundy and Weisstein, introduce a concept called demi-regular tilings, and claimed that there are only 14 such tilings. It was already pointed out by Steinhaus that there are in fact infinitely many such tilings, but I elaborated on the exact definition. For some strange reason, although the authors who claimed that there are only 14 demi-regular tilings faithfully cite each other (and Steinhaus), they all gave different lists of 14 tilings. I have analyzed the lists and found that they include a total of 18 tilings. I studied these tilings in detail and compared them to the results of Grunbaum and Shephard, and Krotenheerdt.

CONTENTS

1 INTRODUCTION

1.1 Background Information about Tiling

The design of geometric shapes, which fit together to cover a surface without gaps or overlapping, has a long history. Far more than mere decoration, the symmetry of repetitive patterns has evoked a deep aesthetic, emotional and even spiritual response from humans since the beginning of recorded history. As soon as man began to build, he would use stones to cover the floors and walls of his house, and started to select the shapes and colors of his stones to make a pleasing design.

Every known human society has made use of tilings or patterns in some form or another. The Sumerians (about 4000 B.C) in the Mesopotamian Valley built homes and temples decorated with mosaics in geometric patterns. Later, Persians showed that they were masters in tile decorations. Similarly, the Moors used congruent, multicolored tiles on the walls and floors of their buildings. Moslem and Islamic tile patterns with striking colors still survive. Roman buildings, floors, and pavements were decorated with tiles which the Romans called tessellate. The Roman word tessellate is the root of our English word tessellation. Examples of some of these designs are shown in Fig 1.1.

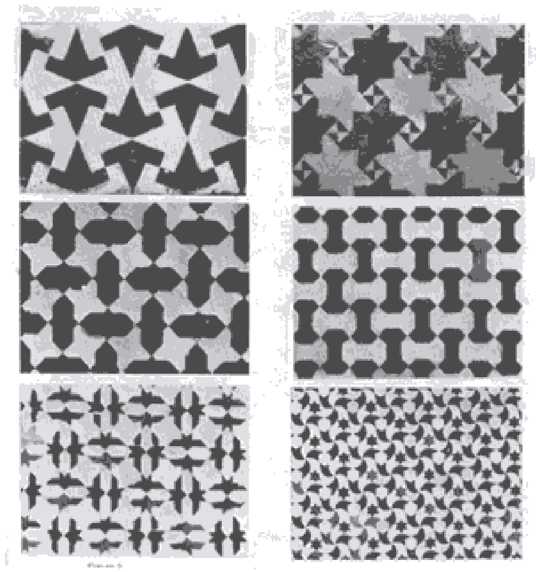


Fig 1.1

A tiling is a geometric pattern made up of one or more shapes which fit together to completely cover an infinite plane region or surface without any gaps or overlapping. The history of tiling has come a long way.

The mathematical definition of tiling will be given in Section 1.2. In mathematical literature, the words tessellation, paving, mosaic and parquetting are used with similar meanings as tilings.

1.2 Mathematical Definition of Tiling

Mathematically, a plane tiling T is a **countable** family of **closed** sets which covers the plane **without any gaps** or **overlaps**,

$$T = \{T_1, T_2, \ldots\ldots..\}$$

where T_1, T_2,….are known as tiles of T.

The terms in bold are the conditions required of a tiling as explained below:

(a) Countable: The number of tiles in the tiling can be counted.
However, there can be infinitely many tiles.

(b) Closed: Each tile is enclosed by its boundary.

(c) Without gaps: The union of all the sets T_1, T_2,….. is to be the whole plane, i.e.
$\{T_1 \cup T_2 \ldots \cup T_n\}$= whole plane.

(d) Without overlaps: The interior of the sets are to be pairwise disjoint, i.e.
{interior of $T_i \cap$ interior of T_j } = Φ , where $i \neq j$ $\forall$ any i and j.

The countability condition (a) excludes families in which every tile has zero area (such as points or line segments). This is because tiles with zero area are uncountable when they are arranged without gaps between them or when condition (c) is satisfied.

Nevertheless, the definition admits tilings in which some tiles have bizarre shape and properties.

A few examples of tiles, which are not exhaustive, are shown in Fig 1.2.1.

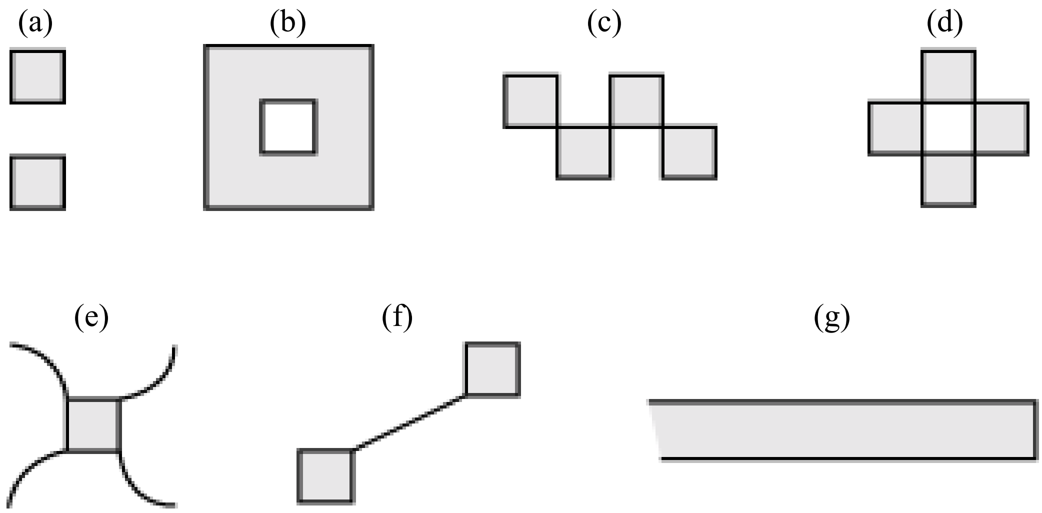


Fig 1.2.1

From Fig 1.2.1, one can see that

(a) is not connected, that means it has two or more separate pieces.

(b) is not simply connected, which means that it encloses at least one hole.

(c) & (d) becomes disconnected upon the deletion of a suitable finite set of points.

(e) & (f) are made up partly of figures of zero area like line segments and arcs.

(g) is unbounded since it cannot be enclosed by a finite circle, no matter how large the circle is. In this particular example, this tile is taken from a part of an infinite strip. Note that it contains all its boundary points.

In this project, we only consider tiles which are closed topological disks, that is, its boundary is a single simple closed curve. Single means a curve whose ends join up to form a "loop". Simple means that there are no crossings or branches.

In Fig 1.2.1(a) and (b), the boundary is not a single curve. (c), (d), (e) and (f) are not simple because they have branches or crossing points. (g) is not a closed curve.

Therefore, the condition that 'tiles are closed topological disks' eliminates tiles which have bizarre shape and properties as shown above. We then include this condition in the definition of tilings given earlier on Page 7 as follows:
(e) Each tile T_i in the tiling is a closed topological disk.

1.3 Terminology Used in Tiling and Tiles

Tiling:

Edge: It is a boundary which separates the two tiles.

Vertex: It is an endpoint of an edge.

If you consider only one tile alone, especially if the tile is a polygon, we would usually denote its elements by edges and vertices. However, it is clear that it would lead to confusion if we use the same notations for both the tiling and its tiles. This leads to the following.

Tiles:

Side: It is an edge of a tile.

Corner: It is a point where its sides intersect.

To get a clearer picture, we consider the following example (Fig 1.3.1):

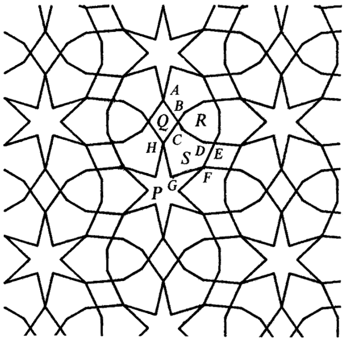


Fig 1.3.1

BCDE is an edge of the tiling as it separates tile R and S, but it is not a side of any tile. BC, CD, DE, HG and GF are sides of tile S but are not edges of the tiling. G is a corner of tile S and tile P but it is not a vertex of the tiling.

Notice that a vertex has to be a corner of some tiles but it does not have to be a corner of every tile surrounding it.

In the subsequent chapters, for most of the part, we shall focus our attention to the special case of tilings in which each tile is a polygon.

We say that the tiling by polygons is edge-to edge if the corners and sides of the polygons coincide with the vertices and edges of the tiling, otherwise we say that the tiling by polygons is not edge-to-edge. It also means that each pair of tiles in the tiling intersects along a common edge, at a vertex, or none at all. This is shown in Fig 1.3.2 and 1.3.3 below.

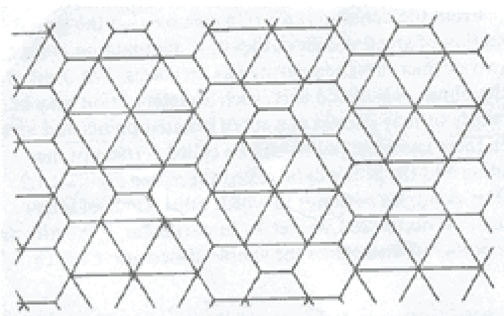


Fig 1.3.2

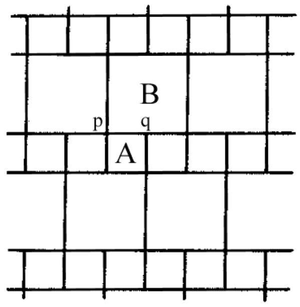


Fig 1.3.3

It can be easily seen that Fig 1.3.2 is edge-to edge. Fig 1.3.3 is not edge-to edge. Let's take an edge *pq* for example. *pq* separates tile A from tile B. Though *pq* is a side of A, it is not a side of B. Therefore, Fig 1.3.3 is not edge-to-edge.

Now, we shall see what we mean by saying two tilings are congruent. Two tilings T_1 and T_2 are congruent if T_1 may be made to coincide with T_2 by a rigid motion of the plane. Often, we are interested in "equal tilings". Two tilings are said to be equal or the same if one of them can be changed in scale (magnified or contracted equally throughout the plane) so as to be congruent to the other. In other words, the tilings have to be the same size to be congruent while size does not matter for them to be equal.

For example, consider

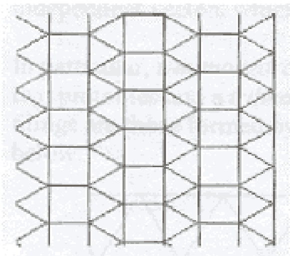


Fig 1.3.4

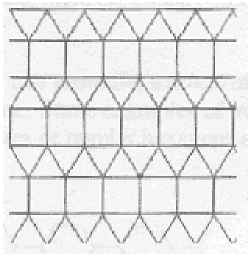


Fig 1.3.5

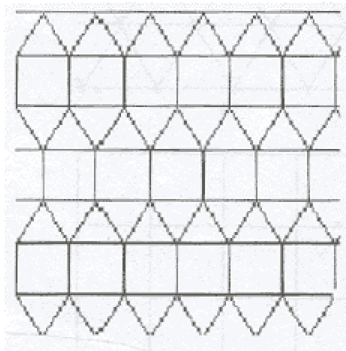


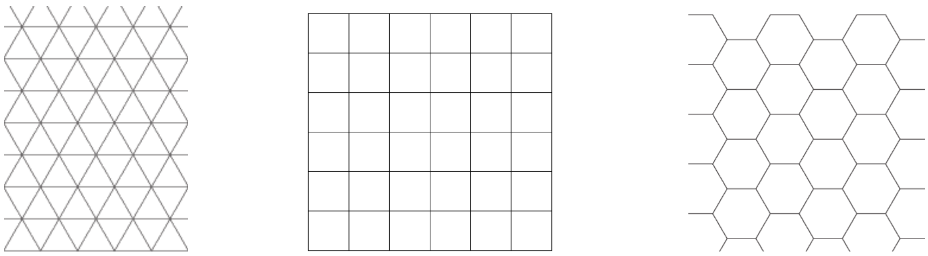
Fig 1.3.6

One can easily see that Fig 1.3.4 is congruent to Fig 1.3.5 while Fig 1.3.5 is equal to Fig 1.3.6.

1.4 Tilings with Tiles of a Few Shapes

We have already mentioned that we only consider tiles which are closed topological disks and as a further simplification, for the most part, we shall be concerned with monohedral tilings.

