The Sun in the Church

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"How do you know if you have understood something? First you know it, then you can explain it, and finally you can explain it in a simpler way for others to understand." – a quote from A/P Aslaksen that I will always remember.

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SUMMARY

There are many indications that the sole important application of arithmetic in Europe during the Middle Ages was the calculation of Easter date. At that time, the Catholic Church played an active role in calendar research in order to determine a systematic way of finding the date to celebrate Easter.

J.L Heilbron's book, *The Sun in the Church*, contains a succinct historical account of how the calculation for Easter had influenced the design for a calendar. He highlighted the main problems and concerns for the design, the key personnel behind calendar refinements, and the resultant method of finding a fixed date for Easter. This is a broad mix of topics which Heilbron had managed to condense into a single chapter. Inevitably, readers may find it hard to fully grasp the issues packed behind the few lines of explanation. This paper is thus compiled as a supplement to Heilbron's book, to

1. provide more background for Easter and its calculation;
2. give details on the calendar reforms;
3. clarify certain points raised in the book.

Chapter 1 gives a quick overview of what Easter is about, the ancient Roman dating system and the Julian calendar.

Chapter 2 introduces lunar cycles, a key factor that influences the design of a calendar. Examples of lunar cycles, including the important Metonic cycle, are furnished.

Chapter 3 is broadly divided into two parts. The first describes general tools used for determining Easter Sunday and the second gives the methods themselves.

Chapter 4 outlines the problems with the Julian calendar and the calendar reforms that were made to the Julian calendar

Chapter 5 is similar to Chapter 3 in structure. It contains descriptions of the modified tools and methods used in finding Easter Sunday in the Gregorian calendar.

Chapter 6 highlights Easter tables found in the Book of Common Prayer and other established Easter algorithms by famous people such as Gauss and Knuth.

It is hoped that readers would have a better appreciation of the first part of Heilbron's book after going through these various chapters.

AUTHOR’S CONTRIBUTIONS

Here is a list of my inputs to this project:

1. I expanded on Heilbron’s explanation of the term “bissextile” by introducing the ancient way of Roman dating.
2. In Chapter 2, I provided information on lunar cycles to enable readers to understand Heilbron’s algebraic method of finding luni-solar cycles. Pertaining to his method, I have also rearranged it to improve clarity.
3. In Table 2, I added the entries corresponding to $J = 84$ and thereafter found 2 discrepancies between the entries and those quoted by Heilbron.
4. The concept of *saltus lunae* is highlighted through the introduction of the Metonic cycle and Table 2.
5. With the help of my supervisor, I have managed to identify interesting properties about Table 4 and provided explanations for their occurrence accordingly. Through the discussion of these properties, the concept of golden numbers can be better appreciated.
6. To demonstrate how the regression of dominical letters takes place, I have included an example through Table 7.
7. The derivation of the formulae for dominical letters for both Julian and Gregorian calendars are given in greater detail than that in the source, [8].
8. I have brought up a confusing feature in Table 10, taken from [3]. Here I have suggested for the epact “0” to be interpreted as “30” and the epact range used for that table to be taken as 1 to 30 instead of 0 to 29.
9. The Easter algorithms described in [8] are explained in more detail. For instance, the uniqueness of epact to golden number has been justified mathematically.
10. Flow-charts have been drawn to summarize the algorithms for determining Easter Sunday for both the Julian and Gregorian calendars, proposed in [8].
11. Table 11, extracted from [3], has been treated differently from that by Heilbron to provide a better appreciation of how the solar and lunar equations influence the epacts over a millenium.
12. Based on one of the meetings I have had with my supervisor, I have come up with a schematic diagram to illustrate how golden numbers corresponding to new moons may be shifted out of the usual range of dates into 6 April.

Chapter 1

Preliminaries

1.1 About Easter

Even though Christmas seems to draw more enthusiasm from most people, it is Easter that carries far greater significance for the meaning of Christianity. The commemoration of the Crucifixion on Good Friday and of the Resurrection on Easter Sunday is the recognition of the central mysteries and promises of the religion: the forgiveness of sin and the enjoyment of eternal life.

There is a chronological discrepancy over when the Crucifixion took place. The Synoptic Gospels differ from the Gospel of John in their indication of when the Last Supper – the last meal that Jesus had shared with his disciples before his Crucifixion – took place. In order to appreciate the account for this discrepancy better, let us first have an overview of what the Passover Festival is.

The Passover Festival is the most important event in the Jewish calendar. It is an 8-day festival, held from the 14th to the 21st day of the Jewish month of Nisan, to remember the exodus of the Israelites from Egypt under the leadership of Moses. The Passover meal is consumed on the first evening of the Passover on 14 Nisan (or technically, 15 Nisan, since the Jewish day begins after 6pm). During this meal, the food served contains food which symbolizes the events during the exodus: the paschal lamb which was slain earlier in the evening and unleavened bread.

All four gospels are agreed that Jesus died on a Friday and the Last Supper was taken on a Thursday. According to the Synoptic Gospels, the Last Supper was a Passover meal, which falls on the evening of 14 Nisan (start of 15

Nisan). However, for the Gospel of John, Jesus was the new paschal lamb (the lamb of God) and sacrificed at the same moment of the slaughter of the Passover lamb. Hence for John, the Last Supper would only be a normal meal taken on the evening of 13 Nisan instead of 14 Nisan. The discrepancy over when the Last Supper was held in turn leads to the discrepancy of dates for when Jesus' Crucifixion took place. This is seen clearly from the following table.

Table 1

Date	Synoptics' Chronology	John's Chronology
13 Nisan	–	Thursday: Last Supper (normal meal)
14 Nisan	Thursday: Last Supper (Passover meal)	Friday: Crucifixion
15 Nisan	Friday: Passover & Crucifixion	Saturday
16 Nisan	Saturday	Sunday: Resurrection
17 Nisan	Sunday: Resurrection	–

Therefore, Jesus died on 15 Nisan according to the Synoptic Gospels and 14 Nisan according to John. Nevertheless, all 4 Gospels agree that Jesus Christ was resurrected three days after Friday, which by inclusive counting, was a Sunday.

By the middle of the second century, the problem of exactly which date that Easter should be celebrated on had arisen. There were two main groups of Christians who had different Easter celebration practices. One group, known as the Quartodecimans, insisted on celebrating Easter on 14 Nisan. These included Christians who converted from Judaism and others in Asia Minor. The other, called the Quintadecimans, emphasized the day of the week by celebrating the Resurrection on the Sunday which followed 14 Nisan. This group comprised Roman and Alexandrian Christians. Initially, both groups received great support. However, the unity among the Quartodecimans started to break down. In addition, there was a rising desire among the Christians to break away from their dependency on the Jewish authorities in order to determine the date for Easter. Eventually, the Quintadecimans' practice spread.

A significant event that reaffirmed the dominance of the Quintadecimans' or Western Church's practice was the first ecumenical council of the Church, the Council of Nicaea, convened in AD 325. The Nicene Creed was established then. It declared the Quartodeciman practice heretical and that Easter was to be celebrated on the same day by all Christians based on the practices of the Western Church. The following rule was incorporated into their practice: *Easter Sunday, the anniversary of the Resurrection, was to be celebrated on the 1st Sunday after the paschal full moon that fell on or after the vernal equinox; the paschal full moon was taken to fall on the 14th day from the paschal new moon.* This rule was designed to ensure, among other things, that Easter Sunday was never on or before the Jewish Passover.

Despite having a common rule, there was still a significant disparity in the dates of Easter Sunday as determined by members of the Western Church – the Romans and the Alexandrians. The Romans considered 25 March to be the vernal equinox but the Alexandrians used 21 March. They also had different dates for the paschal full moon because they had used different lunar cycles. The issue of lunar cycles is discussed in detail in Chapter 2.

1.2 Roman dating system

The Romans had an unusual way of dividing their months. The three chief points of the month were the Kalends, the Ides and the Nones. The Kalends represented the 1st day of a month and originally marked the appearance of a new moon. The Ides represented the full moon and occurred in the middle of the month. For March, May, July and October, Ides fell on the 15th day of the month, and for the remaining months, Ides fell on the 13th day. The Nones occurred 9 days before the Ides, counting inclusively in Roman style. Hence it fell on the 7th day of March, May, July and October and the 5th in other months. In addition to inclusive counting, the Romans used backward counting when naming the days within each division. This is illustrated by the Table 2 and the explanation follows after that.

Table 2

Date	Term in Roman dating
1 March	Kal. Mar.
2 March	VI Non. Mar.
⋮	⋮
5 March	III Non. Mar.
6 March	pridie Non. Mar.
7 March	Non. Mar.
⋮	⋮
13 March	III Idus Mar.
14 March	pridie Idus Mar.
15 March	Idus Mar.
⋮	⋮
29 March	IV Kal. Apr.
30 March	III Kal. Apr.
31 March	pridie Kal. Apr.
1 April	Kal. Apr.

The Latin word “pridie” refers to “the day before”. The last day of March was termed, “pridie Kalendas Aprilis”, meaning day before 1 April, and was abbreviated as “pridie Kal. Apr.” on calendars used then. By inclusive counting, 30 March was the third day before 1 April and so it was termed “ante diem III Kalendas Aprilis” or “III Kal. Apr.” Then 29 March was “ante diem IV Kalendas Aprilis” and so on, back to 15 March which was the Ides of March. The day before the Ides, 14 March, was called “pridie Idus Martius”. 13 March was “ante diem III Idus Martius” and so on, back to the Nones on 7 March. Similarly, 6 March was termed “pridie Nonae Martius”, 5 March termed “ante diem III Idus Martius” and so on until 1 March – the Kalends of March – was reached. With this inclusive and backward dating system in mind, we proceed to understand what “bissextile” refers to and highlight other features of the Julian calendar in the following section.

1.3 The Julian calendar

Prior to the Julian calendar, the Romans used the AUC calendar and numbered years A*b Urbe Condita*, or “from when Rome was founded”. This old calendar was inadequate for the needs of the emerging Roman empire. Hence in 45 BC, equivalent to 709 AUC. of the old calendar, Julius Caesar, the commander of the Roman empire then, accepted astronomer Sosigenes’ recommendations and adopted the Julian calendar.

The Julian calendar introduced a solar year of 365 days and the year was divided into twelve months. April, June, September and November had 30 days, February, 28 days, and all other months, 31 days. In every 4th year, an extra day was added into the calendar, making that year 366 days long. Consequently, the average length of a Julian solar year became 365.25 days. The extra day added in the 4th year was called a “leap day” and the year containing the leap day, “leap year”. The leap day was inserted before 24 February, that is, “ante diem VI Kalendas Martius” or the 6th day before 1 March. Hence, the leap day was the second 6th day before 1 March. The table below illustrates the insertion.

Table 3

Date in non-leap year	Date in leap year	Term in Roman dating	Remarks
23 February	23 February	VII Kal.Mar.	
	24 February	bis. VI Kal. Mar.	*Second* 6th day before 1 March; the inserted leap day
24 February	25 February	VI Kal. Mar.	6th day before 1 March
25 February	26 February	V Kal. Mar.	
26 February	27 February	IV Kal. Mar.	
27 February	28 February	III Kal. Mar.	
28 February	29 February	pridie Kal. Mar.	
1 March	1 March	Kal. Mar.	

In Latin, “sextus” refers to the ordinal “6th”. From this, “sextile” was derived to refer to the 6th day in a calendar month, and in turn, “bissextile” was second 6th day. The leap day, being the second 6th day before 1 March, was thus also termed the “bissextile” and the year containing it, “bissextile year”.

After the Julian calendar was adopted, the vernal equinox was kept on 25 March for many years. However, the average Julian solar year length of 365.25 days was longer than the average astronomical tropical year length. The error accumulated over the centuries and by the 16th century, it had become a significant amount. Eventually, a new calendar, in use till today, evolved to replace the Julian calendar.

Chapter 2

Lunar Cycles

2.1 Background – why the need for lunar cycles?

The *tropical year* is the period between the occurrence of one vernal equinox and the next. Some sources refer to this instead as the vernal equinox year. However for the scope of this supplement, this definition of *tropical year* is adequate. Astronomically, the average length of the tropical year is 365.24219 days. The *synodic month* or *lunation* is the period from one phase of the moon to its next occurrence. Two commonly phases used are the new moon and the full moon. The astronomical average lunation is 29.53059 days long.

Recall that in order to determine when Easter was to be celebrated, it was necessary to identify the dates for both the vernal equinox and the paschal new moon. The vernal equinox is a solar event while the new moon is a lunar event. This meant that to fix a calendar date for Easter, the calendar so constructed had to be able to keep its months synchronous with the moon and its years with the sun. Such a calendar is termed a luni-solar calendar. It would have been easy to construct such a calendar if the ratio of the length of a tropical year to that of a synodic month was a rational number. For example, if a tropical year had 300 days and a synodic month had 30 days, the ratio of the length of a year to a month is a rational number. It is then natural to construct a 300-day calendar which is divided into 10 months of 30 days. Such a calendar would ensure that a lunar event occurs on the same date in the calendar every year.

Yet the fact is that the ratio of length of the average astronomical tropical year to that of the synodic month is not an integer. It is thus not possible to fit a whole number of lunations, forming a lunar year, into a solar year without any remainder. To reconcile the difference between the lunar years and solar years, suitable intercalation or insertion of lunations may be used. The calendar thus produced would bear the feature that after a certain number of solar years has passed, the total length of the solar years is brought into synchrony with that of the lunar years, or it can be said that this whole number of solar years would contain the same number of days as a whole number of lunations would. A lunar cycle is precisely the period of several solar years which has this property. It ensures that a lunar date is returned to the same calendar date after a period of several solar years.

Heilbron highlighted an algebraic method in finding suitable lunar cycles, or what he termed as “luni-solar cycles”, in the Julian calendar. His explanation is reorganized and made clearer below.

We first highlight a few assumptions made of the lunations used in the method. Now, because the average synodic month is 29.53059 days long, it is reasonable to vary the lengths of the lunations between 29 and 30 days; the former is considered as a hollow lunation and the latter, a full lunation. In the computation later, assume that a lunar year has 12 lunations, 6 of which are 29-day lunations and the remaining 6, 30-day lunations. Then the average length of a lunation in the lunar year is 29.5 days. We next assume that all the intercalated lunations are full lunations. Finally, the lunation containing the leap day or bissextile is deemed as a full lunation as well. This implies that the leap day is always inserted into a lunation of length 29 days and never a 30-day lunation. At this point, we may argue that this is an unfair assumption because in the actual lunar calendar, the leap day could have been inserted into a 30-day lunation, thereby making it a 31-day lunation. However since we are only concerned with making sure that the average lunation length is 29.5 days for the lunar year, it does not matter how long each lunation is and how the lunations are arranged in the lunar year. It is thus possible to set the leap day to always fall in a 29-day lunation.

Suppose the lunar cycle is made up of J Julian years, and K lunations. Our aim is to find the number of lunations to intercalate in the lunar years such that the number of days contained in J years is equivalent to that contained in K lunations.

We note that each Julian year is longer than a lunar year before intercalation. Hence in the J-year lunar cycle, the number of lunations, K, contained in the cycle would be greater than the number of lunations contained in J lunar years, $12J$. Mathematically, this is expressed as $K > 12J$. Let the number of lunations required for intercalation be p. Hence, $K = 12J + p$. On the left hand side of the equation, K has the same number of days as J Julian years, by definition of a J-year lunar cycle. The average length of a Julian year is 365.25 days and so K is equivalent to $365.25J$ days. Then on the right hand side of the equation, $12J + p$ is equivalent to $354.25J + 30p$ days. The first term is obtained by assuming that in each of the J Julian years, it contains a lunar year of 12 lunations of 29.5 days each, thereby adding up to 12 x 29.5 = 354 days, and another 0.25 days due to the insertion of a leap day every four Julian years. The second term comes about from the earlier assumption that each intercalated lunation has 30 days. Therefore, the resultant equation is:

$$365.25J = 354.25J + 30p \Rightarrow 11J = 30p.$$

Our original aim would be achieved, subjected to a very small error, once we obtain whole numbers J and p that satisfy the following two conditions:

1. J and p satisfy equation $11J = 30p$ as nearly as possible;
2. J and p give a close approximation of the average length of a calendar lunation to the average astronomical lunation length. That is, the calendar month of length $\frac{354.25J + 30p}{12J + p}$, denoted by Λ closely approximates 29.53059 days.

