A Mathematical Supplement to "The Sun in the Church: Cathedrals as Solar Observatories"

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ABSTRACT

J.L. Heilbron's book "The Sun in the Church: Cathedrals as Solar Observatories" addresses a basic problem: how is time measured? Since the period of Earth's orbit around Sun is not neatly divisible into a whole number of days, it is hard to construct a calendar that will mark a moment in time back at the exact same point after Earth makes a complete revolution around Sun. The Catholic Church wanted a systematic way to determine when Easter should be celebrated and thus became deeply involved in improving the quality of observational data on which calendars were based, thereby improving the accuracy and reliability of the calendars. Heilbron's book records both the history and technicalities of how cathedrals were used as an instrument to measure time and this paper has been written as a mathematical supplement to enable readers to have a clearer comprehension of how certain conclusions have been drawn or how certain values have been obtained. The first part of the paper provides some useful preliminary information whilst the second part gives detailed explanations for selected sections in Heilbron's book. The latter involves calculation of certain values, comparison of solar and planetary models, and use of the meridiana.

PRELIMINARIES

BACKGROUND OF VARIOUS MATHEMATICIANS MENTIONED

This section aims to provide more information on the prominent mathematicians mentioned in this paper, Hipparchus, Claudius Ptolemy, Johannes Kepler and Giovanni Cassini, so that readers may have a more defined impression of their names as they read this paper. The primary source for the contents of this section is:
http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/.

SOME MATHEMATICAL TOOLS

Properties of ellipses are given in this section, including the standard equation of an ellipse, the definition of eccentricity and Cavalieri Principle. In addition, the small-angle approximations for sine, cosine and tangent and the Binomial Theorem are stated without proofs.

FRAME OF REFERENCE

This section explains how the apparent motion between Sun and Earth changes when we shift from a heliocentric, or sun-in-the-centre, frame of reference to a geocentric, or earth-in-the-

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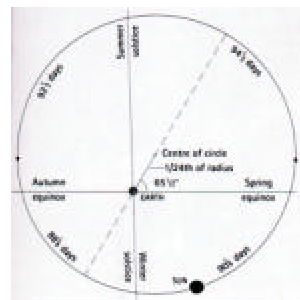
centre, frame of reference. The Earth revolves around the Sun and completes its orbit in a year. By arresting the motion of the Earth, we would also observe the Sun to revolve around Earth on a circular path but in the opposite direction.

The information provided in the later part of the section concerns the Sun, the Earth and the planets where the epicycle-deferent model is introduced and discussed through the use of vectors. In the case of a superior planet, the epicycle corresponds to the orbit of the Earth about the Sun, and the deferent, to the heliocentric orbit of the planet itself; the correspondence is reversed in the case of an inferior planet.

HIPPARCHUS' MODEL OF THE MOTION OF THE SUN

Greek astronomers believe that all orbits of luminaries and planets treated in astronomy should be circles or components of circles. The simplest manner to represent the apparent motion of the Sun as observed from the Earth would be a circle in the plane of the ecliptic, centered on the Earth. if we take y as the number of days in a year, we would expect the interval between the seasons to be exactly of length y/4 days. However, the observed facts show otherwise: the seasons are not equal.



To fix the model, the Earth was displaced from the centre of the Sun's orbit as shown in the figure on the side. The Sun's circle is thus said to be eccentric to the Earth. To generate the longer intervals between solstice and equinox, the Earth had to be removed from the centre in the opposite direction so that the corresponding arcs as seen from the Earth would each be more than ¼ of the circle, and it would take longer than y/4 days to traverse them.

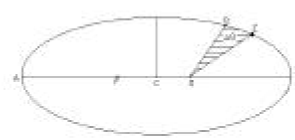
Some terminology frequently used in the paper is defined as well: eccentricity, perigee, apogee and the line of apsides.

PTOLEMY'S SOLAR AND PLANETARY MODELS

Ptolemy's solar model is equivalent to that of Hipparchus'. His planetary theory involved epicycles, deferents and equant points. The development from the ancient zero eccentricity model to the intermediate model and the final planetary model is traced. The distinctive feature of the final planetary model is the insertion of an equant point, defined as a point about which the angular velocity of a body on its orbit is constant. As a result of the equant point, two eccentricities e_1 and e_2 are defined, of which the sum is known as "total eccentricity". Ptolemy always bisects it by putting $e_1 = e_2$.