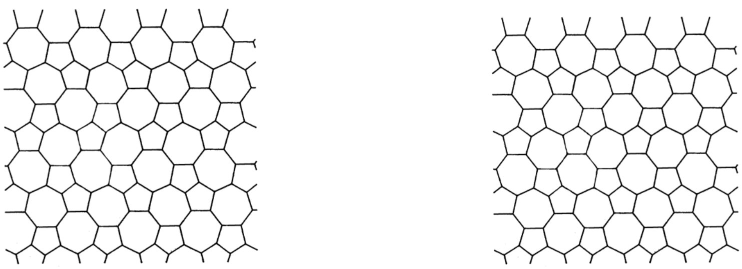
Monohedral: Every tile T_i in the tiling $\mathcal{T}$ is congruent to one fixed set T, meaning all the tiles are of the same shape and size. The set T is called the prototile of $\mathcal{T}$. Some examples of monohedral tilings formed by equilateral triangles, squares or regular hexagons are shown below in Fig 1.4.1(a), (b) and (c).



(a) (b) (c)

Fig 1.4.1

In a similar way, we define dihedral, trihedral, 4-hedral, ……. , n-hedral tilings in which all the tiles in the tiling are congruent to two, three, four,……n distinct prototiles of the set T respectively. Some examples of dihedral tilings are shown below. Fig 1.4.2(a) is formed by squares and heptagons. Fig 1.4.2 (b) is formed by pentagons and heptagons.



(a) (b)

Fig 1.4.2

1.5 Symmetry

Many important properties of tilings depend upon the idea of symmetry. Here, in this project, we explain what is meant by this term and give examples of tilings with various kinds of symmetry. In order to understand symmetry, let us start with the definition of isometry.

Definition of isometry: It is any mapping of the Euclidean plane R^2 onto itself which preserves all distances. Let A, B be any two points, then the distance between A and B is equal to the distances between their images A^1 and B^1.

Here, we will show without proof that every isometry is one of the four types:
(1) Rotation (2) Translation (3) Reflection (4) Glide Reflection

(1) Rotation maps all points of a figure through a given angle θ about a given point O. The point O is called the center of rotation. Let θ be the angle of rotation about a point O that maps a figure onto itself. If $360^0/\theta = n$ (a natural number), then the order of rotation of the figure is equal to n. The point O is then referred to as the center of n-fold rotational symmetry. An example of a rotation is shown below in Fig 1.5.1.

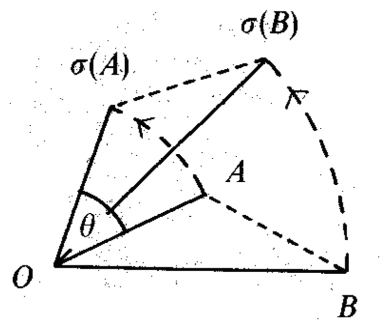


Fig 1.5.1

(2) Translation shifts all points of a figure in a given direction through a given distance. We will use an arrow to represent a translation. The magnitude and direction of the arrow denote the distance and direction of translation, respectively. Translation can take place in any direction. Examples of translations are shown in Fig 1.5.2.

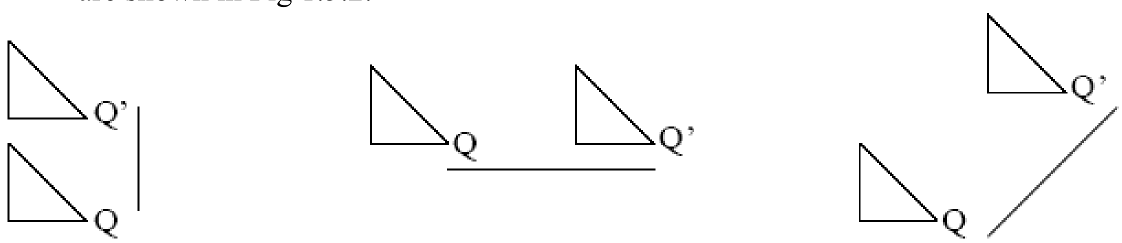


Fig 1.5.2

(3) A reflection maps all points of a figure on one side of any straight line L, to the opposite side of L such that the perpendicular distance between any point P, on one side of L, to the line L is equal to the perpendicular distance of P^1, the image of P on the opposite side of L, to the line L. The line L is called the mirror or the axis of reflection. An example of a reflection is shown in Fig 1.5.3.

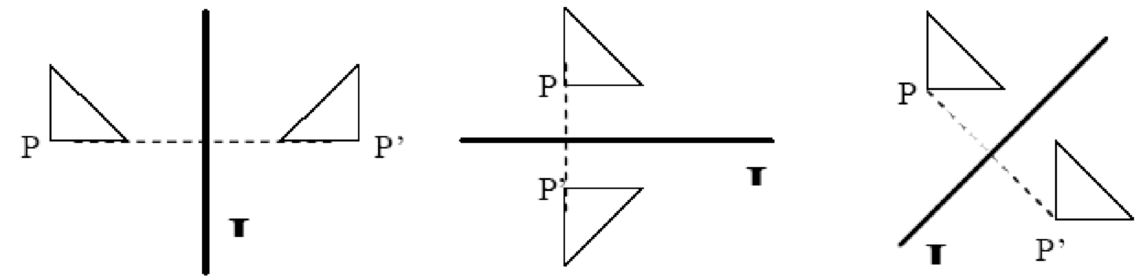


Fig 1.5.3

(4) Glide reflection, as the name suggests, is a combination of the two isometries mentioned earlier, namely, translation and reflection. There are two methods to do a glide reflection. Both methods give the same results.

The first method is translation followed by a reflection. First you translate all points of a figure through a given distance d parallel to a line L, followed by reflection of the image under translation in the same line L.

The second method is reflection followed by translation. You reflect all points of a figure about a line L, followed by translation of the image under reflection through a given distance d parallel to the same line L. An example in Fig. 1.5.4 to illustrate the first and second method is given below.

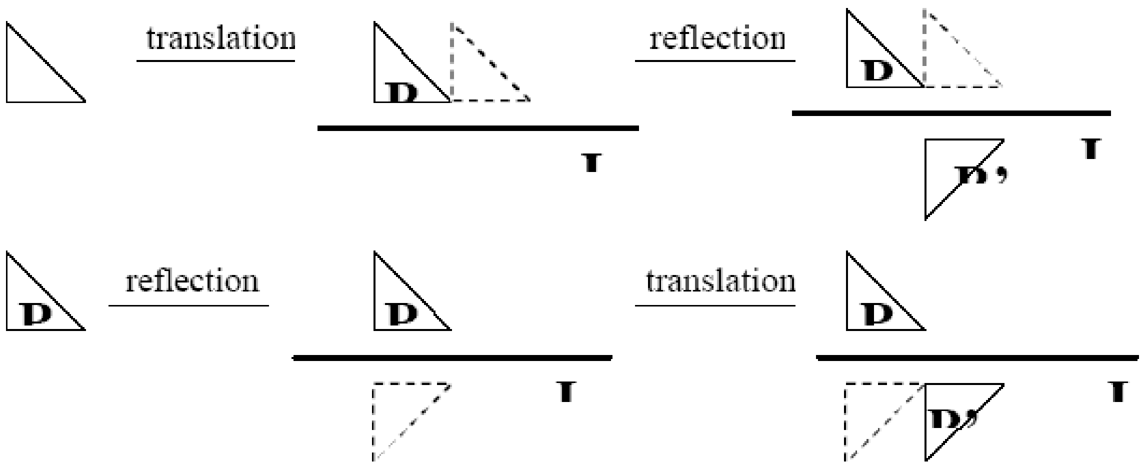


Fig 1.5.4

Definition of a *direct* isometry: If points ABC form the vertices of a triangle named in a clockwise (anticlockwise) direction, then the images of ABC, $A_1B_1C_1$ also form the vertices of a triangle named in a clockwise (anticlockwise) direction.

From previous definition, one can observe that rotation and translation are direct while reflection and glide reflection are indirect.

Given a set S, we define a symmetry of a set S to be an isometry which maps S onto itself. A symmetry group of a set S is the set consisting of all the symmetries of S. The number of symmetries in a symmetry group is called the order of the symmetry group.

Symmetry groups of tiles

(1) Cyclic group: cn (n ≥ 1)

(i) Rotations through angle 360^0j/n (j = 0,1,……,n-1) about a fixed point.

(2) Dihedral group: dn (n ≥ 1)

(i) Rotations through angle 360^0j/n (j = 0,1,……,n-1) about a fixed point.
(ii) Reflections about n lines equally inclined to one another. This means that the angle between any two lines of reflection is the same.

Examples of tiles with cyclic group:

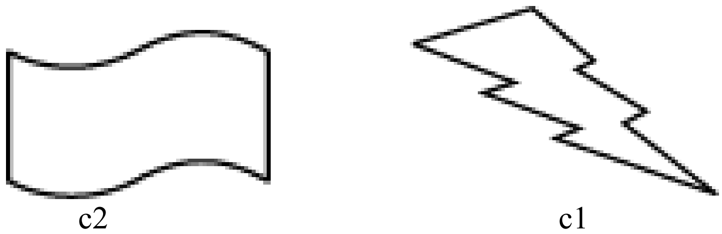


Fig 1.5.5

Examples of tiles with dihedral group:

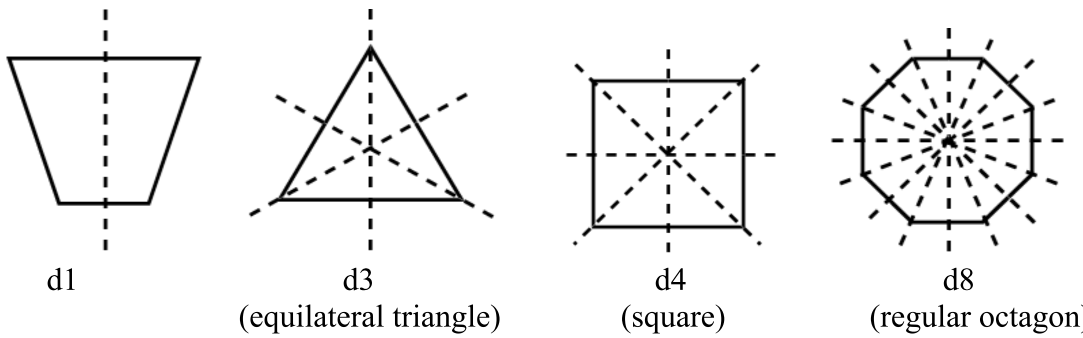


Fig 1.5.6

Note:
a) The dotted lines denote the line of reflection.
b) The order of a cyclic group is n because the cyclic group consists of only n rotations including the identity symmetry (i.e. rotation about 360^0 about a fixed point).
c) The order of a dihedral group is 2n because a dihedral group consists of n reflections and n rotations.

Next, we extend the definition of symmetry to tilings. A symmetry of a tiling is an isometry that maps the tiling onto itself. We will consider an example shown in Fig 1.5.7.

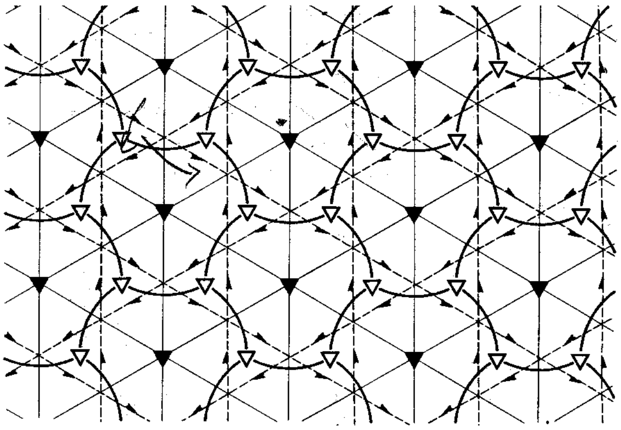


Fig 1.5.7

We can see that some of the symmetries of this tiling are:

(1) Rotations through 120^0, 240^0 about each point in the figure marked by a small triangle (solid or open).
(2) Translations, which take any of the solid black triangles onto one another.
(3) Reflections in each of the solid lines.
(4) Glide reflections consisting of reflection in the dashed lines followed by translation along them. This translation is through half the distance between the solid black triangles and is marked in the diagram by half arrowheads.