The attractive possibilities are as laid out in the table that Heilbron has provided in his book. We reproduce his table with some amendments in the headers to ensure coherence with the explanation above – wherever “month” occurs, it is replaced by “lunation”; and in the final column, “mean” is replaced by “average”. Note that the term embolismic years refer to years in which an extra lunation has been inserted. Since no two or greater intercalated lunations are inserted into the same year, the number of embolismic years is equivalent to the number of intercalated lunations, p, and hence the header for the second column “Embolist[m]ic years, p”. In addition, we also add in the quantities for $J = 84$ in the last row.

Table 4 Solar-lunar cycles

Julian years, J	Embolistic years, p	Total lunations, K	Days in J years, M	Days in K lunations, N	Average lunation, Λ
3	1	37	1095.75	1092.75	29.5338
8	3	99	2922.00	2924.00	29.5353
11	4	136	4017.75	4016.75	29.5349
19	7	235	6939.75	6940.75	29.5351
30	11	371	10957.50	10957.50	29.5350
84	31	1039	30681.00	30687.00	29.5351

From Table 4, we observe that all the combinations of J and p gave average lunations which were all larger than the astronomical average yet very close among themselves. Coupled with the difficulty of computing the average lunation to higher accuracy, the people in the ancient days had a lot of disagreement over which lunar cycle was the most accurate one. In particular, comparing the average lunation lengths of the 19-year and 84-year lunar cycles, we see that up to the fourth decimal place, the lengths were the same. The 19-year lunar cycle was adopted by the Alexandrians and the 84-year cycle was adopted by the Romans; both cycles are discussed in later sections. The equal average lunation lengths of these two lunar cycles lends justification for the conflict between the Romans and the Alexandrians. Even though they produced lunar tables which did not agree with each other, they had believed that their own set of tables was the more accurate one. Both groups adhered to their own set of tables to find the date for Easter Sunday. As a result, the two groups arrived at different dates for Easter celebration.

2.2 Examples of lunar cycles

In the following sections, we give descriptions of some prominent lunar cycles that were proposed and/or implemented in the construction of lunar tables.

2.2.1 Augustalis' cycle

This cycle was created by Augustalis, an unknown Roman, in the early third century. It was 84 years long and contained 1039 lunations, as indicated in Table 4. Heilbron has also mentioned this cycle in his book. However there are two discrepancies between the values he claims in his narrative and what he would have charted in Table 4 had he continued to compute figures for $J = 84$. He wrote, "he took $J = 84$, which, with 6 jumping moons and 30 embolist[m]ic years, agrees to within 1.3 days with 1039 lunations". By comparing these values, we see that the discrepancies lie in the number of embolismic years, p, and the deviation between the Julian years and whole number of lunations, that is, $N – M$. By evaluating p from the rule "$11J = 30p$", with $J = 84$, the resultant value of p is closer to 31 than 30. In addition, the difference between N and M is 6 days, rather than the 1.3 days that he states. However, this second discrepancy can be resolved by using values for the lengths of average astronomical tropical year and synodic month. By these values, the number of days in 84 years, M, is $84 \times 365.2422 = 30680.3448$ and the number of days in 1039 lunations, N, is $1039 \times 29.53 = 30681.67$. The difference between these two values would then be approximately 1.3 days.

Augustalis' cycle was a significant improvement from a previously adopted 8-year cycle but was soon replaced by a more accurate 19-year cycle, the Metonic cycle. After the Alexandrians started using the Metonic cycle, it took some time before the Romans adopted the same cycle. Nevertheless, Augustalis' cycle is significant because it was through this cycle that the idea of the epact was first introduced. The concept of "epact" is discussed in Section 3.1.4.

2.2.2 The Metonic cycle

The Metonic cycle was discovered by a Greek mathematician, Meton. It was a 19-year lunar cycle, containing 12 common years and 7 embolismic years. Note that a lunar year is classified as "common" if it does not have an extra, intercalated lunation, and "embolismic" if otherwise.

In the Metonic cycle, each of the common years consisted of 12 lunations, six of which contained 29 days and the remaining six, 30 days each. Every embolismic year consisted of 13 lunations. Six of the embolismic years had a similar complement of 29 and 30-day lunations as the common year, together with an extra, intercalated full lunation of 30 days. In the last embolismic year, the extra lunation was hollow with 29 days. This omission of a day in the extra lunation was known as the “saltus lunae”, meaning “jump of the moon”. If we refer back to Table 4, we would see why such an omission had to be made. Compare the values for the number of days in 19 years, M, and that in 235 lunations, N. There is a 1-day difference and the “saltus lunae” is precisely added to cover this difference.

Hence, the 12 common years contained 354 days each, and the 7 embolismic years, 384 days each except the last, which contained 383 days. There was no constant alignment between the start of the lunar years and the Julian years. By convention, the first lunation of a Julian year was taken to be the first one that ended in it. Thus the first lunation of a year might have started in December of the previous year.

With such a composition of common and embolismic years, the cycle contained a total of 235 lunations. 120 of them are full lunations and the remaining 115 are hollow lunations. In turn, the number of days contained in these 235 lunations was 6935. This was shorter than the length of 19 Julian years only by 4.75 days. The shortfall is accounted for by the leap days that were inserted at four-year intervals throughout the lunar cycle. Since the original lunation could either have contained 29 or 30 days, the leap day brought the length of this lunation to either 30 or 31 days respectively. After the intercalation of leap days, the lunar cycle was brought into complete harmony with the Julian calendar.

To set up a lunar table based on the Metonic cycle, not only was it necessary to identify the composition of lunations, but it was also important to stick to systematic rules that guided the arrangement of these lunations on the table. The table on the next page, extracted from [8], shows how the 120 full and the 115 hollow lunations had been distributed among the 19-year cycle.

Table 5 Distribution of full and hollow lunations among the 19-year cycle.

Golden number year/cycle	Lunation													Total
	1	2	3	4	5	6	7	8	9	10	11	12	13	
III	30	29	30	29	30	29	30	29	30	29	30	29		354
IV	30	29	30	29	30	29	30	29	30	29	30	29		354
V	30	29	30	29	30	29	30	29	30	30	29	30	29	384
VI	30	29	30	29	30	29	30	29	30	29	30	29		354
VII	30	29	30	29	30	29	30	29	30	29	30	29		354
VIII	30	29	30	30	29	30	29	30	29	30	29	30	29	384
IX	30	29	30	29	30	29	30	29	30	29	30	29		354
X	30	29	30	29	30	29	30	29	30	29	30	29		354
XI	30	30	29	30	29	30	29	30	29	30	29	30	29	384
XII	30	29	30	29	30	29	30	29	30	29	30	29		354
XIII	30	29	30	29	30	29	30	29	30	29	30	29		354
XIV	30	29	30	29	30	29	30	29	30	29	30	29		354
XV	30	29	30	29	30	29	30	29	30	29	30	29		354
XVI	30	29	30	29	30	29	30	29	30	30	29	30	29	384
XVII	30	29	30	29	30	29	30	29	30	29	30	29		354
XVIII	30	29	30	29	30	29	30	29	30	29	30	29		354
XIX	30	29	30	30	29	30	29	29	29	30	29	30	29	383
I	30	29	30	29	30	29	30	29	30	29	30	29		354
II	30	29	30	29	30	29	30	29	30	29	30	29	30	384

Note that the first lunation (of 30 days) begins (except for year III) in the previous calendar year.
The lunations assigned to a golden number define a lunar year; the last column gives the number of days in each such lunar year (which does not include any leap day).
The embolismic years have 13 lunations and the others, 12.

The arrangement of the lunations had been made with respect to the following criteria:

1. The first lunation of each year should have 30 days and begin in the previous year.
2. There should never be more than 1 golden number on the same day in the table.
3. All the paschal lunations should contain just 29 days.
4. Full and hollow lunations should generally alternate, particularly at the beginning and end of each year.

The first 3 rules were followed with no exceptions in the distribution of the lunations. However for the last rule, there were instances in the table where full and hollow lunations did not alternate.

2.2.3 The Great Paschal Cycle

This cycle was attributed to Victorius, Bishop of Aquataine, and sometimes called the “Victorian cycle” or “Dionysian cycle”. Victorius had been requested by the deacon of Rome then, Hilary, who later became Pope, to find a way to reconcile the computation results of Easter dates of Alexandria and Rome. By sheer computation, Victorius found that the dates of Easter Sunday, determined according to the Alexandrian practice, would repeat on the same days every 532 years. In fact, in any year, the days of the month would fall on the same weekday, and the phases of the moon on the same dates, as they did in the year 532 years ago. This 532-cycle was none other than the Great Paschal Cycle.

There is a shorter way of finding the length of the Great Paschal Cycle. This requires us to view the Great Paschal Cycle as having two constituent subcycles, the Metonic cycle and the solar cycle (to be discussed in Section 3.1.3). The former has length 19 and the latter, 28. By the mathematical principle that the length of a cycle is equal to the lowest common multiple of the lengths of its constituent subcycles, we can then compute the length of the Great Paschal Cycle simply by taking the lowest common multiple between 19 and 28. This gives 532, as found by Victorius.

Chapter 3

Determining Easter in the Julian calendar

3.1 Tools

Lunar tables provide the dates of notional new moons throughout a number of years. Such notional moons are predicted without actual observation of the heavens or elaborate astronomical calculations. The Metonic cycle, as introduced in the previous chapter, governs the behaviour of these notional moons by way of stating the length of each lunation and where the lunation is placed. Hence good estimates of the dates of real new moons are obtained from use of the Metonic cycle.

The traditional way of determining the date for Easter Sunday involves lunar tables as well as golden numbers, the paschal full moon and dominical letters/numbers. To calculate Easter Sunday without using tables, the concept of epact is employed. The following sections would give an elaboration for each of these underlined notions.

3.1.1 Golden number

The Metonic cycle was introduced earlier in the chapter about lunar cycles. The length of the Metonic cycle is 19 years and the golden number is defined to be the number that indicates the position of a particular year in the Metonic cycle. The golden number ranges from 1 to 19, often stated in Roman numerals I to XIX. 1 BC was deemed to be the first year of a Metonic cycle, and thus the golden number, G, is given by the formula,

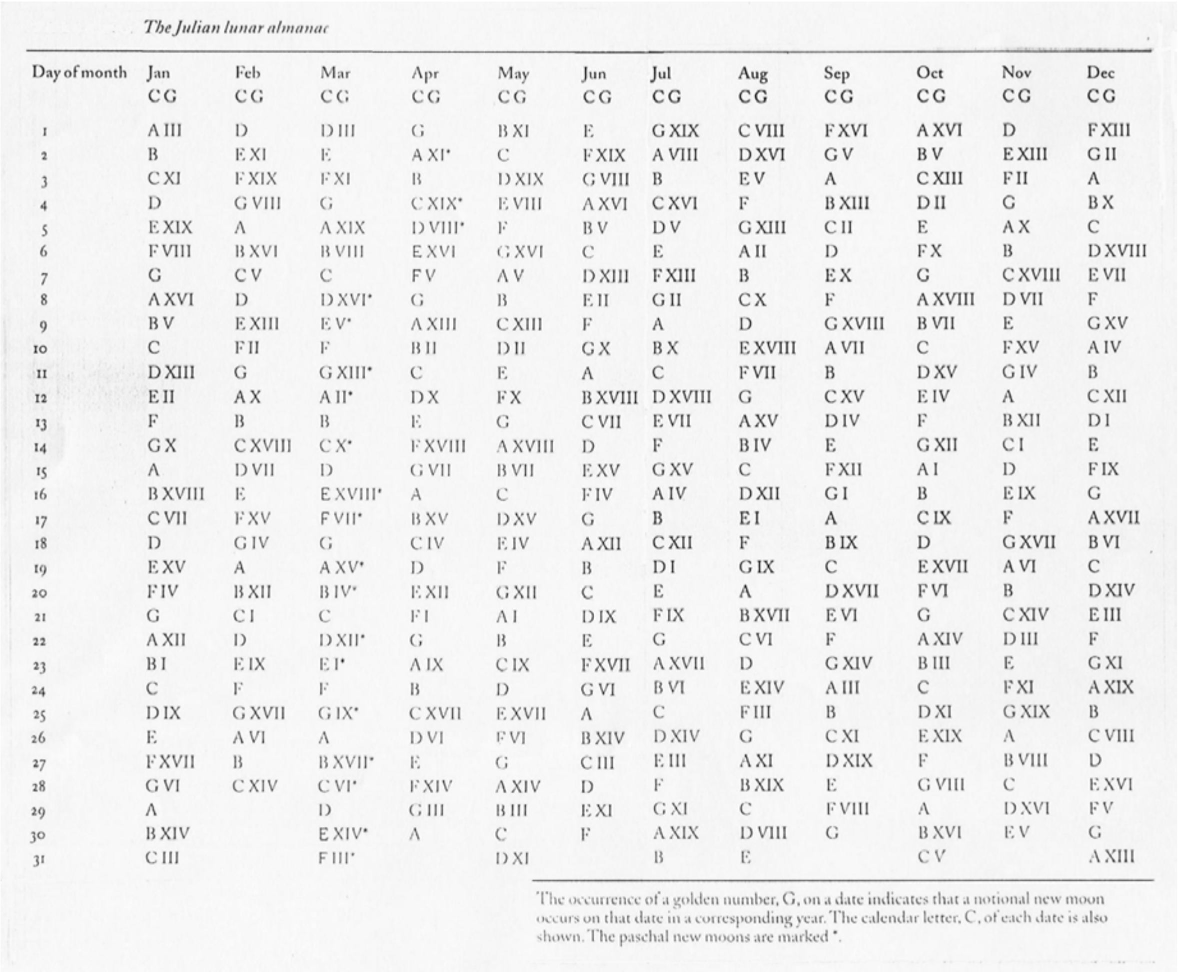
$$G = 1 + \text{mod}\,(Y, 19)$$

where Y is the year number in AD dating.

We note that the expression, mod (Y, 19), represents the remainder of Y after dividing Y by 19. For example, the golden number for AD 2003 is $G = 1 + \text{mod}\,(2003, 19)$. The remainder of 2003 after dividing it by 19 is 8. Therefore the final result for G is 9. Hence AD 2003 is the 9th year in a Metonic cycle.

We next give a sample of a lunar table based on the Julian calendar which is sourced from [8]. Using it, we will study the golden numbers that appear on it in greater detail.

Table 6



The Julian lunar almanac

Day of month	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
	C G	C G	C G	C G	C G	C G	C G	C G	C G	C G	C G	C G
1	A III	D	D III	G	B XI	E	G XIX	C VIII	F XVI	A XVI	D	F XIII
2	B	E XI	E	A XI*	C	F XIX	A VIII	D XVI	G V	B V	E XIII	G II
3	C XI	F XIX	F XI	B	D XIX	G VIII	B	E V	A	C XIII	F II	A
4	D	G VIII	G	C XIX*	E VIII	A XVI	C XVI	F	B XIII	D II	G	B X
5	E XIX	A	A XIX	D VIII*	F	B V	D V	G XIII	C II	E	A X	C
6	F VIII	B XVI	B VIII	E XVI	G XVI	C	E	A II	D	F X	B	D XVIII
7	G	C V	C	F V	A V	D XIII	F XIII	B	E X	G	C XVIII	E VII
8	A XVI	D	D XVI*	G	B	E II	G II	C X	F	A XVIII	D VII	F
9	B V	E XIII	E V*	A XIII	C XIII	F	A	D	G XVIII	B VII	E	G XV
10	C	F II	F	B II	D II	G X	B X	E XVIII	A VII	C	F XV	A IV
11	D XIII	G	G XIII*	C	E	A	C	F VII	B	D XV	G IV	B
12	E II	A X	A II*	D X	F X	B XVIII	D XVIII	G	C XV	E IV	A	C XII
13	F	B	B	E	G	C VII	E VII	A XV	D IV	F	B XII	D I
14	G X	C XVIII	C X*	F XVIII	A XVIII	D	F	B IV	E	G XII	C I	E
15	A	D VII	D	G VII	B VII	E XV	G XV	C	F XII	A I	D	F IX
16	B XVIII	E	E XVIII*	A	C	F IV	A IV	D XII	G I	B	E IX	G
17	C VII	F XV	F VII*	B XV	D XV	G	B	E I	A	C IX	F	A XVII
18	D	G IV	G	C IV	E IV	A XII	C XII	F	B IX	D	G XVII	B VI
19	E XV	A	A XV*	D	F	B	D I	G IX	C	E XVII	A VI	C
20	F IV	B XII	B IV*	E XII	G XII	C	E	A	D XVII	F VI	B	D XIV
21	G	C I	C	F I	A I	D IX	F IX	B XVII	E VI	G	C XIV	E III
22	A XII	D	D XII*	G	B	E	G	C VI	F	A XIV	D III	F
23	B I	E IX	E I*	A IX	C IX	F XVII	A XVII	D	G XIV	B III	E	G XI
24	C	F	F	B	D	G VI	B VI	E XIV	A III	C	F XI	A XIX
25	D IX	G XVII	G IX*	C XVII	E XVII	A	C	F III	B	D XI	G XIX	B
26	E	A VI	A	D VI	F VI	B XIV	D XIV	G	C XI	E XIX	A	C VIII
27	F XVII	B	B XVII*	E	G	C III	E III	A XI	D XIX	F	B VIII	D
28	G VI	C XIV	C VI*	F XIV	A XIV	D	F	B XIX	E	G VIII	C	E XVI
29	A		D	G III	B III	E XI	G XI	C	F VIII	A	D XVI	F V
30	B XIV		E XIV*	A	C	F	A XIX	D VIII	G	B XVI	E V	G
31	C III		F III*		D XI		B	E		C V		A XIII

The occurrence of a golden number, G, on a date indicates that a notional new moon occurs on that date in a corresponding year. The calendar letter, C, of each date is also shown. The paschal new moons are marked *.