KEPLER'S LAWS

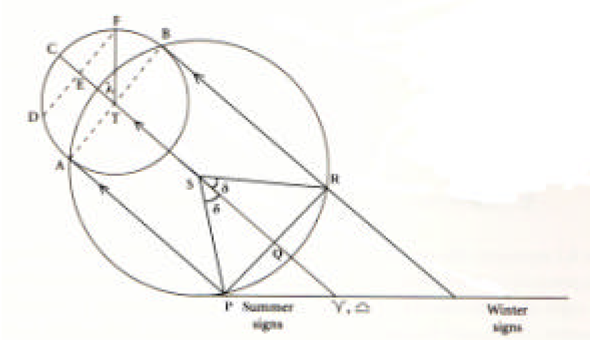
Kepler's First Law states that the path, or orbit, of a planet around the Sun is an ellipse, the position of the Sun being at a focus of the ellipse. Kepler's Second Law states that the radius vector SZ the figure by the side sweeps out equal areas in equal times. r denotes the distance of the planet Z from the Sun and the angle q be the planet's angular distance from perigee, or the true anomaly. Then, the equation of Z's elliptical orbit is known to be $r = \frac{a(1-e^2)}{1+e\cos q}$. The steps to determine the position of the planet in its orbit at any instant, based on Kepler's Second Law, are also given.

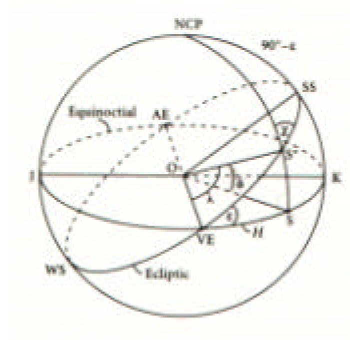


CLARIFYING SELECTED SECTIONS OF HEILBRON'S BOOK

NOTES TO APPENDIX B: THE ANALEMMA IN THE CONSTRUCTION OF SAN PETRONIO

This section evaluates the positioning of the zodiacal plaques along the meridiana geometrically with reference to the diagrams below.





Upon deriving two separate expressions for sin *d*, the following relation involving l, *S*VE* and K (radius of celestial sphere) is obtained:

$$l \approx \frac{S*VE}{K} \Rightarrow S*VE \approx K\,l\,.$$

K l gives the ecliptic longitude and since K is constant, it is sufficient to mark the point where the noon ray falls on the meridiana at an equinox, and then by increasing l in steps of 30°, the rest of the zodiacal plaques could be positioned accordingly.

NOTES TO HEILBRON, PG 105: PTOLEMY'S SOLAR ECCENTRICITY

Ptolemy made use of Hipparchus' model for the Sun's motion around the earth and calculated values for the solar eccentricity, *e*, and the angle y between the line of apsides and the line joining the solstices to be 0.0334 and 12°58' respectively. The actual working steps to arriving at such results are included. Kepler found the value of eccentricity to be only 0.0167 and the section provides an explanation for the factor-of-two difference.

COMPARISON OF MODELS

The defining parameters of a body in its orbit are defined. These include the true anomaly and the radius vector. The table below holds the expressions for these parameters for each of the solar and planetary models as advocated by Hipparchus, Ptolemy and Kepler.

Model	Eccentricities	True anomaly, $q(t)$	Radius vector, $r(t)$
Hipparchus	$e_1 = e$	$wt + 2\left(\frac{e}{2}\right)\sin wt + 2\left(\frac{e}{2}\right)^2 \sin 2wt$	$1 - 2\left(\frac{e}{2}\right)\cos q - 2\left(\frac{e}{2}\right)^2 \sin^2 q$
Ptolemy	$e_1 = e$ and $e_2 = e$	$wt + 2\left(\frac{e}{2}\right)\sin wt + \left(\frac{e}{2}\right)^2 \sin 2wt$	$1 - \left(\frac{e}{2}\right)\cos q - \frac{1}{2}\left(\frac{e}{2}\right)^2 \sin^2 q$

Kepler	$e_1 = e_2 = \frac{e}{2}$	$wt + 2\left(\frac{e}{2}\right)\sin wt + \frac{5}{4}\left(\frac{e}{2}\right)^2 \sin 2wt$	$1 - \left(\frac{e}{2}\right)\cos q - \left(\frac{e}{2}\right)^2 \sin^2 q$

Ignoring the second order terms, then the first discrepancy arises in the radius vector or Hipparchus' model. Ptolemy's equant theory for the planets is an amazingly close approximate to Kepler's planetary model. Up to first order terms in *e*, the empty focus of the Keplerian ellipse is indistinguishable form the equant point in Ptolemy's model; the mathematics involved are explained.

USE OF MERIDIANA AND NOTES TO APPENDIX C

Detailed accounts of Cassini's methodologies and calculations for justifying Kepler's model to be superior over Ptolemy's are given in this section.

BISECTION OF ECCENTRICITY

The double meanings of "bisection of eccentricity" are clarified. It refers to either Ptolemy splitting the total eccentricity into two on his planetary theory or Kepler dividing the solar eccentricity as defined in Ptolemy's (or Hipparchus') solar theory.

NOTES TO HEILBRON PG 114 TO 117 AND APPENDICES D AND E

In this section, the derivation of Kepler's Equation being

$$h - \frac{e}{2}\sin h = M \equiv w(t - t)$$

is given. Seth Ward thought he had discovered a much simpler geomtrical method than Kepler to find the true anomaly. However, there was an error in his method and the mathematics to show this are included.

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