Now, we will compare symmetry of a tiling with that of its tiles. Let T be a tile of any tiling *T*. Then every symmetry of *T* which maps T onto itself is clearly a symmetry of T. But the converse is not true in general. In other words, every symmetry of T may not be a symmetry of *T*. For example in Fig 1.5.8 shown below, the only symmetry of *T* which maps a tile onto itself is the identity symmetry. However, the tile itself, being a square has seven other symmetries. (See examples of tiles with dihedral group).

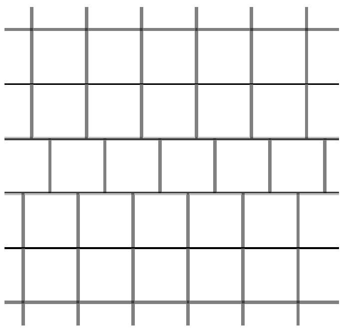


Fig 1.5.8

Hence, we must carefully distinguish between S(T), the group of symmetries of the tile T, and S(*T*|T), the group of symmetries of T which also are symmetries of the tiling *T*. For brevity, we shall refer to S(*T*|T) as the induced tile group or as the stabilizer of T in *T*.

1.6 Transitivity classes

Two tiles T_1, T_2 of a tiling *T* are said to be equivalent if the symmetry group S(*T*) contains a transformation that maps T_1 onto T_2. The collection of all tiles of *T* that are equivalent to T_1 is called the transitivity class of T_1. If all the tiles of *T* form one transitivity class, *T* is said to be isohedral. If *T* is a tiling with k transitivity classes, *T* is called k-isohedral. If a tiling *T* admits only the identity symmetry, then every tile in *T* is a transitivity class on its own. Figure 1.6.1(a) and (b) shows an isohedral tiling. Figure 1.6.1(c) and (d) show a 2-isohedral tiling. All the tiles in these tilings are either equivalent to A or B as denoted in the diagram, meaning these are tilings with two transitivity classes.

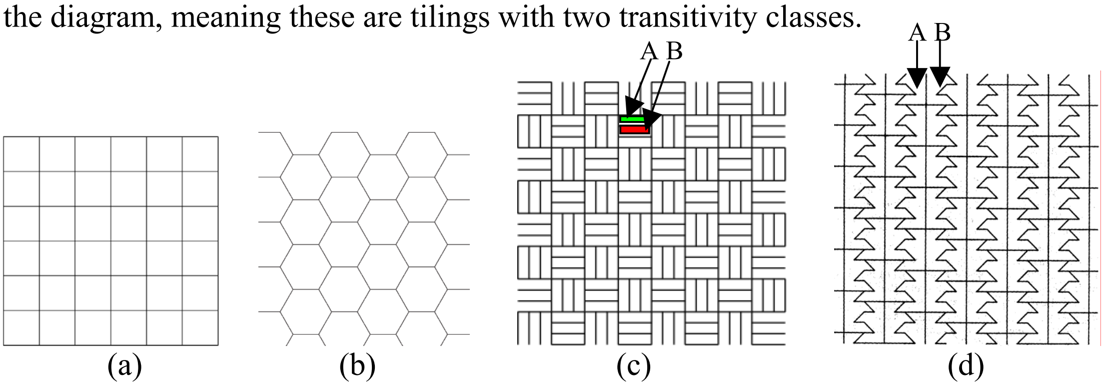


Fig 1.6.1

Compare between monohedral and isohedral:

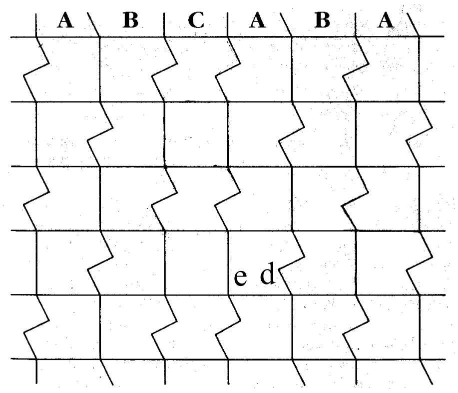
The definition of monohedral is explained in Section 1.3. Both consist of tiles which have same shape and size. In isohedral tiling there is a symmetry that makes all the tiles equivalent. However in monohedral tilings, there might not exist a symmetry which maps one tile onto another.

In Fig 1.6.1(a) and (b), the monohedral tilings are isohedral. However, the monohedral tilings in Fig 1.6.1(c) and (d) are not isohedral as there is no symmetry of the tiling that maps tile A onto tile B. In short, isohedral implies monohedral but not the other way round. A brief explanation is given below.

For Fig 1.6.1(c), some of the ways to map tile A onto tile B is by translation by 1 unit down or reflection. However, you can easily see that these are not symmetries of the tiling. For Fig 1.6.1(d), you may try reflection to map tile A onto tile B at first thought but you can easily see that reflection is not a symmetry of the tiling.

Next, we go on to look at vertices. We will introduce two definitions, monogonal and isogonal. A monogonal tiling is one in which every vertex, together with its incident edges, forms a figure congruent to that of any other vertex and its incident edges. It simply means that vertices with its incident edges form one congruent class. An isogonal tiling is one in which all its vertices form one transitivity class.

The distinction between isogonal and monogonal tilings is analogous to that between isohedral and monohedral tilings. An example of a tiling which is monogonal but not isogonal is shown in Fig 1.6.2.



We analyze that this tiling has the pattern ABCABA…We can see that vertical translations are the only symmetries of this tiling. So, vertex d cannot be mapped onto vertex e. (d and e are denoted in Fig 1.6.2)

Fig 1.6.2

Thus far, we have looked at tiles and vertices. Now, we will look at the last element of the tilings, that is, edges. As usual, we introduce two more definitions: isotoxal and monotoxal. Isotoxal tilings are tilings in which every edge can be mapped onto any other edge by a symmetry of the tiling. In short, it simply means that edges of the tiling form one transitivity class. From the definition of monohedral and monogonal, we try to deduce the meaning of monotoxal. We come up with the following two definitions.

(1) analogy with monohedral: all edges are congruent, meaning all edges have the same length.
(2) analogy with monogonal: all edge figures are congruent. This means that every edge, together with its incident vertices and edges form one congruent class. Note that an edge has two vertices.

The example (Fig 1.6.3) below shows an isotoxal tiling and a monotoxal tiling. It is monotoxal because all edges have the same length or it can be easily seen that all edge figures are congruent.

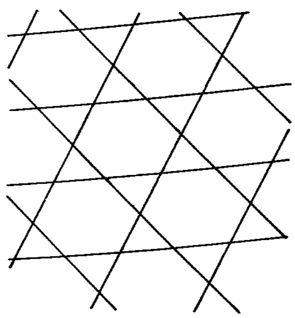


Fig. 1.6.3

To summarize, ‘hedral’ refers to tiles, ‘gonal’ refers to vertices and ‘toxal’ refers to edges. Note that if a tiling has k transitivity classes and m congruent classes, $m \leq k$. In other words, transitivity classes are contained in congruent classes, or one congruent class can be split into many transitivity classes.

2 TILINGS BY REGULAR POLYGONS

2.1 Regular and Uniform Tilings

In this chapter, we are going to study tilings by regular polygons. We will also consider tilings that are edge-to-edge. Tilings by regular polygons were the first kinds of tilings to be the subject of mathematical research. It is important to study them as they form the basis for pattern construction in many traditions of sacred and decorative art. They can be found in Celtic and Islamic patterns, and in the natural world, they appear as crystal and cellular structures. Many people used them widely for their wallpaper and fabric designs.

For the remainder of this project, all our tilings are assumed to be edge-to-edge tilings of regular polygons. We denote the regular n-gon by {n}.

Theorem 1 The only edge-to-edge monohedral tilings by regular polygons are the first three regular tilings of Fig 2.1.1 below, that is (3^6), (4^4) and (6^3). They consist of equilateral triangles, squares and regular hexagons respectively.

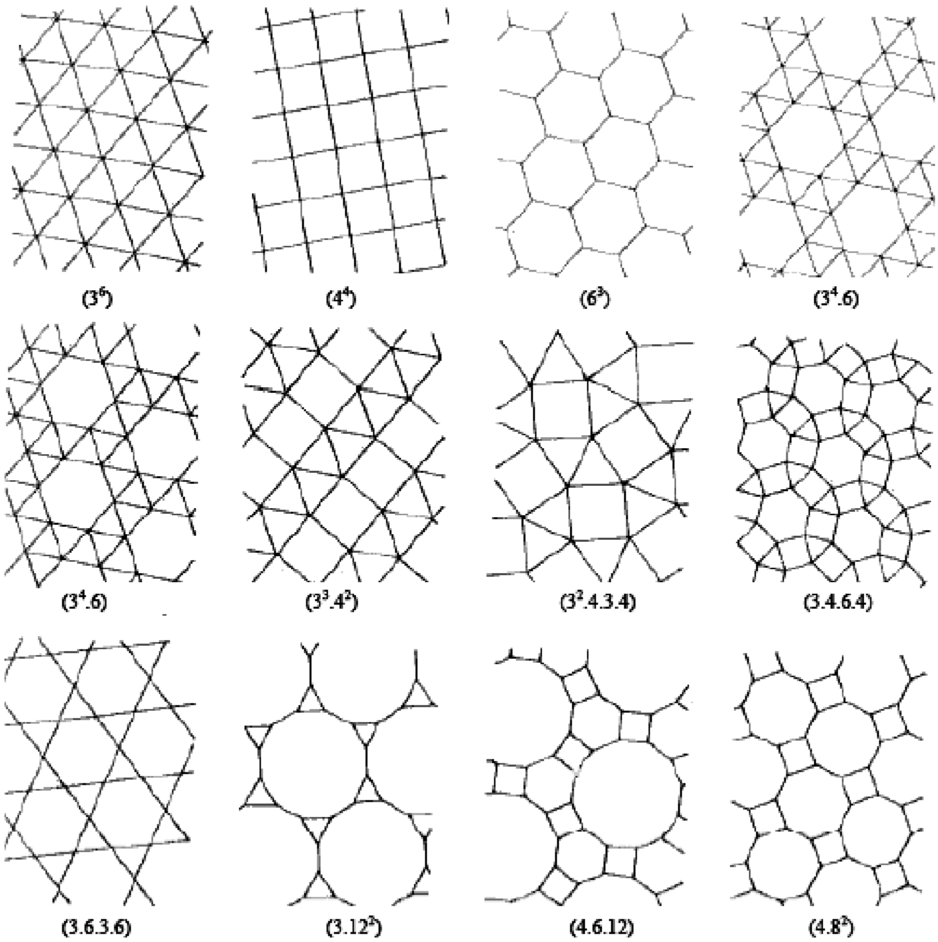


Fig 2.1.1

***Proof**:* Consider a surface to be covered with regular polygons, leaving no spaces between the meeting points of their vertices. If n denotes the number of sides each regular polygon will have, then the interior angles at each vertex of each polygon will be $(n-2)180^0/n$.

At each vertex, there will be $360/[(n-2)180/n] = 2n/(n-2) = 2 + 4/(n-2)$ such polygons. For this to be a whole number for $n > 2$, n must have values equal to 3, 4 or 6. Therefore, this corresponds to tilings which consist of regular hexagons, squares and equilateral triangles respectively.

Next, we consider a vertex which is surrounded by more than one type of polygon. Firstly, it is noted from above that there cannot be less than 3 polygons nor more than 6 polygons around a vertex. Secondly, it is also noted that there cannot be four or more different types of polygons around a vertex. This is because if we add up the four different polygons with the smallest angles, that is triangle with angle of 60^0, square with angle of 90^0, pentagon with angle of 108^0 and hexagon with angle of 120^0, we get a total of 378^0 which is greater than 360^0.

Let us consider case by case.

Case 1: 3 polygons

The sum of the interior angles of the 3 polygons around a vertex add up to 360^0, i.e.
$[(n_1-2)/n_1 + (n_2-2)/n_2 + (n_3-2)/n_3]\ 180^0 = 360^0$,

By simplifying the above equation, we get
$1/n_1 + 1/n_2 + 1/n_3 = ½$

We find that the solutions which satisfy the above equation are as follows: (Solutions indicated with ** will be explained later)

(1) 3.7.42 **
(2) 3.8.24 **
(3) 3.9.18 **
(4) 3.10.15 **
(5) 3.12.12
(6) 4.5.20 **
(7) 4.6.12
(8) 4.8.8
(9) 5.5.10 **
(10) 6.6.6 (note that this is the regular tiling by hexagons)

Case 2: 4 polygons

Following the same procedure as above, we have

$1/n_1 + 1/n_2 + 1/n_3 + 1/n_4 = 1$

Thus, the solutions which satisfy the above equation are as follows: (Solutions indicated with * will be explained later.)