All 365 days of a year are listed in the table. The golden numbers of the years in which a new moon is scheduled to occur on various dates are indicated against those dates. (Ignore the letters for the time being; they will be discussed in section 3.1.3.) Therefore, the table presents the dates of all 235 notional new moons in the Metonic cycle. For example, according to the table, on 12 May, there is a new moon occurring on this date in the 10th year of the Metonic cycle.

How have the golden number entries been inserted into the table? Firstly, the new moons in the year of golden number III are entered into the table, with the first occurring on 1 January. Let us pause to think over why such a choice had been made. This is related to the choice of 1 BC as the start of a Metonic cycle. If we are lunar table makers, there are three natural choices for a year in AD dating to begin a Metonic cycle. The first is AD 1. It is a good choice because it would simplify the calculations for determining golden numbers. AD 1 would have golden number I, AD 2 would have golden number II,… and AD 19 would have golden number XIX. The following year, AD 20, would return to having I as the golden number. Such a relation can be described by $G = 1 + \text{mod}\ (Y - 1, 19)$ or equivalently, $G = \text{mod}\ (Y + 1, 19) - 1$, where G is the golden number and Y is the year in AD dating. This choice would lead to golden number II on 1 January in the lunar table. A second choice for a year to start the Metonic cycle is AD 2. This would translate to having golden number I on 1 January of the lunar table, making the table look neater. Finally, 1 BC is another valid choice since it is the year that was taken to be when Jesus Christ was born in.

Among the three choices, the third was picked. It is unknown as to what exactly had made this arrangement seem most desirable. However we can speculate that mathematics aside, historical, religious and political reasons have made an equal, if not larger, influence on the decision. The simplicity of golden number calculations and neatness of lunar tables could then have been compromised for these other reasons. By such a choice, the notional new moon on 1 January came to belong to year 3 of the Metonic cycle and this fixes the rest of the construction of the lunar table.

After the golden number of 1 January is fixed as III, the subsequent golden numbers III are filled in according to the distribution of full and hollow lunations as listed in Table 5. For example, the second new moon in the third year of the Metonic cycle is scheduled after a full lunation, that is, a 30-day interval, from the first new moon. Hence, the second golden number III is inserted at 31 January. When the counting passes into the following year, the next golden number is introduced. For instance, after the last new moon of year III occurs on 21 December, the next new moon would occur 29 days later. This day coincides with 20 January of the next year and thus the new moon entered on 20 January has golden number IV. The rest of the golden numbers are entered in similar fashion. In particular, since the Metonic cycle is a 19-year cycle, golden number I follows XIX.

Having discussed the construction of the lunar table, we highlight some properties of the table.

Firstly, there are dates that have no golden numbers attached. We call them “empty dates”. This phenomenon is expected because there are only 235 new moons in the Metonic cycle, which translates to at most 235 date slots of a 365-day table being filled. Furthermore, since each date slot can only hold 1 golden number, exactly 235 date slots are required to contain all 235 new moons. This leaves 130 “empty dates” in a common year lunar table. What happens to the lunar table during a leap year? No doubt there are 366 days in a leap year. Yet, since the dates marked with golden numbers indicate notional new moons which may not correspond to the real astronomical new moons, it is therefore a legitimate choice for deeming 29 February, the leap day, not to carry any notional new moon at all throughout the Metonic cycle. Hence the given lunar table with 365 date slots is sufficient to list the new moons that occur in a certain year of the Metonic cycle, regardless of whether the year is a common or a leap year.

Secondly, in the table, the golden numbers repeat in a fixed sequence throughout the year; the sequence of golden numbers runs as follows: III, XI, XIX, VIII, XVI, V, XIII, II, X, XVIII, VII, XV, XII, I, IX, XVII, VI, XIV. This happens because the insertion of golden numbers is based on a fixed distribution of full and hollow lunations, as listed in Table 5. Remember that the construction of the lunar tables begins from golden number III and the next golden number is brought in when the lunation ends in the following year. After the last new moon of the second year in the Metonic cycle occurs on 2 December, the next new moon occurs on 1 January of the next year. In this way, after every nineteen years, golden number III would be returned to 1 January and not to some other date. Since the distribution of the lunations remains unchanged, that is, Table 5 is unaltered, the golden numbers of a second Metonic cycle would be repeated identically on the same dates as before. By similar argument, the golden numbers of any subsequent Metonic cycle would also repeat on the same dates.

The third observation concerns the interval between two successive golden numbers, and a distinct pattern in the values of the golden numbers. For each golden number, the next golden number occurs either on the next day or on the second day following. This gives 1 as the maximum number of "empty dates" between any two golden numbers. Furthermore, the next golden number would always be increased by 8, as shown in Table 7.

Table 7

G	**Subsequent *G***	***G***	**Subsequent *G***
III	XI	XIII	II
IV	XII	XIV	III
V	XIII	XV	IV
VI	XIV	XVI	V
VII	XV	XVII	VI
VIII	XVI	XVIII	VII
IX	XVII	XIX	VIII
X	XVIII	I	IX
XI	XIX	II	X
XII	I		

We observe that golden number XII, after adding 8, is followed by golden number I instead of XX. This is expected since the golden numbers run from I to XIX only. Any golden number, G, bigger than XIX would bear the golden number, $1 + \text{mod}(G - 1, 19)$. Then by adding 8 into the argument of the function, we have $1 + \text{mod}(G + 7, 19)$. That is, given any golden number G, the following golden number is of value, $1 + \text{mod}(G + 7, 19)$.

The regular pattern of occurrence of the golden numbers can be explained by the *Octaeteris* Cycle. This is a period of 8 solar years after which a new moon occurs on the same calendar date plus 1 or 2 days. Suppose a new moon occurs on day d. Then the next new moon listed on the table would occur on either day $d + 1$ or $d + 2$. Respectively, this brings about either zero or one "empty date" on the lunar table between the two golden numbers in question.

Let us understand the *Octaeteris* Cycle in greater detail. Recall that the Metonic cycle has the property that 19 solar years contains approximately the same number of days as 235 lunations do. Embedded within the Metonic cycle, we find that there is an 8-year cycle that gives a fairly close approximation too. The number of days contained in 8 solar years is 2922 days, and that in 99 lunations, assuming that the average length of a lunation is 29.5 days, is 2920.5 days. By a simple comparison, there is only an average difference of 1.5 days. Such a difference amounts to having a subsequent new moon occurring on either one or two days after the date of the initial new moon. We call this the *Octaeteris* Cycle.

Finally, we observe that all nineteen different golden numbers could be found in either a 29-day or 30-day period. The sequence of the numbers is unchanged even though the length of days that accommodates the golden numbers varies between 29 and 30. This is made possible by reducing the number of "empty dates" in the period. The golden numbers are in turn "squeezed" together. For instance, in certain lunations, golden number VIII is followed immediately by XVI, yet in others, there is an "empty" date between the two golden numbers. For later discussion, let us note that for golden numbers XII and bigger, the subsequent golden number would always occur without any "empty" date in between.

3.1.2 Paschal full moon

The word "paschal" is used in any context that relates to the celebration of Easter. As highlighted earlier on, Easter is to be celebrated on the Sunday following the paschal full moon. The paschal full moon is defined to be the first full moon that occurs on or after the vernal equinox that is traditionally taken to be 21 March.

Now, a full moon occurs on the 14th day from its corresponding new moon, where the day of new moon is considered as the 1st day. Hence the paschal new moon occurs no earlier than 13 days before 21 March; it falls on or after 8 March. Referring back to Table 6, we note that there is a new moon on 8 March in year XVI of the Metonic cycle. Moving down the table from 8

March, we find that the last paschal new moon of the remaining 18 years of the Metonic cycle, falls on 5 April in year VIII. This new moon on 5 April has its corresponding full moon on 18 April. In summary, the range of dates for a paschal new moon is between 8 March and 5 April while that for a paschal full moon is between 21 March and 18 April.

Taking the range of dates for paschal full moon into consideration, we now seek to find out the range of dates possible for Easter celebration itself. Suppose 21 March has a full moon, the earliest Sunday after 21 March would be on 22 March. Suppose 18 April has a full moon, and it is a Sunday, then Easter celebration would take place on 25 April, but no later. Hence Easter Sunday can occur on any date between 22 March and 25 April.

On Table 6, dates with paschal new moons have been marked with asterisks. Within the period of 29 days, all 19 golden numbers are represented. This indicates that the paschal new moons of different years in the Metonic cycle fall on different dates. If a lunar table is constructed such that it marks out the dates of full moons, then it is possible to identify a similar 29-day period that contains the paschal full moons. Such a period also has 19 distinct golden numbers occurring in it.

3.1.3 Dominical letter/number

Since Easter is to be celebrated on a Sunday, there must be a way to find out which day of the week a date falls on. Both calendar letters/numbers and dominical letters/numbers are used to determine the day of the week. We first discuss the former.

Seven letters, A to G, are used to specify the different days of a week. In addition, there is also an associated set of seven numbers to the days of a week. The letter or number corresponding to Sunday in any year is called the "dominical letter/number" for that year. The associations are laid out clearly in Table 8.

Table 8

Day of week	Letter	Number
Sunday	A	1
Monday	B	2
Tuesday	C	3
Wednesday	D	4
Thursday	E	5
Friday	F	6
Saturday	G	7

Imagine that the seven letters A to G are allocated sequentially to every day of the common year: A is allocated to 1 January, B to 2 January and so on, with 7 January receiving G. Thereafter, 8 January receives A and the allocation continues till the end of the year. In a leap year, 29 February receives the same letter as 1 March. These letters are called "calendar letters". Referring back to Table 6, the letters indicated beside the golden numbers are in fact the calendar letters.

With such an allocation in place, we then have a fixed calendar letter for the first day of each month, regardless of whether it is in a common or leap year. The leap day is given the same letter as that on 1 March and thus it does not upset the sequence of the calendar letters. From January to December, the fixed calendar letter of the first day of each month in that order is A, D, D, G, B, E, G, C, F, A, D, and F. The letter entries in the sequence are not arbitrarily chosen. Since January has 31 days, counting on from A on 1 January, we arrive at D on 1 February. February has 28 or 29 days. So 1 March has calendar letter D. Similarly, the subsequent letters are derived. Given any date, it is then easy to count on from the first day of the particular month that the date is in, to find out the day that the date falls on.

To work out the day via numerical calculations, we first convert the calendar letters into calendar numbers. Let F denote the calendar number of the first day of the month, D denote the day in the month and C denote the calendar number of the date in question. Then,

$$C = 1 + \text{mod}\ (F + D - 2, 7)$$

The basic formula for calculating the calendar number of a date is mod ($F + D - 1, 7$). However this gives 0 to 6 as the range for calendar numbers. To augment the range to yield 1 to 7, the formula receives a "-1" within the argument of the modulo function and a "+1" outside. For later discussion, let us take R as the day of March, that is, $R = F + D$. Then the calendar number of this day is,

$$C = 1 + \text{mod}\ (R - 2, 7).$$

We now turn towards the dominical letters/numbers. As defined earlier, the dominical letter/number is a letter/number which specifies Sunday. This letter/number changes from year to year due to two reasons.

Firstly, since each solar year comprises 52 weeks and 1 day, the last day of the year would bear the same calendar letter as the first day of the year. In the following year, the calendar letters/numbers would have "regressed" by one. By this, it means that Sunday has letter G instead of A, Monday is given A instead of B, Tuesday is given B instead of C, and so on; the allocation of letters to the days has been shifted "backwards" by one place. Consequently, the dominical letter regresses by one every succeeding year.

Secondly, the inserted leap day, 29 February, also causes the set of calendar letters after this date to regress by one. Hence, in a leap year, each day has two different calendar letters to represent it. In turn, there are also two dominical letters in a leap year. Every Sunday before 1 March has a certain letter while every Sunday on or after 1 March has another letter. The letter for the later Sundays is one letter back from that for the earlier Sundays. Adding the one-letter regression brought about by the leap day and the usual one-letter regression brought about by the yearly transition, we can thus understand why the calendar letters for a year succeeding a leap year regresses by two.

The following example helps to illustrate the regression pattern of the calendar letters just described.

Table 9

Day	Calendar letters in Year				
	1998	1999	2000	2001	
Sunday	D	C	B, A	G	← Dominical letter
Monday	E	D	C, B	A	
Tuesday	F	E	D, C	B	
Wednesday	G	F	E, D	C	
Thursday	A	G	F, E	D	
Friday	B	A	G, F	E	
Saturday	C	B	A, G	F	

The pattern of shift for every calendar letter in Table 9 is the same. For simplicity, we just look at the dominical letter. AD 1998 has dominical letter D. In AD 1999, the dominical letter is C, upon a regression of 1 letter. Since AD 2000 is a leap year, there are two dominical letters; letter B prior to 1 March and A on and beyond 1 March. Finally, compared to the initial dominical letter of AD 2000, the dominical letter of AD 2001 has regressed by two (from B to G).

From the above, we see that the dominical letter/number is closely tied to the number of regressions that has occurred prior to the year in question. Indeed, we can work out the dominical letter/number N of a certain year Y from the number of regressions Γ made by the end of year Y. Here, the Julian leap year rule is considered.

The number of regressions Γ made by the end of year Y, starting from AD 1, is $Y - 1$; AD 1 has 0 regressions. In addition, in the period between AD 1 and year Y, there are $\left\lfloor Y/4 \right\rfloor$ regressions. Therefore, by the end of year Y, the total number of regressions is,

$$\Gamma = Y - 1 + \left\lfloor Y/4 \right\rfloor$$

1 January of AD 1 fell on a Saturday. This means that the dominical letter of AD 1 was B and dominical number, 2. Therefore, whenever mod $(\Gamma, 7) = 0$, the dominical number $N = 2$; when mod $(\Gamma, 7) = 1$, the dominical number would have regressed by one to $N = 1$; and when mod $(\Gamma, 7) = 2$, the dominical number would have further regressed to $N = 7$. We can in turn express the

dominical number N as 7 – mod (Γ – 2, 7). Substituting $\Gamma = Y - 1 + \left\lfloor Y/4 \right\rfloor$, we have $N = 7 - \operatorname{mod}\left(Y - 1 + \left\lfloor Y/4 \right\rfloor - 2, 7\right) = 7 - \operatorname{mod}\left(Y - 3 + \left\lfloor Y/4 \right\rfloor, 7\right)$. This is equivalent to

$$N = 7 - \operatorname{mod}\left(Y + 4 + \left\lfloor Y/4 \right\rfloor, 7\right)$$

For instance, the dominical number for AD 2003 is

$$7 - \operatorname{mod}\left(2003 + 4 + \left\lfloor 2003/4 \right\rfloor, 7\right) = 7 - \operatorname{mod}(2507, 7) = 6.$$

Hence, the dominical letter for AD 2003 in the Julian calendar is F.

Equipped with the knowledge of how calendar letters and dominical letters are worked out, we can then obtain an easy method to determine the day of the week, W, that a particular date falls on.

Had the calendar number C of the chosen date been the same as the dominical number N of the year from which the date is picked, this date clearly falls on a Sunday. The difference $C - N$ indicates the number of days that the date falls beyond Sunday. That is, if $C - N$ is 1, the day is a Monday. If $C - N$ is 2, the day is a Tuesday and so on. Such a relation can be expressed in a modulo function, 1 + mod (C – N, 7). To avoid negative values of W, a further adjustment to this formula yields,

$$W = 1 + \operatorname{mod}(C - N + 7, 7)$$

Having seen an application of the dominical letters/numbers, let us now study the dominical letters or Sunday letters closer. Dominical letters or numbers occur in a cycle which is called the "solar cycle". The solar cycle reaches the point for restart when an arbitrary date returns to the same day that it fell on during the previous cycle. Let us consider 1 January to be the date in question. Suppose it falls on a Sunday in year x. If the calendar has no leap years, the next earliest year in which 1 January is a Sunday, is $x + 7$. On a Julian calendar, things are being complicated by the insertion of a leap day every four years. The lowest common multiple between the lengths of the Julian leap year cycle and the week is 4 × 7 = 28. Therefore, the length of a solar cycle on a Julian calendar is 28 years.