(11) (i) 3.3.4.12 * (ii) 3.4.3.12 *
(12) (i) 3.3.6.6 * (ii) 3.6.3.6
(13) (i) 3.4.4.6 * (ii) 3.4.6.4
(14) 4.4.4.4 (note that this is the regular tiling by squares)

Similarly, we can obtain solutions for Cases 3 and 4 below:

Case 3: 5 polygons

$1/n_1 + 1/n_2 + 1/n_3 + 1/n_4 + 1/n_5 = 3/2$

(15) 3.3.3.3.6
(16) (i) 3.3.3.4.4 (ii) 3.3.4.3.4

Case 4: 6 polygons

$1/n_1 + 1/n_2 + 1/n_3 + 1/n_4 + 1/n_5 + 1/n_6 = 2$

(17) 3.3.3.3.3.3 (note that this is the regular tiling by equilateral triangles)

As shown above, to cover or surround the vertex without gaps or overlaps, there are 17 solutions/choices of polygons, meaning these 17 solutions/choices satisfy the above 4 equations. We call each choice a *species*. In other words, the number of each kind of regular polygon surrounding a vertex determines the species of the vertex. We say that the vertices in a tiling have the *same species* if all of them are surrounded alike in the number of each kind of regular polygon. For example, 11(i) and (ii) belong to the same species.

Let us first define what is meant by *type* of vertex. A vertex is of type $n_1.n_2….n_r$ if it is surrounded in cyclic order by regular n-gons of order n_1, n_2,…..and n_r. The cyclic order can be taken in clockwise or anticlockwise direction. However, n_1 has to be of the lowest order of all polygons surrounding the vertex. In order to obtain a unique symbol for each type of vertex, we shall always choose the sequence that is lexicographically first among all possible expressions.

It can be seen that in 4 of the species, there are 2 distinct ways in which the polygons may be arranged, meaning the position of the polygons is different, resulting in different look and shape. Take the vertices of types (3.3.6.6) and (3.6.3.6) shown in Fig 2.1.2 for example. Both belong to the same species but they are different. For the type (3.3.6.6), the triangles are adjacent to each other, same for the hexagons. But for type (3.6.3.6), the triangles are opposite to each other, same for the hexagons. Note that for the case with 3 polygons, all have only 1 distinct type by virtue of the fact that no matter how you permutate the 3 polygons around a vertex, the look and shape is still the same. Therefore, there are altogether 21 possible types of vertices.

Fig. 2.1.2 shows the 21 types of vertices possible with regular polygonal tiles.

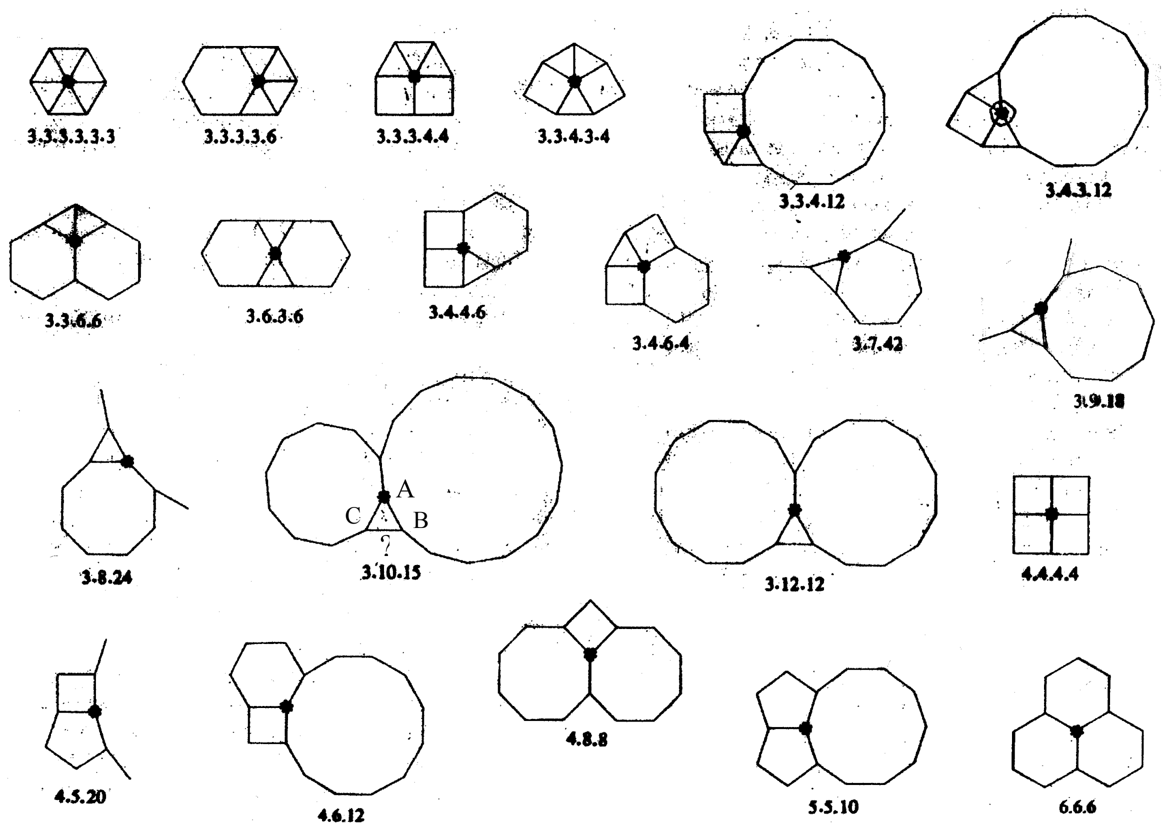
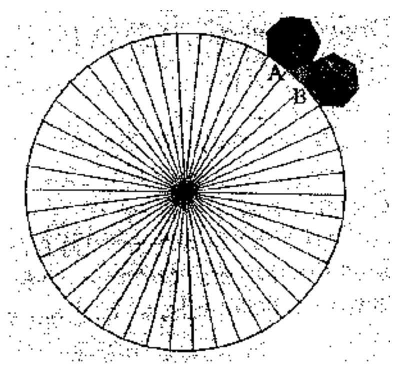


Fig 2.1.2

We shall restrict our study to those tilings whose vertices are of the same type. As shown above, we have found 21 ways to fit regular polygons around a vertex. Unfortunately, not all of these extend to tilings of the plane. We have only found possible tilings as we know only that these vertex combinations add up to 360^0 at each vertex. Now, our goal is to find out what is the total number of edge-to-edge tilings which use polygons as tiles and whose vertices are of the same type.

Let us look at the following example:



If we consider the vertex of type (3.7.42), the pattern starts like the picture above. The 42-gon is drawn with radial lines to help distinguish the sides, since a 42-sided polygon looks very much like a circle. We must have a triangle, a heptagon, and a 42-gon at each vertex. If we arrange this at vertex A and vertex B, then we are forced to have the triangle and two heptagons at vertex C. There is no room to squeeze in a 42-gon at vertex C; therefore, this pattern cannot be extended to a tiling of the plane.

1) Consider a vertex of type 3.x.y, x ≤ y.

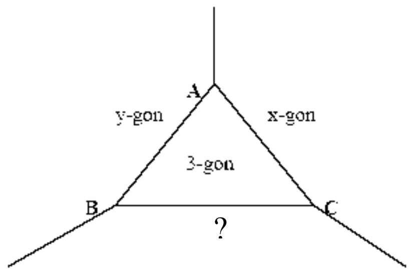


Fig 2.1.3

At vertex A, we have the vertex configuration 3.x.y. A triangle and a y-sided polygon meet at B, so the polygon marked "?" ought to have x sides. But at vertex C, there are already a triangle and a x-sided polygon, so the polygon marked "?" should have y sides. Thus, we cannot have any vertex configuration of the form 3.x.y if y ≠ x. Hence, if 3.x.y is possible, x = y. Therefore, the vertex of type (1), (2), (3), (4) are not possible.

2) Similarly, consider a vertex of type x.5.y, x ≤ 5

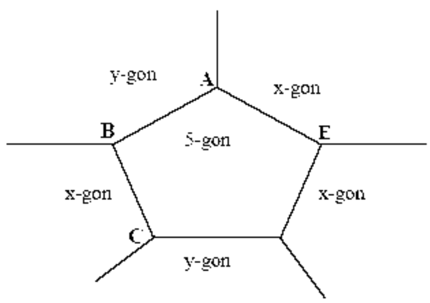


Fig 2.1.4

Consider a regular 5-gon ABCDE. All the vertices are of the type x.5.y, except for vertex E, which is forced to be of the type x.5.x. Therefore, if x.5.y is possible, x = y. Thus, the vertex of type (6) and (9) are not possible.

3) Lastly, consider a vertex of type 3.x.y.z.

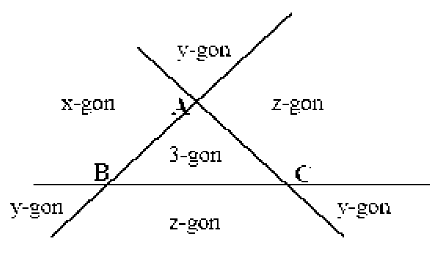


Fig 2.1.5

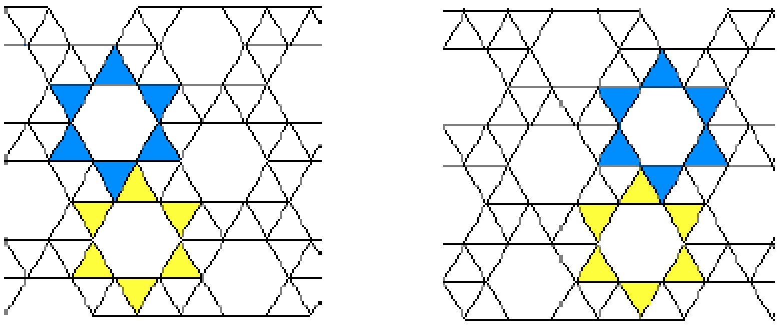
Using the same reasoning as above, we can show that if 3.x.y.z is possible, x = z. Thus, the vertex of type (11)(i) & (ii), (12)(i) and (13)(i) are not possible.

Therefore, all the solutions (altogether 10 of them) marked with * and ** are not possible and 11 types of vertices remain. It can be shown that each of them can be extended to form tilings (See Fig 2.1.1). They are usually called Archimedean tilings (some authors call them homogeneous or semiregular), and they clearly include the three regular tilings. We call them "Archimedean" as it simply means tilings with only one type of vertex. In general, there can be more than one type of vertices in the tiling. If there are k types of vertices, we call it k-Archimedean.

Let us discuss some of the special properties of Archimedean tilings.

Firstly, we observe that the vertex of (3^4. 6) occurs in two forms which are mirror images of one another. We recall from Section 1.3 that two tilings are equal if one may be made to coincide with the other by a rigid motion of the plane (possibly including reflection) followed by a change of scale. In fact, except for the tiling (3^4.6), reflections are never required to establish the equality of two tilings of the same type. But in the case of (3^4.6), reflections may be required and we describe this situation by saying that (3^4.6) occurs in two enantiomorphic (mirror image) forms.

Figure 2.1.6 below shows clearly the two (3^4.6) tilings. In (a), the topmost yellow triangle slides to the right of the bottom-most blue triangle. In (b), the topmost yellow triangle slides to the left of the bottom-most blue triangle. Tiling (a) is known as the right-handed version while tiling b is known as the left-handed version.



(a) (b)

Fig 2.1.6

Secondly, one accidental but very special and important feature of the Archimedean tilings is that each is isogonal. Therefore, we call Archimedean tilings also uniform. The distinction between the meanings of the two words is that "Archimedean" refers only to the fact that the tiling is monogonal, that is, the immediate neighbourhood of any two vertices "look the same", while the term "uniform" implies the much stronger property of isogonality. We will discuss the meaning of k-uniform in the Section 2.2.

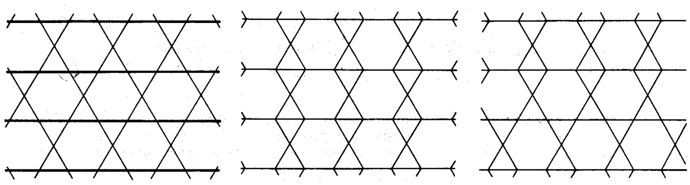
Thirdly, we look at the difference between the solutions marked by * and **. The similarity is that they cannot be extended to form an Archimedean tiling. However, those solutions marked by * can be extended to form a non-Archimedean tiling (A non-Archimedean tiling is a tiling with more than one type of vertices) but those solutions marked by ** cannot be extended to form a non-Archimedean tiling as they give only one vertex each without possibility of repetition as explained below.

Consider the vertex of type (3.10.15) in Fig 2.1.2. As shown in this picture, vertex A is of type (3.10.15). Vertex B has 2 possibilities. Either it has the vertex of type (3.10.15) or type (3.x.y…..15), where x, y….are the sides of polygons around B. Looking at all the possible vertex types that can occur, we can only have the type (3.10.15). The same can be said for vertex C. At vertex B, the polygon marked by "?" has 10 sides. However, at vertex C, the same polygon has 15 sides. This is not possible. That is why solutions marked by ** cannot be extended to form a tiling.

Therefore, there are 21- 6 = 15 kinds of vertex that can occur in a plane tiling with regular polygons. We can see in Section 2.2 and Section 2.3 that there are tilings with combination of vertex of types marked with * and other types of vertex, such as (3^6; **3^2.4.12**), (3^6; **3^2.6^2**), (**3.4^2.6**; 3.4.6.4) and (**3.4.3.12**; 3.12^2), where the vertex marked with * are in bold. There is at least one tiling with a combination of the vertex of type marked with * and other vertex types.

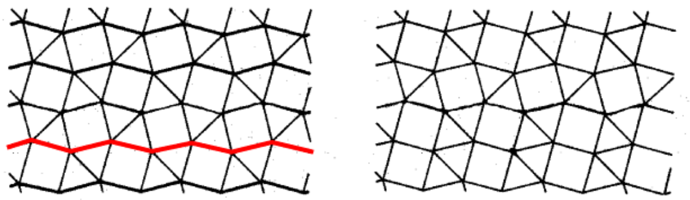
It will be shown below that if we require an edge-to-edge tiling which consists of regular polygons and that all its vertices are of the same species instead of the same type of vertex, then there are infinitely many distinct tilings. This was already noted by Robin[1887]. Let us look at the following examples.

(1)



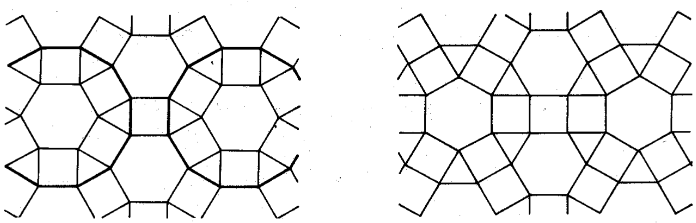
If at each vertex there are two triangles and two hexagons (3.6.3.6), as shown in the pictures above, the uniform tiling (3.6.3.6) may be cut by parallel lines (emphasized on the left) into strips that may be slid independently of one another. In this way we obtain an uncountable infinity of tilings all of whose vertices are of the same species, (3.6.3.6) and (3.3.6.6).

(2)



In the uniform tiling (3^2.4.3.4), there are zigzags of edges as shown on the left. The tiling may be cut along any such zigzag and then one half replaced by its mirror image. This process leads to an uncountable infinity of distinct tilings with all vertices of the same species (3^2.4.3.4) and (3^3.4^2).

(3)

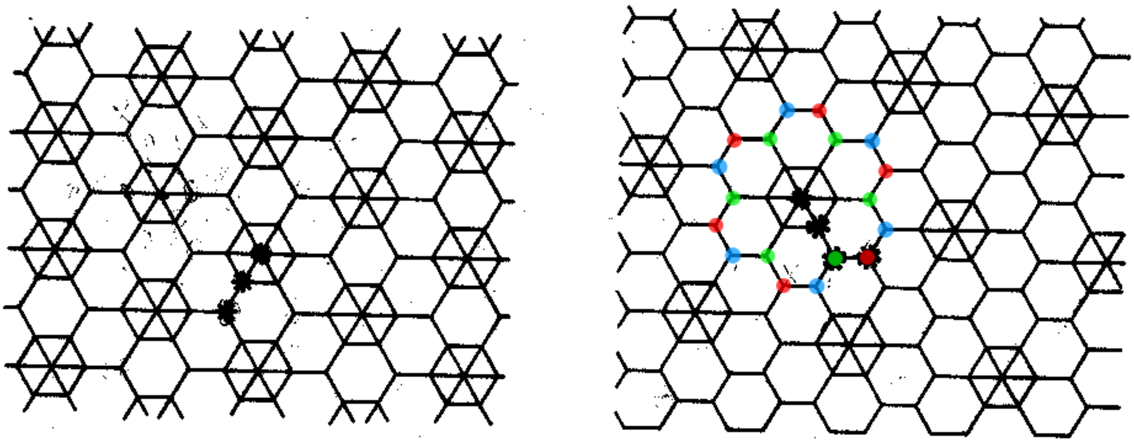


In the uniform tiling (3.4.6.4), there are "disks", each consisting of a hexagon and its neighbours, as indicated on the left. Any such disk may be rotated through the angle 30^0 without altering the species of the vertices. Because pairwise disjoint disks can be independently rotated, this procedure yields an uncountable infinity of distinct tilings with all vertices of species (3.4.6.4) and (3.4.4.6).

2.2 k-Uniform Tilings

For a tiling with tiles of regular polygons, we call it k-uniform if and only if it is k-isogonal (we recall in Section 1.6 that a tiling is k-isogonal if its vertices form precisely k transitivity classes with respect to the group of symmetries of the tiling). In this terminology, uniform tilings are 1-uniform. An example is shown below to illustrate this definition.

Let's take a look at Fig 2.2.1 below.



(a) (b)

Fig 2.2.1

In both Fig 2.2.1(a) and (b), the tilings consist of 3 types of vertices, (3^6), (6^3) and ($3^2.6^2$), thus we know that these 2 tilings must be at least 3-uniform.

For Fig 2.2.1(a), vertices of type (3^6) can be mapped onto one another by translations; therefore they form 1 transitivity class. Vertices of type (6^3) can be mapped onto one another by 60^0 rotation followed by translations, therefore they form 1 transitivity class. Similar to vertices of type (6^3), vertices of type ($3^2.6^2$) also form 1 transitivity class. As there are a total of 3 transitivity classes, we can conclude that Fig 2.2.1(a) is 3-uniform.

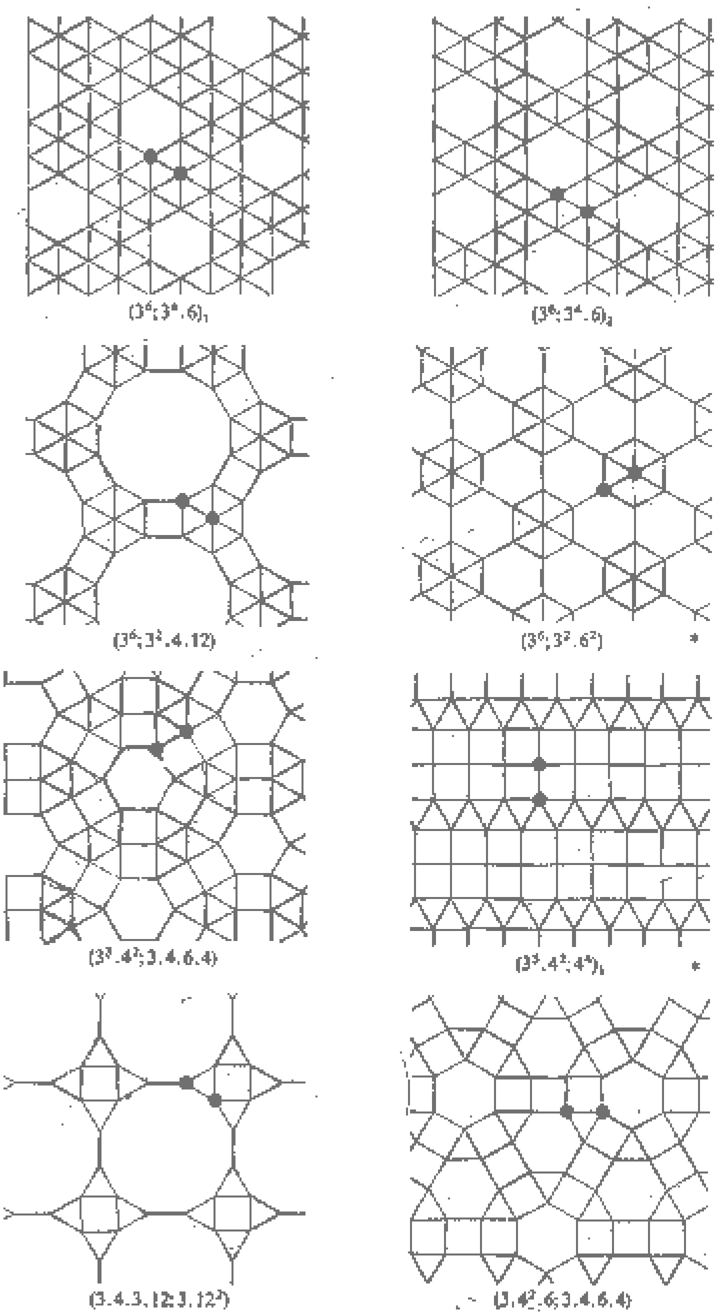
Similarly, for Fig 2.2.1(b), vertices of type (3^6) and vertices of type ($3^2.6^2$) form 1 transitivity class each. However, vertices of type (6^3) form 3 transitivity classes, as the vertices indicated by red dots, green dots and blue dots can be mapped onto one another by 60^0 rotation and they can all be mapped to other corresponding (6^3) vertices by translations. As Fig 2.2.1(b) has altogether 5 transitivity classes, it is 5-uniform.

If the types of vertices in the k transitivity classes are $a_1.b_1.\ c_1$......; $a_2.b_2.c_2$……;….; $a_k.b_k.c_k$…the tiling will be designated by the symbol ($a_1.b_1.\ c_1$......; $a_2.b_2.c_2$……;….; $a_k.b_k.c_k$…), with the obvious shortening through the use of superscripts (e.g. we can write 3.3.3.3.3.3 as 3^6), and with subscripts to distinguish distinct tilings in which the same types of vertices appear.

We will state without proof the following result obtained by Krotenheerdt.

Theorem 2: There exist 20 distinc*t* types of 2-uniform edge-to edge tilings by regular polygons, namely:
$(3^6; 3^4.6)_1$, $(3^6; 3^4.6)_2$, $(3^6; 3^3.4^2)_1$, $(3^6; 3^3.4^2)_2$, $(3^6; 3^2.4.3.4)$, $(3^6; 3^2.4.12)$, $(3^6; 3^2.6^2)$, $(3^4.6; 3^2.6^2)$, $(3^3.4^2; 3^2.4.3.4)_1$, $(3^3.4^2; 3^2.4.3.4)_2$, $(3^3.4^2; 3.4.6.4)$, $(3^3.4^2; 4^4)_1$, $(3^3.4^2; 4^4)_2$, $(3^2.4.3.4; 3.4.6.4)$, $(3^2.6^2; 3.6.3.6)$, $(3.4.3.12; 3.12^2)$, $(3.4^2.6; 3.4.6.4)$, $(3.4^2.6; 3.6.3.6)_1$, $(3.4^2.6; 3.6.3.6)_2$ and $(3.4.6.4; 4.6.12)$.

These tilings are shown in Fig.2.2.2. The solid dots indicate one vertex of each class.