We conclude this section with a table that shows the position of a year in the solar cycle and its related dominical letter(s). The numbers 1 to 28 are called "solar numbers". It is assumed that the first year of a solar cycle should be a leap year and that 1 January is a Monday, yielding dominical letters G and F. Unfortunately, the justification for such a convention is not known.

Table 10 Relation between the position of a year in the solar cycle & its dominical letter

Year of the cycle	Dominical letter	Year of the cycle	Dominical letter
1	G, F	15	C
2	E	16	B
3	D	17	A, G
4	C	18	F
5	B, A	19	E
6	G	20	D
7	F	21	C, B
8	E	22	A
9	D, C	23	G
10	B	24	F
11	A	25	E, D
12	G	26	C
13	F, E	27	B
14	D	28	A

3.1.4 Epact

The epact of a year is defined as the age of the notional moon, in terms of days, on 1 January of the year. If a new moon occurs on 1 January, that year has epact 0. However strictly speaking, such a definition does not apply in the Julian calendar but rather in the Gregorian calendar that developed after. For the Julian calendar, some sources define the epact as the age of the notional moon on 22 March each year.

With reference to [22], the Julian epact E (with epact as the age of notional moon on 1 Jan) may be calculated from the golden number G with the following formula: $E = 8 + \text{mod}\ (11 \times (G - 1), 30)$. This agrees with the neater formula for all values of G running from 1 to 19, as given in [8],

$$E = \mathrm{mod}\ (11 \times (G - 3), 30)$$

This formula would be the one used in the subsequent development of this paper. We now highlight several reasons for why this formula works for finding the epact.

As reflected in Table 6, the epact of year III in the Metonic cycle is 0. The same result is obtained from the formula when $G = 3$ is substituted in it. Furthermore, it encapsulates a characteristic property of the epact that it increases by 11 every succeeding year. This regularity stems from the fact that the lunar year of 354 days is shorter than the solar year by 11 days. However, there is one exception. Recall that *saltus lunae* occurs during the transition from the 19th year of the Metonic cycle to the 1st year of the next cycle. This 1-day skip in the Metonic cycle leads to a 12-day increase in the epact instead of 11 days. Since a lunation has a maximum length of 30, the value of the epact does not exceed 30, which gives rise to the congruence modulo 30 component in the formula. As a result, the epact lies in the range of 0 to 29. This is the traditional way of stating epacts. In astronomical studies however, the epact is usually given in the range 1 to 30, where the age of a new moon corresponds to epact value 1. For the rest of this paper, the epact range should be taken as 0 to 29, unless otherwise stated.

From the formula, we can deduce that there is a unique epact for every golden number. Once we have shown that the given formula is an injective function, we are done.

Suppose $E_1 = E_2$ where $E_1 = \mathrm{mod}\ (11 \text{ x } (G_1 - 3), 30)$ and $E_2 = \mathrm{mod}\ (11 \text{ x } (G_2 - 3), 30)$. Then,

$$\begin{aligned}
&\mathrm{mod}(11 \times (G_1 - 3), 30) = \mathrm{mod}(11 \times (G_2 - 3, 30) \\
&\Rightarrow \mathrm{mod}(11 \times (G_1 - G_2), 30) = 0 \\
&\Rightarrow \mathrm{mod}(G_1 - G_2, 30) = 0 \qquad \text{since } \gcd(11,30) = 1 \\
&\Rightarrow G_1 - G_2 = 0 \\
&\Rightarrow G_1 = G_2
\end{aligned}$$

This shows that the function is injective. Hence, there is a one-to-one correspondence between every golden number G and epact E.

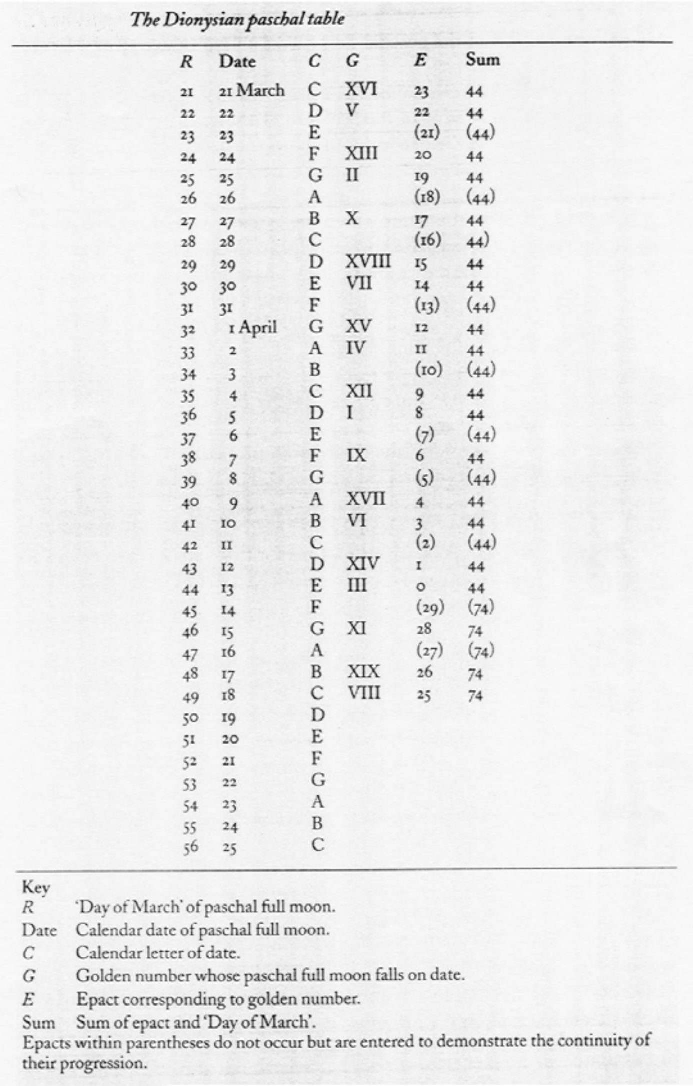
3.2 Determining the Julian Easter Sunday

Easter Sunday may be determined with or without the use of tables. The calculation uses the concept of epact and its link with the day of March of paschal full moon. The following three sections provide a discussion of three methods used in finding Easter Sunday. The first two methods make use of tables and were put into practice in the past. The last method is proposed in [8] as a possible computation algorithm for obtaining the date for Easter.

3.2.1 Using paschal table, dominical letter and golden number

Before the practice of using epacts became common, people used lunar tables to determine Easter. They first extracted information from lunar tables to form a paschal table. The sample paschal table below is extracted from [8].

Table 11



The Dionysian paschal table

R	Date	*C*	*G*	*E*	Sum
21	21 March	C	XVI	23	44
22	22	D	V	22	44
23	23	E		(21)	(44)
24	24	F	XIII	20	44
25	25	G	II	19	44
26	26	A		(18)	(44)
27	27	B	X	17	44
28	28	C		(16)	44)
29	29	D	XVIII	15	44
30	30	E	VII	14	44
31	31	F		(13)	(44)
32	1 April	G	XV	12	44
33	2	A	IV	11	44
34	3	B		(10)	(44)
35	4	C	XII	9	44
36	5	D	I	8	44
37	6	E		(7)	(44)
38	7	F	IX	6	44
39	8	G		(5)	(44)
40	9	A	XVII	4	44
41	10	B	VI	3	44
42	11	C		(2)	(44)
43	12	D	XIV	1	44
44	13	E	III	0	44
45	14	F		(29)	(74)
46	15	G	XI	28	74
47	16	A		(27)	(74)
48	17	B	XIX	26	74
49	18	C	VIII	25	74
50	19	D			
51	20	E			
52	21	F			
53	22	G			
54	23	A			
55	24	B			
56	25	C			

Key

R	'Day of March' of paschal full moon.
Date	Calendar date of paschal full moon.
C	Calendar letter of date.
G	Golden number whose paschal full moon falls on date.
E	Epact corresponding to golden number.
Sum	Sum of epact and 'Day of March'.

Epacts within parentheses do not occur but are entered to demonstrate the continuity of their progression.

Table 11 contains a lot of information. However, in the most primitive way of determining Easter, only a few items from the table are required. These include the date, its calendar letter, and associated golden number, if any. We note that these golden numbers in the paschal table indicate paschal full moons and not paschal new moons. Here is a brief run-through of the steps used in using the paschal table to determine Easter Sunday: firstly, the golden number and dominical letter are found through the formulas as given before:

$$G = 1 + \text{mod}\,(Y, 19)$$

$$N = 7 - \text{mod}\,(Y + 4 + \left\lfloor Y/4 \right\rfloor, 7)$$

Next, the line that contained the golden number is identified. From this line downwards in the date column, the first date that has a calendar letter that matches the dominical letter of the year in question, is Easter Sunday.

3.2.2 Using Easter tables containing epacts

There is an alternative kind of tables that were used to determine Easter. They made use of both golden numbers and epacts. A sample has been taken from [3] and reproduced below as Table 12.

Table 12 Excerpt from the Easter table of Dionysius

A	B	C	D	E	F	G
532b	1	0	4	5 Apr	11 Apr	20
533	2	11	5	25 Mar	27 Mar	16
534	3	22	6	13 Apr	16 Apr	17
535	4	3	7	2 Apr	8 Apr	20
536b	5	14	2	22 Mar	23 Mar	15
537	6	25	3	10 Apr	12 Apr	16
538	7	6	4	30 Mar	4 Apr	18
539	8	17	5	18 Apr	24 Apr	20
540b	9	28	7	7 Apr	8 Apr	15
541	10	9	1	27 Mar	31 Mar	18
542	11	20	2	15 Apr	20 Apr	19
543	12	1	3	4 Apr	5 Apr	15
544b	13	12	5	24 Mar	27 Mar	17
545	14	23	6	12 Apr	14 Apr	18
546	15	4	7	1 Apr	8 Apr	21
547	16	15	1	21 Mar	24 Mar	17
548b	17	26	3	9 Apr	12 Apr	17
549	18	7	4	29 Mar	4 Apr	20
550	19	18	5	17 Apr	24 Apr	21

This table is an excerpt from an Easter table which was prepared by Dionysius Exiguus. It contains information only from AD 532 till AD 550. Let us first familiarize ourselves with the denotations of each column and then find out how epacts were actively used in such tables to determine Easter.

Column A lists the years in the cycle where "b" signifies leap year. The golden numbers are given in column B. In Column C, the epact on 22 March (original definition of the Julian epact) is given. In Heilbron's labeling, "Luna" is used to replace the Christian Nisan. "Luna" refers to the lunation that contains the full moon that determines the holiday, or simply, the lunation that contains the paschal full moon. 1 Luna refers to the day of that particular month whose epact is 1 and 14 Luna refers to that with epact 14. That is, 14 Luna is the day on which the paschal full moon falls. Column D lists the "ferial number" or what we refer to as "calendar number" of 24 March. Column E provides the date for 14 Luna, column F, the date for Easter Sunday and finally, column G, the age of the moon on Easter Sunday.

At this juncture, we highlight a potentially confusing feature of the table. At first glance, the entries in Column C are found within the range 0 to 29. As such, we would likely be influenced to use the traditional way of reading epacts. That is, we take the epacts to range from 0 to 29 and that a new moon has epact 0. However, if we inspect Columns C and E closely, we would discover that the epacts should be taken to run from 1 to 30 instead, in order to make sense of the dates entered for paschal full moon in Column E. We know that the full moon occurs on the 14th day from the day of new moon. If the epact range is 0 to 29, the full moon corresponds to epact value 13. If the range is 1 to 30, the full moon corresponds to epact value 14. Consider the row of entries for AD 532. Its epact on 22 March is given as 0 in Column C. Suppose the epact range adopted is 0 to 29, the full moon should then occur on 12 April, and not 13 April as listed in Column E. Similarly, the rest of the rows would be valid only if the epacts are read from the range 1 to 30. As such, 22 March of AD 532 should be taken as the day before a new moon occurred, and its "0" epact be interpreted as "30" instead.

How do epacts come into play in building the table? As can be inferred from the description above, Column C leads to Column E. If the epact on 22 March is known, then the date for 14 Luna or paschal full moon can be determined. By knowing the day on which 24 March falls, we can then count to the date found for 14 Luna to find out the day that it falls on. Thereafter, the date of Easter Sunday may be obtained by counting on the dates from 14 Luna till the first Sunday immediately after. Column F is thus established. This process of finding the first Sunday after 14 Luna fits in with the rule that Easter is to be celebrated on the first Sunday that follows the paschal full moon. Finally, the difference in dates of 14 Luna and Easter Sunday gives the number of days that the paschal full moon had waned by the time Easter is celebrated. Hence the sum of this number and 14 gives the epact on Easter Sunday and the value is inserted in Column G.

Let us work through an example to better appreciate the steps described above. From the table, the epact on 22 March 546 is 4. Thus the date of 14 Luna is 10 days beyond 22 March. Counting on for 10 days from 22 March, we get 1 April as the date for 14 Luna. The "ferial number" of 24 March 546 is 7 and so it is a Saturday. Counting on in terms of days of the week from 24 March, we find that 1 April falls on a Sunday. Since the paschal full moon falls on a Sunday, Easter is celebrated on the following Sunday, which is 8 April. The age of the moon on 8 April is then 7 days older than the full moon. Its epact is thus 14 + 7 = 21.

3.2.3 By calculation

Referring to Table 11 again, we identify a few properties concerning epacts which enable us to calculate Easter Sunday without the use of tables.

In the paschal table, the epacts associated with each date from 21 March to 18 April are listed in column *E*. This is the range of dates for paschal full moons and all 19 golden numbers are found in this range. As highlighted before, there is a unique epact for each golden number. Therefore, there are also 19 values for the epact. After these 19 epact values are filled up, a regular decreasing sequence of epacts, other than a jump at 14 April, is discernible. The regularity of the sequence makes it possible to allocate suitable epact values to the dates that do not have any golden numbers attached. These epact values are bracketed on Table 11. As a result, we have a sequence of epact values between 21 March and 13 April that decreases steadily from 23 to 0, jumps to 29 on 14 April and begins decreasing again until 25.

Now consider the sum of the epact and the day of March of paschal full moon, R, where R refers to the number of days that a date falls beyond 1 March, taking 1 March as having $R = 1$. For instance, 3 March has $R = 3$ and 1 April, being the 32nd day from 1 March, has $R = 32$. The last column of Table 11 indicates this sum. By observation, we see that the R values increase at the same rate at which the E values decrease, except for a sudden break on 14 April. Therefore the sum between R and E remains constant at 44 between 21 March and 13 April, and at 74 between 14 April and 18 April. The difference between these two constants is 30. This net increase of 30 days arises from the jump in the values of the epact from 0 to 29 at 14 April. We also observed that only those epacts that are greater or equal to 24 are linked to the sum 74 while the rest of the epacts are linked to 44. This neat relation between the epact E, and day of March R provides a quick way to determine the date of the paschal full moon, as highlighted below.

To calculate Easter, first find the dominical letter, N, and golden number, G. Then derive the epact, E from the golden number. The day of March, R, of the paschal full moon can be found from the following,

$$R = 44 - E \qquad \text{if } E < 24$$

$$R = 74 - \mathrm{E} \qquad \text{if } E \geq 24$$

Alternatively, we can have a single formula for R. Since the paschal full moon falls no earlier than 21 March, R has a minimum value of 21. Suppose the paschal full moon falls on a date later than 21 March, then the difference is accounted for by the relation mod $(30 + 24 - E, 30)$, or equivalently, mod $(54 - E, 30)$. The resultant formula for R is thus,

$$R = 21 + \mathrm{mod}\ (54 - E, 30)$$

Using the result for R, we can find the calendar number, C, of that day by an earlier mentioned formula,

$$C = 1 + \text{mod}\,(R + 2, 7)$$

Let d be the number of days after the paschal full moon that Easter is celebrated. Easter Sunday thus falls on the day of March, $R + d$.

If $C = N$, then $d = 7$

If $C < N$, then $d = N - C$

If $C > N$, then $d = 7 - (N - C) = 7 - N + C$

In general, $d = 7 - \text{mod}\,(C - N, 7)$

Therefore, the day of March of Easter Sunday, S, is

$$S = R + 7 - \text{mod}\,(C - N, 7).$$

By following these steps, given any year Y in the Julian calendar, in AD dating, we can find the date of Easter Sunday in Y. Here is an example.

If $Y = 1550$,

$N = 7 - \text{mod}\,(1550 + 4 + \left\lfloor 1550/4 \right\rfloor, 7) = 5.$

$G = 1 + \text{mod}\,(1550, 19) = 12$

$E = \text{mod}\,(11 \times (12 - 3), 30) = 9$

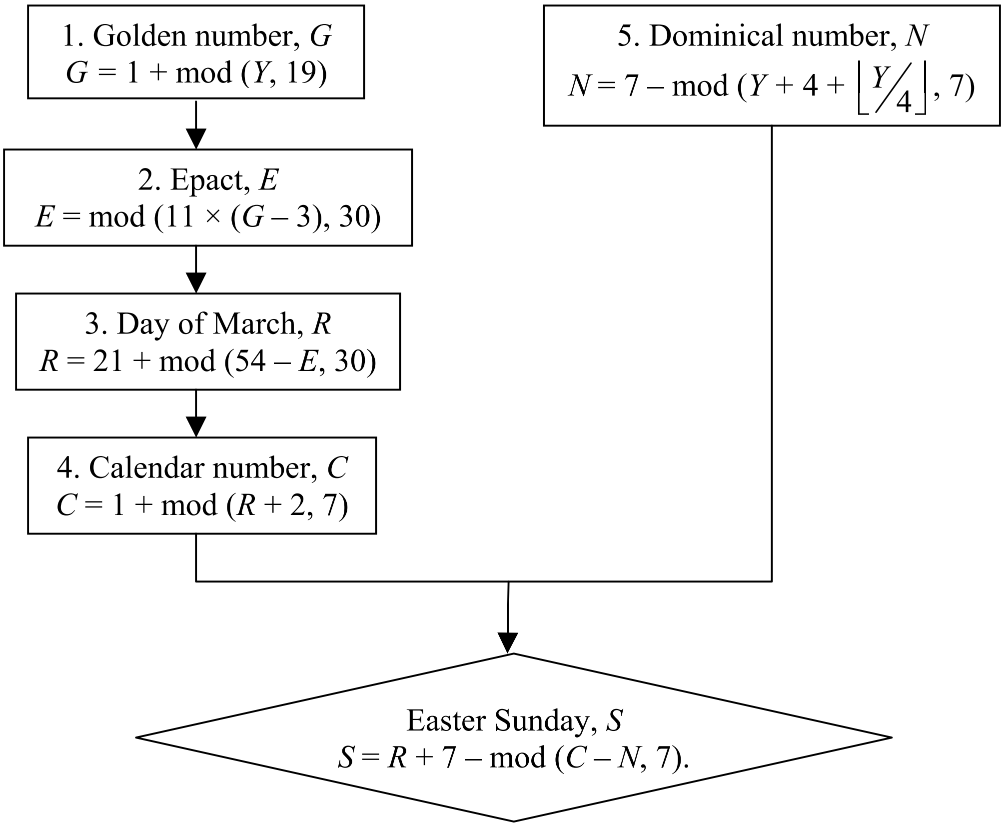
$R = 21 + \text{mod}\,(54 - 9, 30) = 36$

$C = 1 + \text{mod}\,(36 + 2, 7) = 4$

$S = 36 + 7 - \text{mod}\,(4 - 5, 7) = 37.$

So, Easter Sunday in AD 1550 fell on the 37th day of March, which corresponds to 6 April.

Here is a flow-chart to summarize the steps taken to calculate Easter Sunday for any year Y in AD dating.



During the time when the Julian calendar was still in use, the computation steps given in this section no doubt could have been used to determine Easter Sunday. However, in that age, people had still relied mainly on tables to enable them to determine Easter Sunday. It was only after the Gregorian calendar was adopted that determining Easter by calculation became popular. In the next chapter, we see why the Gregorian calendar came to displace the Julian calendar as the calendar of choice.

Chapter 4

From Julian to Gregorian

4.1 Problems with the Julian calendar

Even though the Julian calendar was widely used throughout the Roman Empire and later by various Christian churches, there were problems with it that amounted to a sharp call for amendments to be made.

The average length of a solar year on the Julian calendar is 365.25 days. However, the length of the average astronomical tropical year is about 365.2422 days. Hence, the Julian year is longer than the actual length of the year by 0.0078 days or approximately, 0.0078 x 24 x 60 ≈ 11 minutes. What this amounted to was the gradual shift of day of occurrence of astronomical events, such as vernal equinox, to earlier calendar dates. By AD 1582, the vernal equinox was observed to occur on 11 March, a whole ten days earlier than the scheduled date for vernal equinox, 21 March.

Not only was the calendar year designed to be too long than the length of the astronomical year, the length of the lunation used in building the lunar tables was also longer than the actual length of a lunation. The Metonic cycle contains in all, 19 x 365.25 days or equivalently, 6939.75 days. These days are contained in 235 lunations. Thus each lunation has an average length of 6935.74/235 = 29.53085 days. Yet the value of the average synodic month is about 29.53059 days. By a simple comparison, we find that the notional lunation length has exceeded the true length by about 0.00026 days, or approximately 22 seconds. As a result, the astronomical new moons were observed to occur on earlier calendar dates than the ones listed in the lunar tables. Eventually, by AD 1582, the real paschal full moon was observed to be occurring 4 days before the notional paschal full moon.

Such discrepancies between the dates for astronomical phenomena and the dates for tabulated phenomena were annoying. However, urgent calls for reform only came along when the Catholic Church found that Easter was gradually celebrated on dates that were increasingly removed from those required to fulfill the original intentions of the Council of Nicaea. The Council had ruled the celebration to take place on the first Sunday following the paschal full moon. Yet over time, the celebration was taking place too late because people referred to a delayed notional paschal full moon. It was deduced that Easter would be celebrated in the seasonal summer and eventually winter if no corrections were made to the calendar. Since Easter is a very important event in the Christian faith, the continual slipping of dates for its celebration was not accepted by the Catholic Church. To counter the problem, the Church thought up and implemented amendments to the Julian calendar. These have been brought up in “Papal Reform” of [3] and are further elaborated upon in the next two sections.

4.2 Gregorian reforms

The reforms made to the Julian calendar were spearheaded by Pope Gregory III and thus the new calendar was name after him as the “Gregorian calendar”. He had recruited several astronomers, principally the Jesuit Christopher Clavius, to come up with a solution to stop the slipping of dates for Easter celebration. The astronomers came up with several calendar reform proposals based on those by the astronomer and physician Luigi Lilio. Eventually, Clavius’ proposal was implemented. Pope Gregory III issued a papal bull, Inter Gravissimas, on 24 February 1582 and thereby established what is now known as the Gregorian calendar reform. The reforms made may be broadly divided into two categories: solar and lunar, and are summarized below:

- Ten days were omitted from the calendar.
- The rule for leap years was changed.
- The position of the extra day in a leap year was moved from the day before 25 February to the day following 28 February.
- New rules for the determination of the date of Easter were adopted.

As highlighted before, by 1582, the vernal equinox was observed to be falling on a date that was 10 days before the notional vernal equinox on 21 March. To place the astronomical vernal equinox back in sync with the notional one, ten days were dropped in one fell swoop from October 1582: 4 October 1582 was immediately followed by 15 October 1582. The decision to drop ten days instead of another number has been controversial.

Some argued that ten days was insufficient. With the Julian year being 0.0078 days longer than the mean tropical year, by the end of AD 1582, the vernal equinox had slipped from that in AD 1 by 0.0078 x 1582 ≈ 12.34 days. So why had Pope Gregory III not removed twelve days instead of ten? The reason was because he had chosen the vernal equinox that occurred in AD 325 as the epoch instead. This is the year when the First Council of Nicaea had convened and established the vernal equinox as 21 March, thereby making AD 325 more appropriate than AD 1 as the year of reference for the vernal equinox. Between AD 325 and AD 1582, the vernal equinox had only slipped by 0.0078 x 1257 = 9.81 days, hence justifying the drop of ten days.

Yet there are still points of objection to AD 325 being a suitable epoch. Firstly, the actual vernal equinox in AD 325 had occurred on 20 March and not 21 March. The removal of ten days in AD 1582 had resulted in the vernal equinox occurring mostly on 20 March and less often on 21 March, as observations support. Thus some have argued that nine days should have been dropped instead. There were also times when the vernal equinox was observed to occur on 22 March. This has driven others to argue that an eleven-day amendment is more appropriate than a ten-day one. Presumably Pope Gregory's astronomical advisors had considered all three possibilities and took ten as a compromise. The fact that the omission of ten days required only the insertion of an "X" to correct old calendars perhaps also made the choice of a ten-day drop seem more favourable.

One last objection to AD 325 being a suitable epoch lies in that AD 325 was one year after a leap year, AD 324. Since AD 1580 was a leap year, the year that would have been in a more similar phase as AD 325 from AD 324, was

AD 1581. By such an argument, if changes were to be made in AD 1582, the vernal equinox taken for reference should have been taken from AD 326 instead of AD 325.

Besides the omission of days, there was another solar adjustment. The rule for leap years was changed in order to keep the calendar synchronized with the vernal equinoxes. That is, the new rule in the calendar ensured that the real vernal equinox occurred around the same calendar date every year. In the Julian calendar, any year that was divisible by 4 was a leap year. The new rule for leap years was aimed at shortening the Julian year by skipping some leap years. It made any year divisible by 4, except years divisible by 100 but not 400, a leap year. For instance, 1700, 1800 and 1900 were not leap years but 2000 was. As a result, a period of 400 years contain 400 x 365.25 – 3 = 146097 days, giving the Gregorian year an average length of 365.2425 days. Even though this was still longer than the mean tropical year length by 0.0003 days, or approximately 52 seconds, the vernal equinox would only lag behind 21 March by 1 day in about 1600 years.

The solar adjustments of dropping ten days from the calendar and changing the leap year rule were complemented with lunar adjustments. Just as leap days were dropped to shorten the calendar year, a similar scheme of dropping days was applied to shorten the lunations.

Recall that the average notional lunation had been too long by about 1 day in 312.5 years or 8 days in 2500 years. To accomplish the adjustment, a day is dropped from centurial years, or years divisible by 100, at intervals of 300 years for seven times and once more after another 400 years. The first drop was made in AD 1800. In addition, between AD 325 and AD 1582, the astronomical new moons had diverged from the notional new moons to dates that were (1582 – 325) / 312.5 ≈ 4 days earlier. After the solar adjustment of omitting ten days, these astronomical new moons were occurring 6 days after the notional new moons instead. Hence the net change in the dates of notional moons should have been a rescheduling of their dates to 6 days later. However in the actual adjustment, the dates of notional new moons were moved on to seven days later. The extra day was added to ensure that Easter Sunday will never fall on or before the Jewish Passover, a stipulation believed to have been made at the Council of Nicaea. In [3], it was mentioned that Gregory's committee justified this seven-day drop by "switching the epoch from the Nicene Council to the time of Dionysius Exiguus, which reduced the lunar correction from four days to three." This may be interpreted as: instead of taking AD 325 as the year where the astronomical new moons met the notional new moons, Gregory's committee used AD 532 as the reference year instead. Therefore by AD 1582, the number of days that the astronomical new moons had diverged from the notional ones amounted to (1582 – 532) / 312.5 ≈ 3 days instead of 4.

Another reform made concerns the shift of the leap day to 29 February. It is not clear exactly why such an adjustment was made. However here is a possible explanation. Under the Roman dating, the leap day was the second sixth day from the 1 March. After the reforms were made, the Roman way of dating became obsolete and people started to count dates forward from the start of the month instead. Naturally, rather than consider the extra day to be inserted in the middle of February, people started to treat the last day of the month, 29 February, to be the extra one.

The solar and lunar adjustments described above have led to alterations in the tools used in the determination of Easter Sunday in the Gregorian calendar. These alterations and new rules adopted for determining Easter Sunday in the Gregorian calendar are discussed in detail in the next chapter.

Chapter 5

Determining Easter in the Gregorian calendar

5.1 Modifications of tools

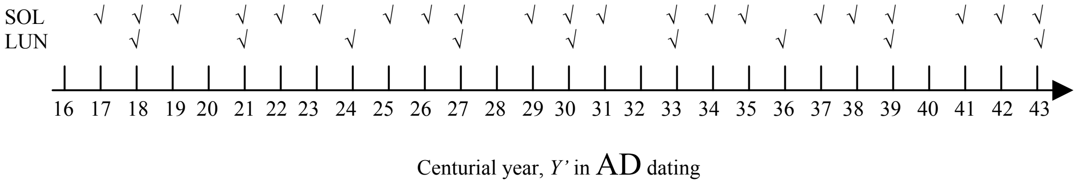
Since the Gregorian calendar is built upon the Julian calendar, the tools used for determining the Gregorian Easter Sunday are essentially the same ones that are used for determining the Julian Easter Sunday. However, some of these tools are modified before being put to use with the Gregorian calendar. The following sections give elaborations for the modifications done to each tool. Any concept or property that was mentioned in Section 3.1, if not brought up in the elaborations, should be taken as applicable to the Gregorian calendar as well.

5.1.1 Golden number and lunar table

Under the solar adjustment, certain leap years are skipped. Each time this is done, the date of a given astronomical event is increased by one day. In particular, the astronomical new moon occurs on a calendar day that is one day later than otherwise expected. This leads to the need to increase the dates of the notional moons by one and in turn, the need to move the golden numbers in the lunar table *down* to a position one day later. This adjustment is known as the solar equation, denoted by SOL. AD 1700 was the first centurial year after AD 1582 that had its leap date skipped and thus SOL was first applied in AD 1700.

Another amendment to the golden numbers arises from the lunar adjustment. In order to remove eight days in 2500 years, the golden numbers are shifted *up* by one day every 300 years for seven times and then once more after an interval of 400 years. This adjustment is known as the lunar equation and is denoted by LUN. AD 1800 was the first year in which LUN was applied.

The following is a timeline to indicate in which centurial years SOL and LUN are applied after AD 1582. Each *Y'* marks a centurial year with the 0's omitted. For instance, AD 1600 is expressed as 16 on the timeline. A " √ " above centurial year *Y'* along the row "SOL" or "LUN" means that SOL or LUN is applied in that centurial year, respectively.



Even though SOL and LUN are applied only in centurial years, the shifts in the golden numbers induced by them are cumulative over every year. SOL and LUN shift the golden numbers in opposing directions; the former shifts them *down*, the latter shifts them *up*. During centurial years, there are four possibilities in the way that golden numbers are shifted in the lunar tables, namely:

1. Both SOL and LUN are not applied. → Golden numbers are not shifted.
2. Both SOL and LUN are applied. → Since their effects cancel each other out, the golden numbers are not shifted.
3. SOL is applied, but not LUN. → Golden numbers are shifted down to a day later.
4. LUN is applied, but not SOL. → Golden numbers are shifted up to a day earlier.

SOL is applied more often than LUN and thus there is a progressive downward drift in the golden numbers. Let us work out the number of centuries required before the sequence of shifts repeats itself. SOL is applied 3 times every 4 centuries; LUN is applied 8 times every 25 centuries. Once more, we apply the principle: the length of a cycle is equivalent to the lowest common multiple of the lengths of its constituent cycles. The sequence of shifts forms a cycle with its constituent cycles being that of SOL and LUN. The lowest common multiple of the lengths of the SOL cycle and LUN cycle is 4 × 25 = 100. Therefore, the sequence of shifts repeats after every 100 centuries.

Over 100 centuries, there are 25 four-century periods and so SOL is applied 25 × 3 = 75 times. In addition, within these 100 centuries, there are 4 twenty-five-century periods. Thus, LUN is applied 4 x 8 = 32 times. The net shift in the golden numbers over these 100 centuries is therefore 75 – 32 = 43 days downward in the lunar table.

Depending on whether SOL or LUN is applied, in different years, a particular golden number could occur on different dates. It is not practical to mark the family of dates of each golden number into a single lunar table, such as the one shown as Table 6. Therefore, for the Gregorian calendar, multiple lunar tables have been constructed based on the Julian lunar table.

As a result of the seven-day increase in the dates of notional new moons, each golden number in the Julian lunar table was shifted to a position seven days later. This gave an initial Gregorian lunar table which was valid from AD 1582 to AD 1699. In AD 1700, SOL was applied but not LUN. Therefore, the golden numbers in the lunar tables were shifted down one more day. In subsequent centurial years, the golden numbers are shifted up and down according to the dictates of SOL and LUN.