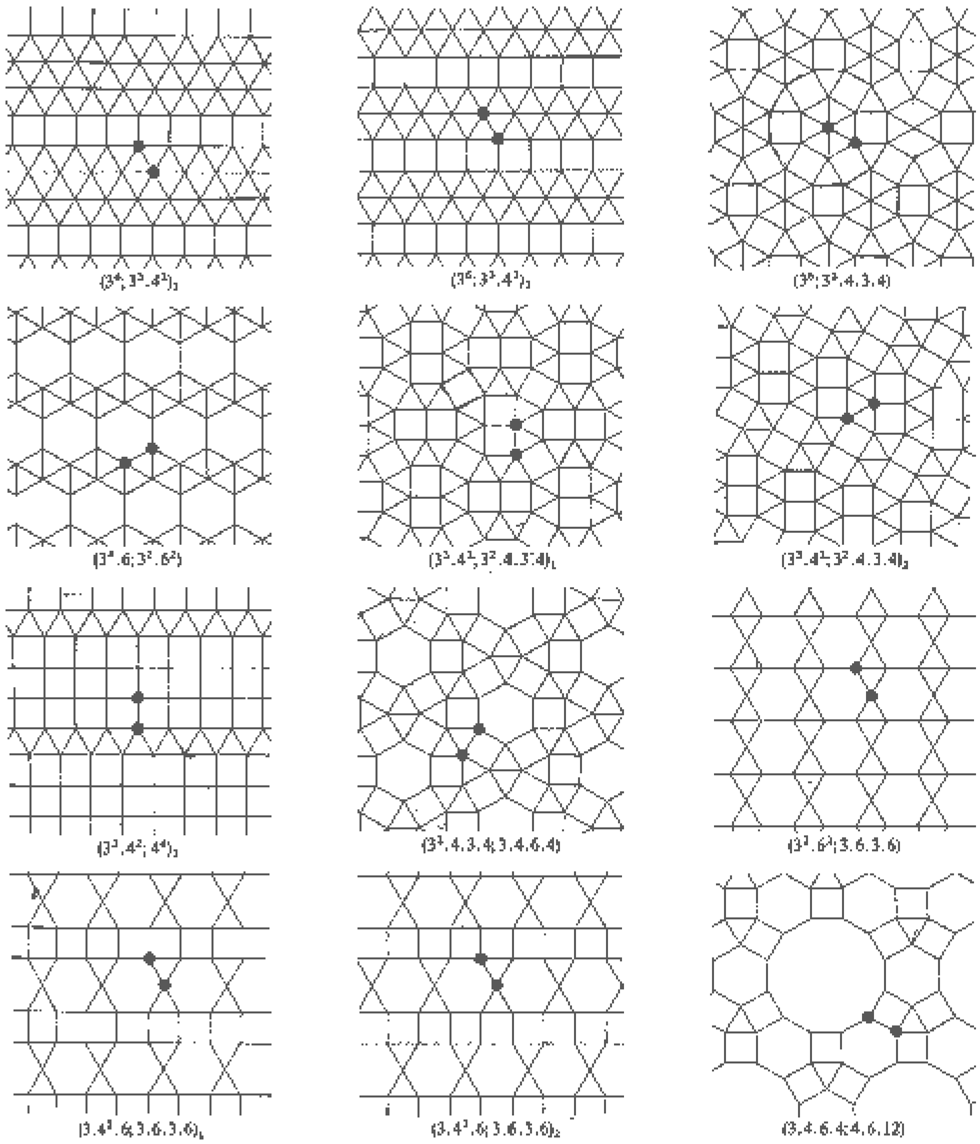


Fig 2.2.2

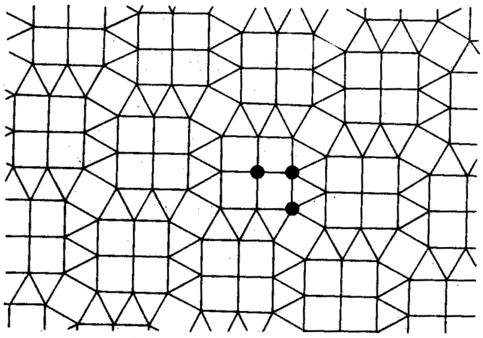
In a series of papers, Krotenheerdt examined a closely related question. He considered those k-uniform tilings in which the k transitivity classes of vertices consist of k distinct types of vertices. If we denote by K(k) the number of distinct Krotenheerdt tilings, his results are:

K(1)=11, K(2)=20, K(3)=39, K(4)=33, K(5)=15, K(6)=10, K(7)=7, and K(k)=0 for each $k \geq 8$.

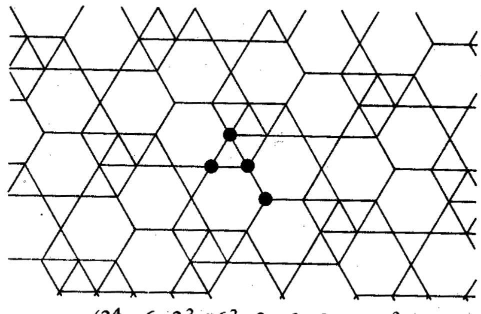
For k=1 and k=2, his results are the same as the tilings shown earlier, meaning that the results coincide with the uniform and 2-uniform ones respectively. (See remarks made at the end of this section).

For k=3 and above, his definition is more restrictive. Note that for k >15, Krotenheerdt finds no tilings, meaning K(15) = 0. As explained in Section 2.1, there are only 15 kinds of vertex that can occur in a plane tiling with regular polygons, so we know for sure that K(k) = 0 for k >15. Vertex 488 can only occur in the 488 tiling, meaning there is no way to substitute any other polygons, so the number of possible different vertices drops to 14. Krotenheerdt showed that in fact K(k) = 0 for $k \geq 8$.

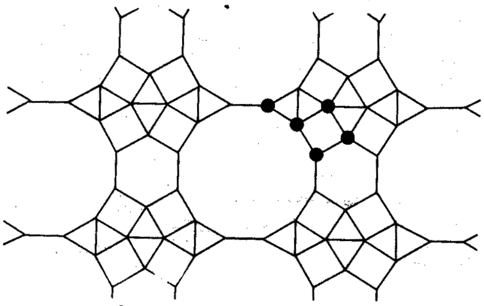
Some examples of Krotenheerdt tilings with k=3,4,5,6 and 7 are shown below.



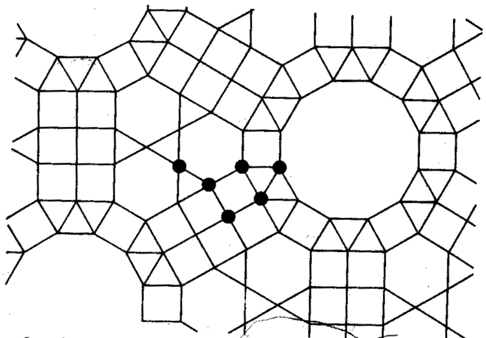
$(3^3 . 4^2; 3^2 . 4 . 3 . 4; 4^4)$



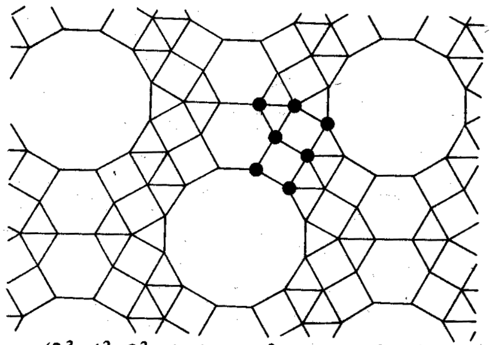
$(3^4 . 6; 3^2 . 6^2; 3 . 6 . 3 . 6; 6^3)$



$(3^2 . 4 . 3 . 4; 3^2 . 4 . 12; 3 . 4 . 6 . 4; 3 . 12^2; 4 . 6 . 12)$



$(3^3 . 4^2; 3^2 . 4 . 12; 3 . 4^2 . 6; 3 . 4 . 6 . 4; 3 . 6 . 3 . 6; 4^4)$



$(3^3 . 4^2; 3^2 . 4 . 3 . 4; 3^2 . 4 . 12; 3^2 . 6^2;$
$3 . 4 . 3 . 12; 3 . 4^2 . 6; 4 . 6 . 12)$

Fig 2.2.3

Examples of k-uniform tilings with k=3,4,5,6 and 7

However, if we do not impose Krotenheerdt condition, even for small k such as k=4, it is not known how many k-uniform tilings exist, nor is any kind of asymptotic estimate available for the number of k-uniform tilings with large k. We just state here that for k=3, Chavey [1984b] determined that there are 61 tilings of this kind.

Remarks about k-uniform and k-Archimedean tilings

1. If a tiling is m-Archimedean and n-uniform, we can say that m ≤ n, meaning that if a tiling has m different types of vertices, it must also be at least m uniform, in other words, each type of vertex form 1 transitivity class. When m = n, the tiling is a Krotenheerdt tiling.

2. We have already seen from Fig 2.1.1 that there are 11 Archimedean tilings (although there are 12 tilings in the figure, $(3^4.6)$ occurs in 2 forms which are the mirror

images). It is coincidental that all these 11 Archimedean tilings are 1-uniform. So, for k=1, there is no difference between 1 Archimedean and 1 uniform tiling.

3. We have also seen that there are 20 distinct types of 2-uniform edge-to-edge tilings by regular polygons. However, unlike for the case where k=1, there are infinite number of 2-Archimedean tilings. It was shown in Section 2.1 that for ($3^3.4^2$ and $3^2.4.3.4$), ($3^2.6^2$ and 3.6.3.6), or ($3.4^2.6$ and 3.4.6.4), there exist infinitely many distinct 2-Archimedean tilings.

2.3 Demi-regular Tilings

Demi-regular tiling is defined as k-uniform where k >1, meaning that vertices form k transitivity classes. In each of the transitivity classes, the vertices can be mapped onto one another by a symmetry of the tiling. This will be illustrated below.

Although the definition implies that there are infinite number of them, many books claimed that there are 14 demi-regular tilings. However, even for k=2, we know from Section 2.2 that there are 20 of them which is already more than the 14 ones which the books have claimed. Another inconsistency is that although many books claimed that there are 14 demi-regular tilings, not all the books gave the same 14 demi-regular tilings. Therefore, in this project, we will analyse all the pictures given by the following 5 sources:

a) “Order In Space” by Keith Critchlow.
b) “Sacred Geometry” by Lundy.
c) “Geometry of Art and Life” by Ghyka.
d) “Mathematical Snapshots” by Steinhaus.
e) “Demi-regular Tessellation from MathWorld” by Eric W.Weisstein.

We will compile a table listing all the pictures in the above 5 references and indicate which are the pictures under each of the references. This will show clearly whether the same pictures appear in the references.

We will also analyze, for each picture,

1. How many types of vertices are there?
2. How many species of vertices are there?
3. Determine its uniformity.
   (i) Identify the symmetries of the tiling;
   (ii) Find out the number of transitivity classes for each type of vertex.
4. Is it a Krotenheerdt tiling?

We first start with the pictures from the book “Order in Space” by Critchlow.

The 14 demi-regular tilings from Critchlow, as shown in pictures 1 to 14 in Appendix 1 are analyzed and found to be made up of:

(i) Krotenheerdt tiling

7 of them are 2 uniform, (6 have 2 types and 2 species and 1 has 2 types and 1 species).

(ii) Non Krotenheerdt tiling

a) 1 has 3 types and 3 species, which is 7 uniform.
b) 4 have 3 types and 2 species, 2 are 4 uniform, 1 is 7 uniform and 1 is 9 uniform.
c) 1 has 2 types and 1 species, which is 4 uniform.
d) 1 has 2 types and 2 species, which is 5 uniform.

The analysis below shows how the above results are obtained. For the 2-uniform ones, the pictures by Critchlow are compared with those pictures shown in Fig 2.2.2.

Picture 1:
- 2 types of vertices (3.4.3.12; 3.12^2).
- 2 species
- 2 uniform.
- Krotenheerdt tiling

Picture 2:
- 2 types of vertices (3^6; 3^2.4.12).
- 2 species
- 2 uniform.
- Krotenheerdt tiling.

Picture 3:
- 3 types of vertices (3^2.4.3.4; 3^2.4.12; 3.4.3.12)
- 2 species
- 4 uniform
- not a Krotenheerdt tiling.

Types of symmetry:
1) Rotation through 90^0 about the centre of a 12-gon and the centre of a square surrounded by 4 triangles.
2) Translation

It is noted that there is no reflection symmetry. We look at one of the 12-gons and denote the vertex (3^2.4.12) by A and (3.4.3.12) by B. The pattern is AABAAB….All the Bs can be mapped onto one another by a 90^0 rotation. All the As denoted in blue can be mapped onto one another by a 90^0 rotation. However, there is no symmetry that maps the As in blue to the As in green. But all the As in green can be mapped by a 90^0 rotation. Therefore, (3^2.4.12) form 2 transitivity classes, and (3.4.3.12) form 1 transitivity class.

It is also noted that (3^2.4.3.4) form 1 transitivity class since the vertices can be mapped onto one another by a 90^0 rotation about the centre of a square surrounded by 4 triangles. Then, these vertices can be mapped onto other (3^2.4.3.4) vertices by translations.

Thus, we can conclude that the vertices form 4 transitivity classes altogether, meaning that picture 3 is 4 uniform. Since it has only 3 types of vertices, it is not a Krotenheerdt tiling.