How long does it take for the golden numbers in the lunar tables to return to their position after the seven-line drop in AD 1582 before the sequence of shifts begins again? This requires us to find out how many centuries it takes before the downward shift of golden numbers amounts to 30 days. 30 days is considered because it is the period that always contains all the 19 different golden numbers. Therefore, after a downward shift that amounts to 30 days, the dates covered by the initial sequence of golden numbers will be covered again by the exact same sequence of golden numbers.

Even though we know that that there is a 43-day shift every 100 centuries, it is not possible to work out the number of centuries a 30-day shift requires by using a direct ratio method. This is because the shift does not occur at regular intervals. Instead, we have to break the calculation up into several components. Firstly, compute the number of days by which the dates of notional new moons are increased by SOL after x centurial years have passed. Secondly, compute the number of days by which the dates of notional new moons are decreased by LUN after x centurial years have passed. Lastly, find x such that the difference of days shifted due to SOL and LUN is 30. Here is a demonstration of the computation process; x is arbitrarily chosen at each stage:

Table 13

x centurial years after AD 1582	No. of days shifted by		Net shift
	SOL	LUN	
25	Down by 18	Up by 8	Down by 11
50	Down by 37	Up by 16	Down by 21
60	Down by 45	Up by 19	Down by 26
70	Down by 52	Up by 22	Down by 30

Note: The first centurial year after AD 1582 is AD 1600. SOL was first applied in AD1700; LUN was first applied in AD 1800.

From the above, we see that it takes 70 centurial years to elapse after AD 1582 before the golden numbers would return to their original positions in AD 1582. That is, the golden numbers between AD 8500 and AD 8599 will be in the same positions as those in AD 1582 after the seven-line drop.

We could have formulae to compute the number of days that the golden numbers are shifted by SOL and LUN. These are discussed in the following.

Let SOL* denote the number of days by which the dates of the notional new moons must be increased on account of the decrease in the average length of the calendar year for year, Y.
Let LUN* denote the number of days by which the dates of the notional new moons must be decreased on account of the lunar adjustment for year, Y.
We first work out a formula for SOL*.

SOL* depends on the number of centurial years passed since AD 1600, that is, it depends on $\left\lfloor Y/100 \right\rfloor - 16$. SOL is applied 3 times in every four centuries, starting from AD 1700. That is, 3 leap days are dropped in every four centuries. By year Y, the number of complete four-century periods is

$\left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 16\right)}{4} \right\rfloor$ and hence the number of leap days dropped over all these four-century periods is $3 \times \left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 16\right)}{4} \right\rfloor$. After all these complete four-century periods are considered, there may also be up to three centurial years remaining before year Y is reached. The number of leap days dropped in this remainder is equivalent to $\mathrm{mod}\left(\left\lfloor Y/100 \right\rfloor - 16, 4\right)$. Therefore, the total number of leap days dropped by the end of year Y is,

$$\begin{aligned}
SOL^* &= 3 \times \left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 16\right)}{4} \right\rfloor + \mathrm{mod}\left(\left\lfloor Y/100 \right\rfloor - 16, 4\right) \\
&= 3 \times \left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 16\right)}{4} \right\rfloor + \left(\left\lfloor Y/100 \right\rfloor - 16\right) - 4 \times \left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 16\right)}{4} \right\rfloor \\
&= \left\lfloor Y/100 \right\rfloor - 16 - \left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 16\right)}{4} \right\rfloor \\
&= \left\lfloor Y/100 \right\rfloor - 16 - \left\lfloor \frac{\left\lfloor Y/100 \right\rfloor}{4} \right\rfloor + 4 \\
&= \left\lfloor Y/100 \right\rfloor - \left\lfloor \frac{\left\lfloor Y/100 \right\rfloor}{4} \right\rfloor - 12.
\end{aligned}$$

We next work out a formula for LUN*.

LUN* depends on the number of centurial years passed since AD 1800, that is, it depends on $\left\lfloor Y/100 \right\rfloor - 18$. LUN is applied 8 times in every 25 centuries, starting from AD 1800. Beginning the count from AD 1800, the number of complete twenty-five-century periods that year Y contains is $\left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 18\right)}{25} \right\rfloor$.

Hence the number of days dropped over all these twenty-five-century periods is $8\times\left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 18\right)}{25} \right\rfloor$. Let Z denote the number of centurial years that has passed by the end of year Y, in the current twenty-five-century cycle of year Y. Z is given by $\operatorname{mod}\left(\left\lfloor Y/100 \right\rfloor - 18, 25\right)$ and thus Z can range from 0 to 24. Recall that LUN is applied at 3-century intervals for the first 21 centuries of the cycle, and its last application is made after another 4 centuries. If $Z < 24$, then $\left\lfloor Z/3 \right\rfloor$ days are removed; if $Z = 24$, then 7 days are removed. However, $\left\lfloor Z/3 \right\rfloor$ leads to 8 days being omitted when $Z = 24$. To fix this expression up to describe the behaviour of LUN, we make use of $\left\lfloor Z/24 \right\rfloor$. Note that $\left\lfloor Z/24 \right\rfloor = 1$ if $Z = 24$; $\left\lfloor Z/24 \right\rfloor = 0$ if $Z < 24$. Hence the number of days omitted from the current 25-century cycle of Y may be expressed as $\left\lfloor Z/3 \right\rfloor - \left\lfloor Z/24 \right\rfloor$. To account for the day omitted in AD 1800, 1 is added to the total lunar correction. Therefore, the total number of days dropped by the end of year Y is,

$$LUN^* = 1 + 8\times\left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 18\right)}{25} \right\rfloor + \left\lfloor Z/3 \right\rfloor - \left\lfloor Z/24 \right\rfloor$$

where $Z = \operatorname{mod}\left(\left\lfloor Y/100 \right\rfloor - 18, 25\right)$.

There is a simpler formula for LUN^*. Consider the following:

$$\left\lfloor \frac{\left\lfloor Y/100 \right\rfloor - 15 - \left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 17\right)}{25} \right\rfloor}{3} \right\rfloor \text{ where } \left\lfloor Y/100 \right\rfloor \geq 18.$$

We check that this exactly describes LUN*. When $\left\lfloor Y/100 \right\rfloor = 18$, the formula gives a value of 1. This corresponds to the fact that LUN is first applied in AD

1800 and 1 day is dropped then. The value given by the formula increases by 1 every time $\left\lfloor Y/100 \right\rfloor$ increases by 3, which corresponds to the fact that LUN is applied at intervals of 300 years. However, when $\left\lfloor Y/100 \right\rfloor = 42,$ we have $\left\lfloor \frac{\left\lfloor Y/100 \right\rfloor - 17}{25} \right\rfloor = 1,$ thereby causing the value of the formula to remain constant at 8 at this stage. The value only increases by 1 more when $\left\lfloor Y/100 \right\rfloor$ becomes 43. Thereafter, the value increases gradually by 1 and becomes constant again when $\left\lfloor Y/100 \right\rfloor = 68.$ This pattern of increases describes the intervals at which LUN is applied: once every 300 years for the first 2100 years of a twenty-five-century cycle, and another time after 400 years. In the first twenty-five-century cycle beginning at AD 1800, a day each is dropped in the years AD 2100, AD 2400, AD 2700, AD 3000, AD3300, AD 3600, AD 3900 and AD 4300. The term $\left\lfloor \frac{\left\lfloor Y/100 \right\rfloor - 17}{25} \right\rfloor$ in the formula serves to postpone the one-day drop from AD 4200 (the 24th year in the first twenty-five-century cycle from AD 1800) to AD 4300, from AD 6700 (the 24th year in the second twenty-five-century cycle from AD 1800) to AD 6800 etc. As a result, we conclude that the refined formula corresponds exactly to the number of days omitted by the end of year *Y*, due to the lunar adjustment. That is,

$$LUN^* = \left\lfloor \frac{\left\lfloor Y/100 \right\rfloor - 15 - \left\lfloor \frac{\left(\left\lfloor Y/100 \right\rfloor - 17 \right)}{25} \right\rfloor}{3} \right\rfloor \text{ where } \left\lfloor Y/100 \right\rfloor \geq 18.$$

Using the formulae for SOL* and LUN*, we can easily verify that when $\left\lfloor Y/100 \right\rfloor = 85,$ SOL* – LUN* gives 30, as found by the computation earlier.

Mentioned earlier, multiple lunar tables are drawn up for the Gregorian calendar. How many such tables are required? Since the golden numbers are repeated in their original positions after a net downward shift of 30 days, we know that there are a total of 30 different patterns of distribution of golden numbers that have to be indicated on the lunar tables. Therefore, 30 different lunar tables are constructed for the Gregorian calendar. These tables are identified by index letters have a one-to-one correspondence to thirty index numbers, 0 to 29. These index numbers are in turn obtained from mod (SOL* – LUN*, 30), giving the number of days by which the golden numbers have been shifted down.

5.1.2 Dominical letter/number

The dominical letter/number in a year regresses by 1 from that in the previous year and by 2 if the previous year is a leap year. In the Gregorian calendar, there is still a strong link between the number of regressions, Γ, made by the end of year *Y* and the dominical letter/number, *N*, of year *Y*. However, the new leap year rule adopted in the Gregorian calendar alters the number of regressions that occur in a given period of time. Hence the formulae for Γ and *N* derived in Section 3.1.3 have to be modified to cater to the Gregorian calendar. The following discussion provides a derivation of these new formulae.

AD 1 has 0 regressions and by the end of year *Y*, starting the count of years from AD 1, the number of regressions Γ made is $Y - 1$. In the Gregorian calendar, a year is a leap year if it is divisible by 4 unless it is a centurial year not divisible by 400. Thus, between AD 1 and year *Y*, there are $\left\lfloor Y/4 \right\rfloor - \left\lfloor Y/100 \right\rfloor + \left\lfloor Y/400 \right\rfloor$ leap years. Then by the end of year *Y*, the total number of regressions is,

$$\Gamma = Y - 1 + \left\lfloor Y/4 \right\rfloor - \left\lfloor Y/100 \right\rfloor + \left\lfloor Y/400 \right\rfloor$$

By the Gregorian calendar, 1 January AD 1 is a Monday. Thus the dominical letter of AD 1 was G and dominical number, 7. Therefore, when mod (Γ, 7) = 0, the dominical number $N = 7$; when mod (Γ, 7) = 1, the dominical number would have regressed by one to $N = 6$; and so on. In this way, we can express the dominical number N as 7 – mod (Γ, 7). Substituting $\Gamma = Y - 1 + \lfloor Y/4 \rfloor - \lfloor Y/100 \rfloor + \lfloor Y/400 \rfloor$, we have

$$N = 7 - \text{mod}\,(Y - 1 + \lfloor Y/4 \rfloor - \lfloor Y/100 \rfloor + \lfloor Y/400 \rfloor, 7)$$

For instance, the dominical number for AD 2003 is

$$\begin{aligned} N &= 7 - \text{mod}\left(2003 - 1 + \lfloor 2003/4 \rfloor - \lfloor 2003/100 \rfloor + \lfloor 2003/400 \rfloor, 7\right) \\ &= 7 - \text{mod}(2487, 7) \\ &= 5 \end{aligned}$$

Therefore, the dominical letter for AD 2003 in the Gregorian calendar is E.

To determine the day of the week that a particular date falls on in the Gregorian calendar, the same formula given in Section 3.1.3 may be used but with one exception: N is obtained from the new formula given above.

What is the length of the solar cycle in the Gregorian calendar? We know that the Gregorian leap year cycle is 400 years. However, contrary to the Julian case, the length of the solar cycle in the Gregorian calendar is not equivalent to the lowest common multiple of 400 and 7. Instead, its length is simply 400 years. By the end of 400 years, the number of leap years skipped during this period is 3. This makes the number of days contained in a period of 400 Gregorian years to be 146 097, which is a figure divisible by 7. Thus an arbitrary date bearing a particular dominical (or calendar) letter would return to bear the same dominical (or calendar) letter after every 400 years.

5.1.3 Paschal full moon

Since there are 30 different lunar tables in the Gregorian calendar, 30 different paschal tables may be extracted from them. In determining which date that the paschal full moon falls on in a certain year, it is important to use the paschal table that has the same index number as that of the lunar table applicable to that year.

In general, the paschal full moon dates fall in the range 21 March to 18 April, however there are exceptions in the Gregorian paschal lunar tables. This adds an additional consideration in the calculation of Easter Sunday in the Gregorian calendar. This issue is further discussed in Section 5.2.

5.1.4 Epact

In the Gregorian lunar tables, the solar equation, SOL, and lunar equation, LUN, come into play in shifting the golden numbers down and up respectively. Recall that the epact of a year is the age of the notional moon on 1 January of that year, taking the epact of a new moon as 0. As such, the value of the epact is directly linked to when a new moon is scheduled.

Let us consider only the first lunation of the year. This refers to the lunation that first ends in that year and may begin from the previous year. If SOL is applied, golden numbers are shifted down by 1 day. Hence, the new moon for this first lunation would occur 1 day later than what would have been the scheduled date had SOL not been applied. As a result, the age of the moon on 1 January is reduced by 1 day. That is, when SOL is applied once, the epact is decreased by 1. If LUN is applied, the golden numbers are shifted up by 1 day on the lunar tables. Hence the new moon for the first lunation would occur 1 day earlier than the original date. This leads to the age of the moon on 1 January to increase by 1. That is, when LUN is applied once, the epact is increased by 1.

The table on the next page has been extracted from [3] and would show how SOL and LUN affect epacts over a millennium in the Gregorian calendar.

Table 14 Epacts through the third millennium

Cycle year (Golden number)	Epacts valid for 100 years beginning in						
	1500, 1600	1700, 1800	1900, 2000, 2100	2200, 2400	2300, 2500	2600, 2700, 2800	2900, 3000
1	1	0 (30)	29	28	27	26	25
2	12	11	10	9	8	7	6
3	23	22	21	20	19	18	17
4	4	3	2	1	0 (30)	29	28
5	15	14	13	12	11	10	9
6	26	25	24	23	22	21	20
7	7	6	5	4	3	2	1
8	18	17	16	15	14	13	12
9	29	28	27	26	25	24	23
10	10	9	8	7	6	5	4
11	21	20	19	18	17	16	15
12	2	1	0 (30)	29	28	27	26
13	13	12	11	10	9	8	7
14	24	23	22	21	20	19	18
15	5	4	3	2	1	0 (30)	29
16	16	15	14	13	12	11	10
17	27	26	25	24	23	22	21
18	8	7	6	15	4	3	2
19	19	18	17	26	15	14	13

Table 14 lists the values of epact for every year in the third millennium. An epact in row *n* is the epact of an *n*th year in the Metonic cycle. Recall that there is a unique epact for every golden number. Since the rows correspond to the 19 golden numbers, there are 19 different epact values found in each column. In this table, the epacts range from 1 to 30, with epact of value 30 being listed as “0 (30)” on the table. The epact between succeeding years increases by 11, as can be readily observed from the sequence of epacts in each column. This is because the lunar year is 11 days shorter than the solar year. The exception arises when we consider the epact change from the 19th year to the 1st year of the Metonic cycle. The *saltus lunae* takes place at this juncture, thereby increasing the epact by 12 instead of 11. So far, the features described are familiar ones which apply also to Julian epacts. What is peculiar to Gregorian epacts is this: at every turn of a century, there is a possibility for the sequence of epacts to change.

The epact values in a column are valid for 100 years beginning in the centurial year that is stated at the top of the column. The sequence of epacts runs through the column from the top to the bottom and back to the top again. After 100 years, whether the sequence of epacts remains within the existing column or switches to that in the next column depends on whether SOL or LUN has been applied in the centurial year in question. For example, in the first column, the epacts run in the sequence: 1, 12, 23, 4, 15, 26, 7, 18, 29, 10, 21, 2, 13, 24, 5, 16, 27, 8, and 19. This sequence is valid from AD 1500 to AD 1599, and again from AD 1600 to AD 1699. No switch of columns occurs at AD 1600 because both SOL and LUN are not applied in that centurial year. However, SOL is applied in AD 1700 and thus epact values are reduced by 1 in the following century. For example, after the last entry "19" in the first column of epacts, had SOL not been applied in AD 1700, the 1st entry for AD 1700 would be mod (19 + 12, 30) = mod (31, 30) = 1. However SOL reduces the epact by 1. Hence a new sequence of epacts beginning with "0 (30)" is generated for 100 years beginning at AD 1700. The sequence of epacts thus switches from running through the first column to running through the second column. We observe that the entries in the second column are indeed smaller by 1 when compared to the epacts in the first column within their respective rows. The second column holds true for years from AD 1700 to AD 1899. No change occurs at AD 1800 because both SOL and LUN are applied then and their effects cancel out, thereby not shifting the epacts at all. At the onset of AD 1900, there is a reduction of epact values by 1 again for the 100 years beginning from then. This is the result of applying SOL in AD 1900.