Picture 4:
- 2 types of vertices (3.4.6.4; 4.6.12)
- 2 species
- 2-uniform
- Krotenheerdt tiling.

Picture 5:
- 3 types of vertices (3^6; 3^2.4.12; 3^2.4.3.4)
- 3 species
- 7 uniform
- not a Krotenheerdt tiling.

Types of symmetry:
1) Horizontal and vertical reflection.
2) Horizontal and vertical translation.
3) Rotation through 180^0 about the centre of a 12-gon/ square surrounded by 4 triangles.

All the vertices of the type (3^2.4.12) lie around a 12-gon. They form 3 transitivity classes: A, B and C. Note that vertex A is different from vertex B as A is incident to a 'connecting' square (a square which connects 2 12-gons) while B is not. Similarly, B is different from C as C is incident to a 'connecting hexagon' while B is not. (we view the 6 triangles together as a hexagon).

It can be seen that vertices of the type (3^6) form 2 transitivity classes: D and E. Also, vertices of the type (3^2.4.3.4) form 2 transitivity classes: F and G. All the vertices can be translated horizontally or vertically to the corresponding vertices in other parts of the whole picture.

Thus, we can conclude that all the vertices together form 7 transitivity classes, meaning picture 5 is 7 uniform. Since it has only 3 types of vertices, it is not a krotenheerdt tiling.

Picture 6:
- 2 types of vertices ($3^2.6^2$; 3.6.3.6)
- 1 species
- 2 uniform
- Krotenheerdt tiling

Picture 7:
- 2 types of vertices (3^2.4.3.4; 3.4.6.4)
- 2 species
- 2 uniform
- Krotenheerdt tiling

Picture 8:
- 3 types of vertices (3.4.6.4; 3^2.4.3.4; $3^3.4^2$)
- 2 species
- 9 uniform
- Not a Krotenheerdt tiling

Types of symmetry:
1) Translations
2) Rotations through 180^0 about the centre of any hexagon
3) Vertical and Horizontal reflections

It can be easily seen that the vertices of type ($3^3.4^2$) form 1 transitivity class A as denoted in the diagram.

The vertices of type (3.4.6.4) forms 2 transitivity classes: B, C, D and E. By looking at the hexagon shown in the picture whose vertices are denoted by B and C, it is noted that vertex B is different from vertex C as B is incident to a connecting diamond while C is not. There is no symmetry to map these vertices to the vertices of the hexagon on its right. This explains why there are another 2 transitivity classes, D and E.

Similarly, vertices of type (3^2.4.3.4) form 4 transitivity classes: F, G, H and I. (by looking at the vertices surrounding the 12-gon)

Thus, we can conclude that the 3 types of vertices together form 9 transitivity classes. Therefore, Picture 8 is 9 uniform. Since it has only 3 types of vertices, it is not a Krotenheerdt tiling.

Picture 9:
- 2 types of vertices (3^6; 3^2.4.3.4)
- 2 species
- 5 uniform
- not a Krotenherdt tiling

Types of symmetry:
1) Horizontal and vertical reflections
2) Horizontal and vertical translations
3) Rotational symmetry of 180^0

It is obvious that vertices of type (3^6) forms 1 transitivity class, E. We view the 6 triangles together again as a hexagon. The vertices lying around a hexagon form 2 transitivity classes: C and D. The vertices surrounding the 12-gon (the outer layer). form another 3 transitivity classes: A, B and C. (as shown in the picture).

Since all other vertices are equivalent to these vertices by horizontal and vertical translations, we can conclude that the 2 types of vertices together form 5 transitivity classes. Therefore, Picture 9 is 5 uniform. Since it has only 2 types of vertices, it is not a Krotenheerdt tiling.

Picture 10:
- 2 types of vertices (3^6; 3^2.4.3.4)
- 2 species
- 2 uniform
- Krotenheerdt tiling

It is noted that although Picture 10 has exactly the same type of vertices and species as Picture 9, Picture 10 is 2 uniform and a Krotenheerdt tiling unlike Picture 9. By observing Picture 9 and 10 again and viewing the 6 triangles together as a hexagon, it is noted that in Picture 10, the hexagons are linked only by squares while in Picture 9, the hexagons are linked either by squares or diamonds.

Picture 11:
- 3 types of vertices (3^6; 3^2.4 3.4; $3^3.4^2$)
- 2 species
- 7 uniform
- Not a Krotenheerdt tiling

Types of symmetry:
1) Rotation through 180^0 about the vertex type of 3^6.
2) Horizontal and vertical reflections.
3) Horizontal and vertical translations.

Obviously, the vertices of type (3^6) form 1 transitivity class G as they can be mapped onto one another by horizontal or vertical translations.

The vertices of type ($3^3.4^2$) form 2 transitivity class: E and F. There is no symmetry to map class D onto Class E. Another way you can view it is as follows: Vertices of class D are incident to squares which are connected to hexagons while vertices of class E are incident to squares which are connected to triangles. Thus, we can see that vertices belonging to class D and E are different.

The vertices of type (3^2.4.3.4) form 4 transitivity classes (shown in the picture), A, B, C and D. This is because by looking at the vertices along the hexagons (think of the 6 triangles together as a hexagon) as well as a 12-gon, we can see that these can be translated horizontally and vertically to the corresponding vertices in the picture.

Thus, we can conclude that the 3 types of vertices together form 7 transitivity classes. Therefore, Picture 11 is 7 uniform. Since it has only 3 types of vertices, it is not a Krotenheerdt tiling.

Picture 12:
- 3 types of vertices (3^6; 3^2.4.3.4; $3^3.4^2$)
- 2 species
- 4 uniform
- Not a Krotenheerdt tiling

Types of symmetry:
1) Horizontal and vertical translations
2) 180^0 rotations
3) Reflections along the axis shown in bold in the picture.

The vertices of type (3^6) form 2 transitivity classes: A and B. (View the 6 triangles connected together as a hexagon). The hexagon in the centre between the 2 other hexagons has its vertices at the top and bottom while the other 2 hexagons have their edges at the top and bottom. There is no symmetry to map the vertices from class A onto class B. All the vertices of type (3^2.4.3.4) lying around the hexagon (shown in the picture) form 1 transitivity class by reflections. All the vertices of type ($3^3.4^2$) lying around the 12-gon also form 1 transitivity class by reflections. They can be translated horizontally and vertically to the corresponding vertices in other parts of the picture.

We can conclude that the 3 types of vertices together form 4 transitivity classes. Therefore, Picture 12 is 4 uniform. Since it has only 3 types of vertices, it is not a Krotenheerdt tiling.

Picture 13:

- 2 types of vertices (3.4.6.4; 3.4.4.6)
- 1 species
- 4 uniform
- Not a Krotenheerdt tiling

Types of symmetry:

1) 60^0 rotation about the centre of a hexagon with “arms” sticking out.
2) 180^0 rotation about the centre of another hexagon surrounded by 4 squares and 2 triangles.
3) Horizontal and vertical translations.
4) Horizontal and vertical reflections.

Let’s look at a hexagon with arms sticking out. All the vertices around this hexagon are of the type (3.4.6.4). They can be mapped onto one another by the above-mentioned symmetry. Let’s call them ‘A’.(shown in the picture). However, A cannot be mapped onto B and D using any symmetry above. Another way of looking at it is that A is incident to a triangle which is connected to a hexagon while B is incident to a triangle connected to a square. Therefore the vertices of type (3.4.6.4) form 3 transitivity classes, A, B and D.

The vertices of type (3.4.4.6) form 1 transitivity class denoted by ‘C’. It is noted that Cs is at the outer layer and can be mapped onto one another by using the above-mentioned symmetry.

We can conclude that there are 2 types of vertices and they form 4 transitivity classes. Therefore, Picture 13 is 4 uniform. Since it has only 2 types of vertices, it is not a Krotenheerdt tiling.

Picture 14:

- 2 types of vertices ($3^3.4^2$; 3.4.6.4)
- 2 species
- 2 uniform
- Krotenheerdt tiling

Now, we look at the other references mentioned above. “Sacred Geometry” by Lundy also gives 14 pictures. All the pictures are exactly the same as “Order in Space” by Keith Critchlow except for one picture. This picture replaces Picture 8 by Critchlow.

Picture 8 by Critchlow has 3 types of vertices ($3^2.4.3.4$; 3.4.6.4; $3^3.4^2$) and 2 species while the new one given by Lundy as shown in Picture 15 in Appendix 1 has 2 types of vertices ($3^2.4.3.4$; 3.4.6.4) and 2 species. It does not have the vertex of type ($3^3.4^2$). Analysis show that the vertices of type (3.4.6.4) form 2 transitivity classes and vertices of type ($3^2.4.3.4$) form 3 transitivity classes. Therefore, Picture 15 is 5 uniform. It is also not a Krotenheerdt tiling.

Ghyka, in his book “The Geometry of Art and Life” listed 14 demi-regular tilings as shown in Appendix 2. By studying the listing, we manage to identify 13 pictures. These

13 pictures are the same as those given by Critchlow except for Plate XXV. Plate XXV has 3 types of vertices (3^6; 3^2.4.12; 3^2.4.3.4) and 2 species. (3^6) and (3^2.4.3.4) each forms 1 transitivity class and (3^2.4.3.4) forms 2 transitivity classes. Therefore, Plate XXV is 4-uniform. It is not a Krotenheerdt tiling. It is noted that Plate XXV is similar to Picture 4 by Critchlow. We denote this new tiling as Picture 16 in Appendix 1.

In “Demi-regular Tessellation from MathWorld” by Weisstein (http://mathworld.wolfram.com/DemiregularTessellation.html.), there are 7 pictures shown. Out of the 7 pictures, 4 are from Critchlow (Pictures 2, 4, 7 and 12 in Appendix 1), 1 is from Ghyka (Plate XXV or Picture 16 in Appendix 1) and 2 are from Mathematical Snapshots by Steinhaus (Picture 17 and Picture 18 in Appendix 1). Analysis of Picture 17 and Picture 18 shows that both are 3-uniform.

Steinhaus, unlike the authors in other references, mentioned in his book “Mathematical Snapshots” that the number of demi-regular tilings is infinite. He illustrated 5 pictures, 3 of which are found in the book by Critchlow and the other 2 are also found in the website by Weisstein (Picture 17 and Picture 18 in Appendix 1).

To summarize, the references mentioned above presented altogether 18 distinct demi-regular tilings . These 18 distinct demi-regular tilings are shown in Appendix 1.

The table below shows a list of the 18 pictures and indicates which of them are found in each of the above references. Gh, C, L and S refer to the books by Ghyka, Critchlow, Lundy and Steinhaus respectively and W refers to Weisstein’s website.

Table 1:

Picture	**Gh**	**C**	**L**	**W**	**S**
1	**Listed**	√	√		
2	**Listed**	√	√	√	√
3	**Listed**	√	√		
4	√	√	√	√	
5		√	√		
6	**Listed**	√	√		
7	√	√	√	√	
8	**Listed**	√			√
9		√	√		
10	**Listed**	√	√		
11	**Listed**	√	√		
12	√	√	√	√	
13	**Listed**	√	√		
14	**Listed**	√	√		√
15			√		
16	√			√	
17				√	√
18				√	√
not listed, no pictures	**1**				
Total	**14**	**14**	**14**	**+(see below)**	**++(see below)**

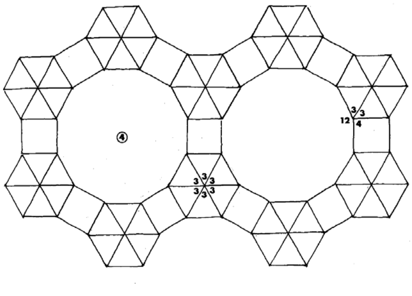
+ Weisstein mentioned that there are 14 demi-regular tilings but only illustrated 7 of them.
++ Steinhaus mentioned that there are infinite number of demi-regular tilings but only illustrated 5.

Note:
Picture 1, 2, 4, 6, 7, 10 & 14 are 2 uniform.
Picture 17 & 18 are 3 uniform.
Picture 3, 12, 13 & 16 are 4 uniform.
Picture 9 & 15 are 5 uniform.
Picture 5 & 11 are 7 uniform.
Picture 8 is 9 uniform.

Conclusion:
As already pointed out by Steinhaus, there are infinitely many demi-regular tilings. We do not understand why Ghyka believed there were only 14. For some reason, this number stuck, and has been repeated many times. The incredible thing is that nobody gives the same 14! All the sources that repeat Ghyka's claim that there are only 14, cite him and usually several other sources. However, when they try to illustrate, they always ignore some pictures from previous sources and include some new ones. Yet they still claim that there are only 14. Maybe this is an example of "sacred geometry"?