As we progress through the centurial years 1500, 1600, 1700, 1800,… , the columns are read from left to right in general. The first change comes about at AD 2400. This is the centurial year where there is an increase in the epacts by 1 because only LUN is applied but not SOL. After AD 2399, we read the columns from right (5th column) to left (4th column) instead. By comparison, the epacts in the fourth column are bigger than those in the fifth column by 1, as expected. By interpreting the rest of the table in the same way, we would see how SOL and LUN influence epact values over a period of 1000 years.

Having seen how SOL and LUN affect Gregorian epacts on a table, we now focus on studying how epacts are manipulated when used in the computation of Easter Sunday.

From Section 3.4, we recognize that one of the key steps in the computation makes use of the sum of the day of March of a paschal full moon and the epact for the corresponding year. This sum either added up to 44 or 74 according to the golden number. Suppose the golden numbers have drifted downwards by x days due to SOL and LUN. This results in an epact which is x days smaller. Yet at the same time, the day of March of any golden number in the lunar table increases by x. By considering the action of SOL and LUN on the epact and day of March themselves, the above explanation suggests that the sum remains constant at either 44 or 74. From knowing the Gregorian epact, the date of the paschal new moon, and in turn paschal full moon, may then be determined easily. Unfortunately, there are exceptions to the sum being 44 or 74. Such exceptions stem from a problem in the paschal tables. The problem is discussed in the next section.

5.2 Gregorian paschal lunar table

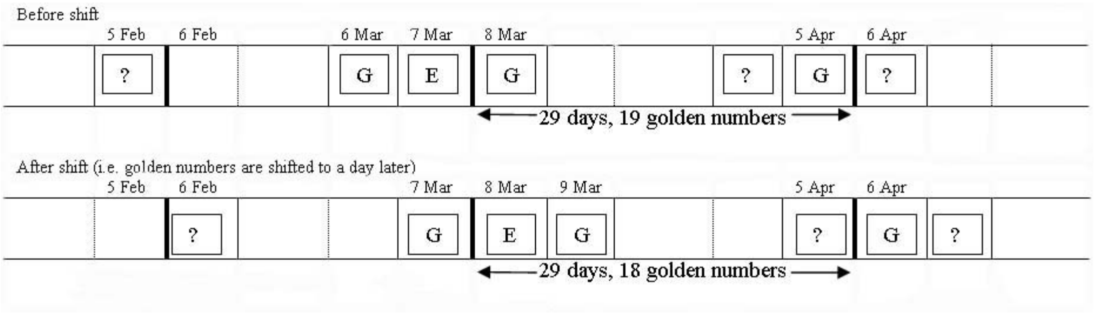
5.2.1 Problem

In the Julian paschal table, the paschal new moon occurs between 8 March and 5 April. All 19 golden numbers can be found within this range of dates. However this range is not kept in all the newly constructed Gregorian paschal tables. In some, only 18 of the golden numbers fell within this range of 29 days while the last golden number fell on the 30th day instead, that is, on 6 April.

Why has such an anomaly occurred? Recall the paschal lunation is the lunation that contains the paschal new moon. The Julian lunar table was set up with a condition that the paschal lunation contained only 29 days. Therefore all the 19 paschal new moons, and in turn, the 19 golden numbers, can be found within a period of 29 days. However, the lunation preceding the paschal lunation is 30 days long. This means that two consecutive identical golden numbers, with the latter corresponding to the paschal new moon, are 30 days apart. The 30-day period before the period 8 March till 5 April therefore also contains the full set of 19 golden numbers in the exact same sequence as that in the latter period. Comparing the lengths of these 2 periods, we find that there is an additional “empty date” in the 30-day period.

With the applications of SOL and LUN, the golden numbers are progressively shifted down in the lunar table. We thus have a situation of 19 golden numbers, which were originally spread over a period of 30 days, having to be squeezed into a period of 29 days. When the downward shift occurs, either an “empty date” or a golden number is being shifted out of the period between 8 March and 5 April into 6 April. Simultaneously, either an “empty date” or a golden number from the previous 30-day period is being shifted into the gap created on 8 March. The anomaly highlighted at the beginning of this section occurs when the item that was shifted out of the period 8 March to 5 April was a golden number but that which was shifted into the gap on 8 March was an “empty date”. In such a case, the period 8 March to 5 April would only contain 18 golden numbers, with the last one occurring on 6 April.

Here is a schematic diagram to illustrate how the anomaly has been created:



Legend: G golden no. E “empty date” ? golden no./ “empty date”

As the diagram suggests, the first row of entries is shifted to the right to obtain the second row of entries. This corresponds to a downward shift of the golden numbers in the Gregorian lunar table. In particular, an empty date is shifted into the 29-day period as a substitution for the golden number that is shifted out of the same period. Hence only 18 golden numbers remain in the period 8 March till 5 April while the 19th golden number falls on 6 April.

It is not obvious why the missing golden number invariably lands on 6 April and not any later. The underlying reason for this is that the number of "empty dates" in the lunar table is at most 1 day long. We can also view it as: every "empty date" is always sandwiched between 2 golden numbers. After an empty date is shifted into the 29-day period on 8 March in the first round, the subsequent entry to be shifted into the period in the second round is certainly a golden number. This new entry either substitutes an "empty date" or a golden number that was shifted out of the 29-day period during the second round. If the new entry replaces an "empty date", the end result is that the 29-day period resumes its feature of containing all the paschal new moons. Equivalently, all 19 golden numbers would be returned into the range 8 March till 5 April after this second round of shifts. If the new entry replaces a golden number, the end result is that the 29-day period continues to contain only 18 golden numbers and the missing golden number still falls on 6 April. Note however that the golden number that is missing from the 29-day period after the second round of shifts is different from the original golden number that was missing after the first round. Recall that we are shifting golden numbers from a 30-day period into a 29-day period. The original golden number had been omitted because its earlier identical golden number had been blocked from entering the range at 8 March by the extra day slot. After the second round of shifts, the new golden number being shifted into the 29-day period is identical to the original missing golden number. Hence, we conclude that the new missing golden number after the second round of shifts is different from the original.

Nonetheless, Pope Gregory III and Christopher Clavius wanted to keep the range of dates for the paschal new moon to be within 8 March and 5 April across all the Gregorian lunar tables. They wanted to stick to the traditional 29-day period for which Easter celebration had been held in, for the past thousand years. To achieve this, they made further adjustments which are discussed in the next section.

5.2.2 Adjustments

To maintain the dates 8 March till 5 April as the range for the paschal new moons, whenever a golden number fell out of this range of dates into 6 April, it was shifted back up into 5 April. However this was not always possible because 5 April is not an "empty date". In making adjustments to the table, Pope Gregory III and Christopher Clavius were not prepared to forgo another condition which the lunar tables had been designed to satisfy: no date can bear more than 1 golden number. Let us find out when it is possible to shift the golden number into 5 April, when not, and what additional adjustment is then required to deal with the second case.

In Section 3.1.1, it was mentioned that for golden numbers XII and bigger, the subsequent golden number always occurred without any "empty" date in between. These subsequent golden numbers are in the range I to VIII. Should any of these numbers between I to VIII fall on 6 April, there would certainly be another golden number on 5 April from the range XII to XIX. Therefore, we can divide the missing golden numbers that land on 6 April into two classes: the first class contains golden numbers between IX and XIX, and the second class contains golden numbers between I to VIII.

When golden numbers between IX and XIX fell on 6 April, one shift is required to shift them back to 5 April. When the golden numbers in the range I to VIII fell on 6 April, the golden numbers in the range XII to XIX fell on 5 April. Therefore two shifts are needed to complete the adjustment: the golden number on 5 April was shifted to 4 April and that on 6 April was shifted to 5 April. We note that for the second case, 4 April would always be an "empty date" and therefore can accommodate the golden number shifted up from 5 April.

To understand why 4 April is invariably an "empty date", we revert to inspection of the Julian lunar table from which the Gregorian lunar tables are derived. Recall that due to the influence of the solar equation (SOL) and lunar equation (LUN), there is a net downward drift in the golden numbers in the original Julian lunar table. A total of 30 different Gregorian lunar tables are then derived to account for the 30 possible distributions of golden numbers. These are then cycled through for use over the years. From the Julian lunar table, the golden numbers, which are shifted into the period 8 March till 5

April, come from the period between 6 February and 7 March. There are no strings of golden numbers which are longer than two within this period. Therefore, among the 30 generated lunar tables, there is no string of golden numbers exceeding two in length within the period between 8 March and 5 April. Equivalently, the 30 paschal lunar tables would only have strings of golden numbers of length at most two. Hence whenever a golden number falls on 6 April in the paschal lunar table, and another that falls on 5 April, the date 4 April would always be an “empty date”.

5.2.3 Effect of the adjustments on Gregorian epact

We have seen how the problem of the paschal lunar tables is solved through an occasional adjustment of the golden numbers. However, such a solution has led to cases where the sum of the day of March of the paschal full moon and the epact for the corresponding year does not add up to either 44 or 74. This section addresses these exceptions.

When a paschal new moon falls outside the period 8 March till 5 April into 6 April, the corresponding paschal full moon occurs on 19 April. 19 April is the 50th day of March and thus for that year, the epact is 74 – 50 = 24. However, by rule of the adjustments to the golden numbers on a paschal lunar table as given in the previous section, this paschal new moon must be shifted back by one day to 5 April. The paschal full moon is thus pushed back to 18 April, the 49th day of March. This results in the sum of epact and ‘Day of March’ of paschal full moon for that year to be 24 + 49 = 73 instead of the usual 74. To patch up the shortfall, the epact for that year is increased from 24 to 25.

In addition, if the golden number being pushed back from 6 April to 5 April is between golden numbers I to VIII, there is a golden number in the range XI to XIX that falls on 5 April. Correspondingly, there is a paschal full moon from the 9th to 19th years of the Metonic cycle occurring on 18 April, the 49th day of March. The epact for this year is thus 74 – 49 = 25. By rule of the adjustments again, the golden number on 5 April is shifted back to 4 April so that 5 April can accommodate the golden number shifted up from 6 April. Thus the paschal full moon on 18 April is shifted to 17 April, the 48th day of March.

After the adjustment, the sum of epact and day of March of paschal full moon for that year becomes 25 + 48 = 73. To change this sum back to 74, whenever a golden number between XII and XIX occurs with a calculated epact of 25, the epact is increased to 26 instead.

Here is a summary of the two rules, henceforth to be referred to as “Clavius’ adjustments”, by which the calculated Gregorian epacts are adjusted:

1. If the calculated epact is 24, increase it to 25;
2. If the golden number is greater or equal than XII and the calculated epact is 25, increase it to 26.

5.3 Determining the Gregorian Easter Sunday by calculation

After the Gregorian calendar was adopted, the primary way of determining the date for Easter celebration involved calculations and not tables. The flow of steps used is very similar to that in the Julian calendar. Comparing both schemes of calculations, the main difference lies in how the final epact values are derived. We first give a detailed elaboration of how the Gregorian epacts are found before summarizing the calculation steps of Gregorian Easter via a flow chart.

With the seven line drop of the golden numbers in the lunar tables, the Julian epact formula, E = mod (11 x (G – 3), 30) is modified to, E = mod (11 x (G – 3) – 7, 30). The net downward drift of the golden numbers is given by mod (SOL* – LUN*, 30). Note that SOL* denotes the number of days by which the dates of the notional new moons is increased on account of the decrease in the average length of the calendar year for year Y, and LUN* denotes the number of days by which the dates of the notional new moons must be decreased on account of the lunar adjustment for year Y. With the downward drift, the epact value is reduced by the same amount, mod (SOL* – LUN*, 30). Therefore, the formula for the Gregorian epact after accounting for both solar and lunar adjustments is given by,

$$\begin{aligned} E &= \operatorname{mod}\big(11\times(G-3)-7,30\big)-\operatorname{mod}(SOL*-LUN*,30) \\ &= \operatorname{mod}\big(11\times G-40-(SOL*-LUN*),30\big). \end{aligned}$$

Substituting the formulae for SOL* and LUN* accordingly into the above, we get,

$$E = \operatorname{mod}\left(11\times G-40-\left(\left\lfloor Y/100\right\rfloor-\left\lfloor\frac{\left\lfloor Y/100\right\rfloor}{4}\right\rfloor-12-\left\lfloor\frac{\left\lfloor Y/100\right\rfloor-15-\left\lfloor\frac{\left(\left\lfloor Y/100\right\rfloor-17\right)}{25}\right\rfloor}{3}\right\rfloor\right),30\right)$$

$$=\operatorname{mod}\left(11\times G-\left\lfloor Y/100\right\rfloor+\left\lfloor\frac{\left\lfloor Y/100\right\rfloor}{4}\right\rfloor-28+\left\lfloor\frac{\left\lfloor Y/100\right\rfloor-\left\lfloor\frac{\left(\left\lfloor Y/100\right\rfloor-17\right)}{25}\right\rfloor}{3}-5\right\rfloor,30\right)$$

$$=\operatorname{mod}\left(11\times G-\left\lfloor Y/100\right\rfloor+\left\lfloor\frac{\left\lfloor Y/100\right\rfloor}{4}\right\rfloor-33+\left\lfloor\frac{\left\lfloor Y/100\right\rfloor-\left\lfloor\frac{\left(\left\lfloor Y/100\right\rfloor-17\right)}{25}\right\rfloor}{3}\right\rfloor,30\right)$$

To ensure that the argument of the above modulo function is not negative, a multiple of 30 is added into the argument to yield,

$$E=\operatorname{mod}\left(57+11\times G-\left\lfloor Y/100\right\rfloor+\left\lfloor\frac{\left\lfloor Y/100\right\rfloor}{4}\right\rfloor+\left\lfloor\frac{\left\lfloor Y/100\right\rfloor-\left\lfloor\frac{\left(\left\lfloor Y/100\right\rfloor-17\right)}{25}\right\rfloor}{3}\right\rfloor,30\right)$$

However, this is not the final formula for the Gregorian epact; Clavius' adjustments have to been fitted in. Here are two possible formulae for calculating the value of the correction.

Let V denote the correction to be made in the epact as a result of Clavius' adjustments. Consider,

$$V = \left\lfloor E/24 \right\rfloor - \left\lfloor E/25 \right\rfloor + \left\lfloor G/12 \right\rfloor \times \left(\left\lfloor E/24 \right\rfloor - \left\lfloor E/25 \right\rfloor \right)$$

where E is the epact value calculated by the earlier formula, and G is the golden number for the year in concern.

V attains 1 either when $E = 24$, or $E = 25$ and $G \geq 12$; V is 0 otherwise. Hence V may be added to E, or to achieve the same effect, subtracted from the 'Day of March' of the paschal full moon to give E_{final} where E_{final} indicates the final epact value after considering Clavius' adjustments.