Appendix 1

2



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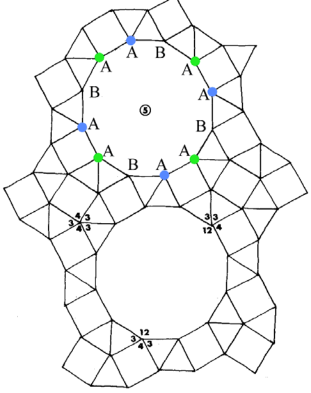

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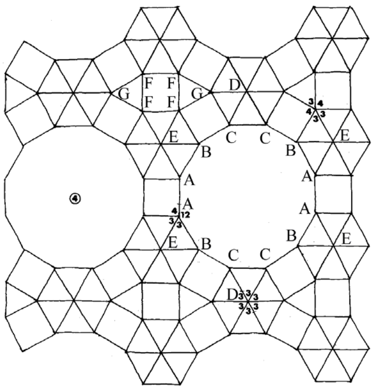

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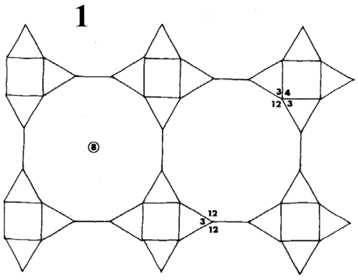

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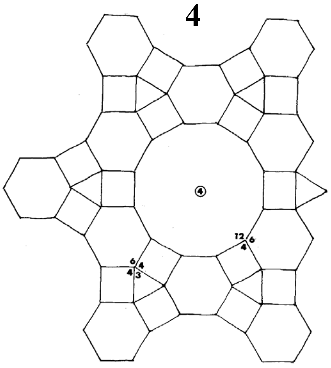

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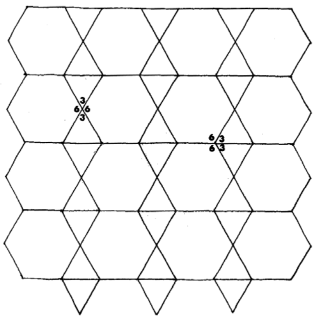

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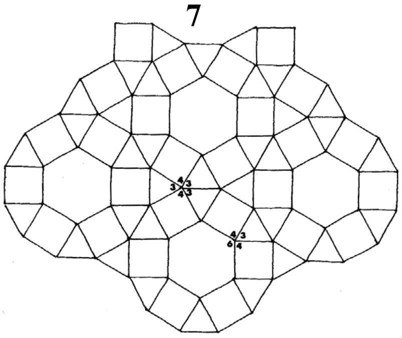

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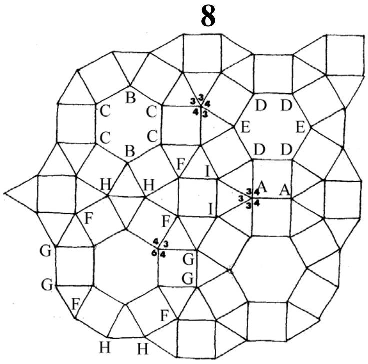

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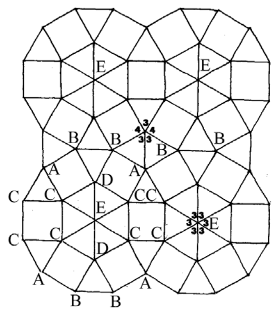

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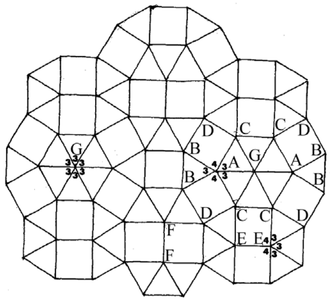

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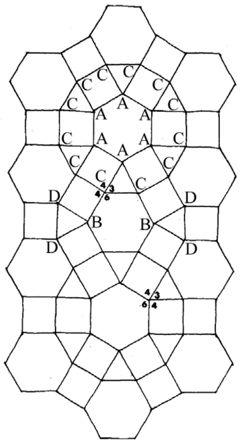

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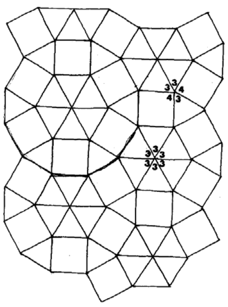

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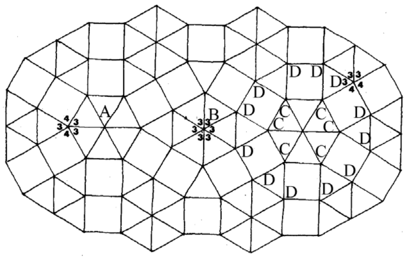

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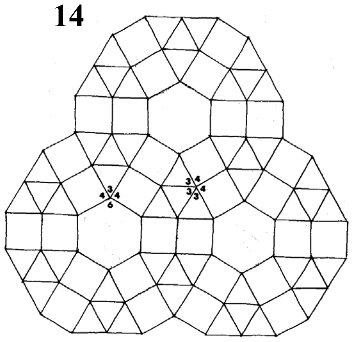

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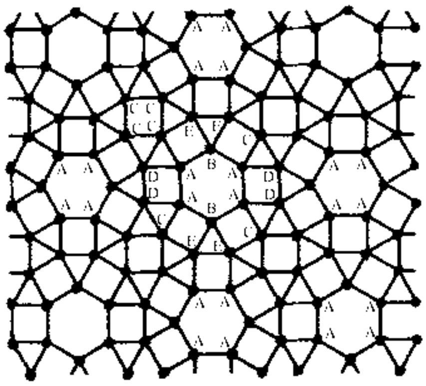

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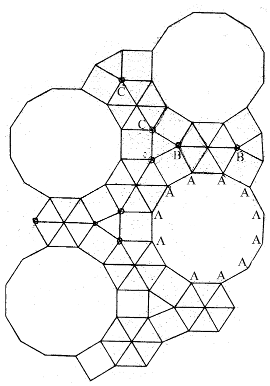

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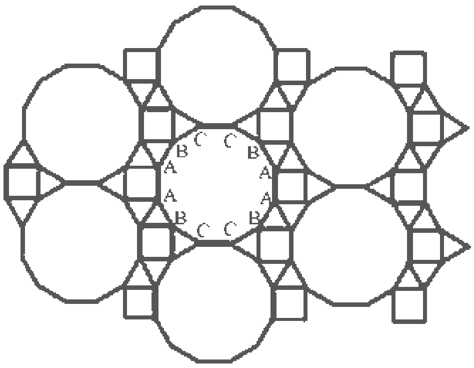

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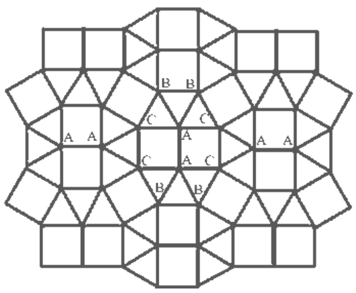

Appendix 2

Analysis of Ghyka's Listings: (The horizontal numbers show the species of the vertices. The vertical rows of angles show the clockwise succession of angles for each type of vertex.)

Listing	Details		Type Of Tiling	Remarks
No.1	3-12-12	150° 150° 60°	Archimedean	
No.2	4-6-12	150° 150° 120° 90° 90° 120°	Archimedean	There are 2 different orientations of the same type of vertex.
No.3	4-8-8	135° 135° 90°	Archimedean	
No.4	6-6-6	120° 120° 120°	Archimedean	
No.5	3-3-4-12	150° 150° 150° 60° 60° 90° 60°	Demi-regular	There could be an error/misunderstanding in the book. From the 2 columns of angles given, there should be another species besides 3-3-4-12, i.e. 3-12-12. This could be Picture 1.
No.5 bis	3-3-4-12	150° 150° 90° 60° 60° 60° 60° 90° 60° 90° 60° 90° 60°	Demi-regular	Same as No. 5, there could be an error/misunderstanding in the book. From the 3 columns of angles given, there should be another species which is 3-3-4-3-4. This could be Picture 3.
.No.6	3-3-6-6	120° 60° 120° 60°	Archimedean	
No.6 bis	3-3-6-6	120° 120° 120° 60° 60° 120° 60° 60°	Demi-regular	The types of vertices are 3-3-6-6 and 3-6-3-6. This could be Picture 6.
No.7	3-4-4-6	120° 120° 60° 60°	Archimedean	

No.7 bis	3-4-4-6	120º 90º 60º 90º	120º 90º 90º 60º	120º 60º 90º 90º	Demi-regular	The types of vertices are 3-4-4-6 and 3-4-6-4. This could be Picture 13.
No.8	4-4-4-4	90º 90º 90º 90º			Archimedean	
No.9	3-3-3-4-4	90º 90º 90º 60º 60º			Archimedean	
No.9 bis	3-3-3-4-4	90º 60º 60º 90º 60º			Archimedean	
No.10	3-3-3-3-6	120º 60º 60º 60º 60º			Archimedean	
No.11	3-3-3-3-3-3	60º 60º 60º 60º 60º 60º			Archimedean	
No.12	4-6-12 3-4-4-6	150º 90º 120º	150º 120º 90º	120º 90º 60º 90º	Demi-regular	This is Plate XXVII as shown in Ghyka's book, which is Picture 4.
No.13	3-4-4-6 3-3-3-4-4	120º 90º 60º 90º	90º 90º 60º 60º 60º	90º 60º 60º 90º 60º	Demi-regular	The 3 types of vertices are 3-4-6-4, 3-3-4-3-4 and 3-3-3-4-4. This could be Picture 8.
No.13 bis	3-4-4-6 3-3-3-4-4	120º 90º 60º 90º	90º 60º 60º 90º 60º	90º 90º 60º 60º 60º	Demi-regular	Ghyka interchanges the 2nd and 3rd column of No.13 and No.13 bis. We have no idea why he did this. Though No.13 and No.13 bis have the same type of vertices, they could be different tilings.
No.13 ter.	3-4-4-6 3-3-3-4-4	120º 90º 60º 90º	90º 90º 60º 60º 60º		Demi-regular	The 2 types of vertices are 3-4-6-4 and 3-3-3-4-4. This could be Picture 14.

No.13 quatuor	3-4-4-6 3-3-3-4-4	120º 90º 60º 90º	90º 60º 60º 90º 60º			Demi-regular	The 2 types of vertices are 3-4-6-4 and 3-3-4-3-4. This is plate XXIV as shown in Ghyka’s book, which is Picture 7.
No.14	3-3-3-4-4 3-3-3-3-3-3	90º 90º 60º 60º 60º	90º 60º 60º 90º 60º	60º 60º 60º 60º 60º 60º		Demi-regular	The types of vertices are 3-3-3-4-4, 3-3-4-3-4 and 3-3-3-3-3-3. This could be Picture 11. Note again that the columns of No 14 and No 14 bis are interchanged.
No 14 bis	3-3-3-4-4 3-3-3-3-3-3	90º 60º 60º 90º 60º	90º 90º 60º 60º 60º	60º 60º 60º 60º 60º 60º		Demi-regular	This is plate XXVI shown in Ghyka’s book, which is Picture 12**.**
No 14 ter	3-3-3-4-4 3-3-3-3-3-3	90º 60º 60º 90º 60º	60º 60º 60º 60º 60º 60º			Demi-regular	The types of vertices are 3-3-4-3-4 and 3-3-3-3-3-3. This could be Picture 10.
No 15	3-3-4-12 3-3-3-3-3-3	150º 90º 60º 60º	150º 60º 60º 90º	60º 60º 60º 60º 60º 60º		Demi-regular	This could be Picture 2.
No 16	3-3-4-12 3-3-3-4-4 3-3-3-3-3-3	150º 90º 60º 60º	150º 60º 60º 90º	90º 60º 90º 60º 60º	60º 60º 60º 60º 60º 60º	Demi-regular	This is plate XXV shown in Ghyka’s book, which is Picture 16.

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1. B. Grunbaum and G.C. Shephard: *Tilings and Patterns*, W. H. Freeman and Company, New Yoke, 1987

2. Matila Ghyka: *The Geometry of Art and Life*, Sheed and Ward, New Yoke, 1962.

3. Miranda Lundy: *Sacred Geometry*, Walker and Company, New Yoke, 2001.

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6. http://mathworld.wolfram.com/DemiregularTessellation.html