Consider an alternative formula:

$$V' = \left\lfloor \frac{G - 1 + 11 \times U}{319} \right\rfloor$$

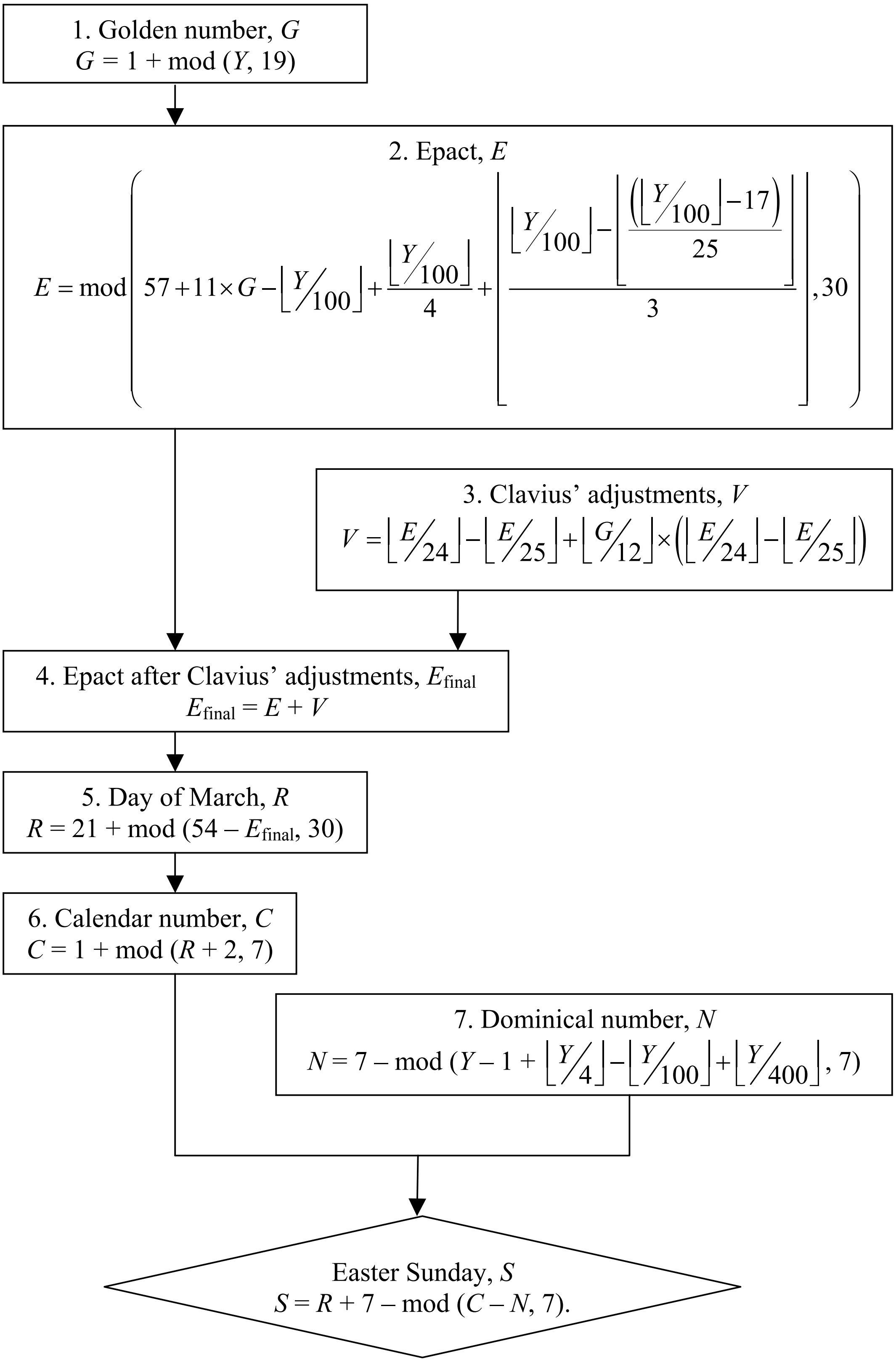
where $U = \text{mod}\,(53 - E, 30)$.

V ' attains 1 only when

$$G - 1 + 11 \times U \geq 319 \Rightarrow G - 1 \geq 319 - 11 \times U \Rightarrow G - 1 \geq 11(29 - U).$$

This relation holds in two cases. Firstly, when $E = 24$, we have $U = 29$ and thus the above relation holds for all values of G. Secondly, when $E = 25$, we have $U = 28$ and the above relation holds for $G \geq 12$. Therefore, V' may be used in place of V to account for Clavius' adjustments.

We present all the steps involved in the computation of the Gregorian Easter Sunday in a flow chart next page.



In this flow chart, the first formula for Clavius' adjustments, V, has been used and it is added to E to yield E_{final} instead of being subtracted from R. By following these steps, we obtain the date for the Gregorian Easter Sunday for any year Y in AD dating.

Chapter 6

Easter algorithms

6.1 A selection of Easter algorithms

In chapters 3 and 5, we have described procedures for finding the date of Easter Sunday by means of tables and calculations in both Julian and Gregorian calendars. We note that the given algorithms are but a few which have been constructed based on the rules found in Easter tables. In this section, we furnish readers with a couple more examples of tables that people have used to determine Easter. Then, a review of several other algorithms would be provided.

The tables have been taken from the Book of Common Prayer and the source for each algorithm is mentioned explicitly in its respective section. To honour the sources of the algorithms, minimal changes have been made to their manner of presentation as they are adapted into this supplement. As such, there are differences between the notations used in the supplement so far, and those that are to appear in the following algorithms. These notation differences would be clarified accordingly.

6.1.1 Easter tables in the Book of Common Prayer

The first edition of the Book of Common Prayer was produced in AD 1549 and since then, the book has exerted major influence on the religious lives of many. The AD 1549 edition of the prayer book was in use only for three years, until extensive revision in AD 1552. However, much of its tradition and language remains in the prayer books of today. See Appendix A for a translation of the contents page of the second edition of the book to get a sense of the sort of issues addressed in this book. Tables for determining the date of Easter Sunday can be found in The Book of Common Prayer. We next reproduce these tables from the AD 1662 edition of the Book of Common Prayer found in [14].

Table 15

A Table to Find Easter-Day,

From the Present Time till the Year 2199 Inclusive
According to the Foregoing Calendar.

Golden Number.	Days of the Month.		Sunday Letters.
	March	21	C
XIV.	”	22	D
III.	”	23	E
	”	24	F
XI.	”	25	G
	”	26	A
XIX.	”	27	B
VIII.	”	28	C
	”	29	D
XVI.	”	30	E
V.	”	31	F
	April	1	G
XIII.	”	2	A
II.	”	3	B
	”	4	C
X.	”	5	D
	”	6	E
XVIII.	”	7	F
VII.	”	8	G
	”	9	A
XV.	”	10	B
IV.	”	11	C
	”	12	D
XII.	”	13	E
I.	”	14	F
	”	15	G
IX.	”	16	A
XVII.	”	17	B
VI.	”	18	C
	”	19	D
	”	20	E
	”	21	F
	”	22	G
	”	23	A
	”	24	B
	”	25	C

THIS Table contains so much of the Calendar as is necessary for the determining of *Easter*; to find which, look for the Golden Number of the year in the first Column of the Table, against which stands the day of the Pascal Full Moon; then look in the third Column for the Sunday Letter, next after the day of the Full Moon, and the day of the Month standing against that Sunday Letter is *Easter Day*. If the Full Moon happens upon a Sunday, then (according to the first rule) the next Sunday after is *Easter-Day*.

To find the Golden Number, or Prime, add one to the Year of our Lord, and then divide by 19; the remainder, if any, is the Golden Number; but if nothing remaineth, then 19 is the Golden Number.

To find the Dominical or Sunday Letter, according to the Calendar, until the Year 2099 inclusive, add to the Year of our Lord its Fourth Part, omitting Fractions; and also the Number 6: Divide the sum by 7; and if there is no remainder, the A is the Sunday Letter: But if any number remaineth, then the Letter standing against that number in the small annexed Table is the Sunday Letter.

0	A
1	G
2	F
3	E
4	D
5	C
6	B

For the next Century, that is, from the year 2100 till the year 2199 inclusive, add to the current year its fourth part, and also the number 5, and then divide by 7, and proceed as in the last Rule.

Note, that in all Bissextile or Leap-Years, the Letter found as above will be the Sunday Letter, from the intercalated day exclusive to the end of the year.

Table 16

Another Table to Find Easter
Till the Year 2199 Inclusive

	Sunday Letters						
Golden Number.	**A**	**B**	**C**	**D**	**E**	**F**	**G**
I.	April 16	—— 17	—— 18	—— 19	—— 20	—— 21	—— 15
II.	April 9	—— 10	—— 4	—— 5	—— 6	—— 7	—— 8
III.	Mar. 26	—— 27	—— 28	—— 29	—— 30	—— 24	—— 25
IV.	April 16	—— 17	—— 18	—— 12	—— 13	—— 14	—— 15
V.	April 2	—— 3	—— 4	—— 5	—— 6	—— 7	—— 1
VI.	April 23	—— 24	—— 25	—— 19	—— 20	—— 21	—— 22
VII.	April 9	—— 10	—— 11	—— 12	—— 13	—— 14	—— 15
VIII.	April 2	—— 3	—— 4	Mar. 29	—— 30	—— 31	Apr. 1
IX.	April 23	—— 17	—— 18	—— 19	—— 20	—— 21	—— 22
X.	April 9	—— 10	—— 11	—— 12	—— 6	—— 7	—— 8
XI.	Mar. 26	—— 27	—— 28	—— 29	—— 30	—— 31	April 1
XII.	April 16	—— 17	—— 18	—— 19	—— 20	—— 14	—— 15
XIII.	April 9	—— 3	—— 4	—— 5	—— 6	—— 7	—— 8
XIV.	Mar. 26	—— 27	—— 28	—— 29	—— 23	—— 24	—— 25
XV.	April 16	—— 17	—— 11	—— 12	—— 13	—— 14	—— 15
XVI.	April 2	—— 3	—— 4	—— 5	—— 6	Mar. 31	Apr. 1
XVII.	April 23	—— 24	—— 18	—— 19	—— 20	—— 21	—— 22
XVIII.	April 9	—— 10	—— 11	—— 12	—— 13	—— 14	—— 8
XIX.	April 2	—— 3	Mar. 28	—— 29	—— 30	—— 31	Apr. 1

T**O** make use of the preceding Table, find the Sunday Letter for the Year in the uppermost Line, and the Golden Number, or Prime, in the Column of Golden Numbers, and against the Prime, in the same Line under the Sunday Letter, you have the Day of the Month on which *Easter* falleth that year. But note, that the Name of the Month is set on the Left Hand, or just with the Figure, and followeth not, as in other Tables, by Descent, but Collateral.

Both Tables 15 and 16 include clearly stated instructions of how they are to be used to find Easter day. Notice that the same tools – day of month, golden number, and dominical letter – have been employed. Pertaining to the dominical letter and golden number, the Book of Common Prayer has included general tables for finding them. Samples of these general tables can be found in Appendix B. Notice that these tables are only valid between AD 1662 and AD 2199; separate tables are drawn up for later periods of time. In modern times, the computing approach is preferred to table look-ups. The next section furnishes some established algorithms.

6.1.2 Gauss' Easter algorithm

This algorithm is due to Carl Friedrich Gauss, a famous German mathematician. It may be considered as an incomplete algorithm because it involves the use of a table; nonetheless, it is fairly easy to use. When the algorithm was first published, it contained an error. Recall that the lunar equation, LUN, is applied eight times in every twenty-five century period. The first seven applications are made at 300-year intervals and the last is made after 400 years. LUN was first applied in AD 1800 and subsequently, it would be applied at AD 2100, AD 2400, … , AD 3600, AD 3900 and finally AD 4300. Gauss' mistake in the algorithm was that he had ignored the final 400-year interval between LUN applications. By his algorithm, the date for Easter Sunday in AD 4200 would be 13 April instead of the correct 20 April. The corrected version of Gauss' algorithm, taken from [12], is given below.

MM	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
M	22	22	23	23	24	24	24	25	26	25	26	27	27	27	28	28	29

MM	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
M	29	29	0	1	0	1	2	2	2	3	4	4	4	5	5	6	6

MM	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65
M	6	7	8	7	8	9	9	9	10	10	11	11	11	12	13	12	13

MM	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80	81	82
M	14	15	14	15	16	16	16	17	17	18	18	18	19	20	19	20	21

MM	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99
M	21	21	22	22	23	23	23	24	25	25	25	26	27	26	27	28	28

$$M = MM = \left\lfloor {}^{Year}\!/_{100} \right\rfloor$$

$$N = \left(4 + \left\lfloor {}^{Year}\!/_{100} \right\rfloor - \left\lfloor {}^{Year}\!/_{400} \right\rfloor\right) \bmod 7$$

$$A = Year \bmod 19$$

$$B = Year \bmod 4$$

$$C = Year \bmod 7$$

$$D = (19 \times A + M) \bmod 30$$

$$E = (2 \times B + 4 \times C + 6 \times D + N) \bmod 7$$

$$F = 22 + D + E$$

If $F = 57$, or ($D = 28$ and $E = 6$ and $A > 10$), then $F = F - 7$.

MM refers to the century that the year concerned belongs to. For example, this year, AD 2003, belongs to the 20th century. It thus corresponds to $MM = 20$. In turn, we find the matching value for M from the table. This gives M as 24. The remaining sequence of steps is straightforward. The end result F gives the date of Easter Sunday in terms of day of March. When F exceeds 31, it would mean that Easter falls on $(F - 31)$ of April.

6.1.3 O'Beirne's algorithm

An anonymous American had presented the first flawless, purely arithmetical algorithm in AD 1876 in the journal *Nature*. In AD 1965, Thomas H. O'Beirne published two algorithms based on the one in *Nature* in his book "Puzzles and Paradoxes". His algorithms differed slightly from that in *Nature* but retain the feature of being wholly arithmetic. To begin the computation, only the Gregorian year number is required. Then, through a series of ten successive division operations, and use of the quotients and remainders thus generated, the final outcome gives the day and month that Easter Sunday falls in. The steps of both procedures are tabled below.

Algorithm 1

Step	*Dividend*	*Divisor*	*Quotient*	*Remainder*
(1)	x	19	–	a
(2)	x	100	b	c
(3)	b	4	d	e
(4)	$8b + 13$	25	g	–
(5)	$19a + b - d - g + 15$	30	–	h
(6)	$a + 11h$	319	μ	–
(7)	c	4	i	k
(8)	$2e + 2i - k - h + \mu + 32$	7	–	λ
(9)	$h - \mu + \lambda + 90$	25	n	–
(10)	$h - \mu + \lambda + n + 19$	32	–	p

In step (1), the position of the year in a 19-year cycle is identified. That is, the remainder a is one unit less than the golden number. Step (2) gives the values for which solar and lunar corrections are made: b increases by 1 at centurial years only and c is used later in step (7). Step (3) gives d which increases only in centurial leap years. It also gives remainder e, the number of centurial years which have not been leap years, subsequent to the previous centurial leap year.

From step (4), g is obtained. The value of g increases only when there is an increase of the epact because of the correction of the month. In step (5), the value of $b - d$ is required. This has a value which increases when a reduction of the epact is required because of the omissions which gave the Gregorian correction of the year. At step (5), the value h obtained is equivalent to the epact. In fact, the epact is either $23 - h$ or $53 - h$, whichever places the epact in the range between 1 and 30. Step (6) results in a value μ which is used in steps (8), (9) and (10) to account for Clavius' adjustments. With step (7), it is possible to find the day of the week for the Easter full moon date at step (8). However, an intermediate step is required to find the dominical letter. This is achieved by dividing $2e + 2i - k$ or $2j - k + 4$ by 7. Multiples of 7 can be added to this to attain the smallest non-negative remainder. The letter is A if the remainder is 0, and it moves on in the alphabet as the remainder increases. For the case of leap years, the follow dominical letter is added as a prefix. Steps (9) and (10) give the month and day of Easter Sunday respectively.

Algorithm 2

Step	*Dividend*	*Divisor*	*Quotient*	*Remainder*
(1)	x	100	b	c
(2)	$5b + c$	19	–	a
(3)	$3(b + 25)$	4	δ	ε
(4)	$8(b + 11)$	25	γ	–
(5)	$19a + \delta - \gamma$	30	–	h
(6)	$a + 11h$	319	μ	–
(7)	$60(5 - \varepsilon) + c$	4	j	k
(8)	$2j - k - h + \mu$	7	–	λ
(9)	$h - \mu + \lambda + 110$	30	n	q
(10)	$q + 5 - n$	(32)	(0)	p

Algorithm 2 is arithmetically equivalent to Algorithm 1; the steps have been modified to simplify the computation of various dividends. The result gives Easter Sunday as the pth day of the nth month, as in the first algorithm.

6.1.4 Gregorian algorithm by Lilius and Clavius

Together with Christopher Clavius, Aloysius Lilius was the primary designer of the Gregorian calendar. The following algorithm, adapted from [5], was developed by them in the late 16th century for determining the date of Easter in the Gregorian calendar. As such, this algorithm is only valid for years after AD 1582.

```
Algorithm E. (Date of Easter.) Let $Y$ be the year for which the date of Easter is desired.
E1. [Golden number.] Set $G \leftarrow (Y + 19) + 1$.
E2. [Century.] Set $C \leftarrow \lfloor Y/100 \rfloor + 1$.
E3. [Corrections.] Set $X \leftarrow \lfloor 3C/4 \rfloor - 12$, $Z \leftarrow \lfloor (8C+5)/25 \rfloor - 5$.
E4. [Find Sunday.] Set D $\leftarrow \lfloor 5Y/4 \rfloor - X - 10$.
E5. [Epact.] Set $E \leftarrow (11G + 20 + Z - X) \bmod 30$. If $E = 25$ and the golden
    number $G$ is greater than 11, or if $E = 24$, then increase $E$ by 1.
E6. [Find full moon.] Set $N \leftarrow 44 - E$. If $N < 21$ then set $N \leftarrow N + 30$.
E7. [Advance to Sunday.] Set $N \leftarrow N + 7 - ((D + N) \bmod 7)$.
E8. [Get month.] If $N > 31$, the date is $(N - 31)$ APRIL; otherwise the date is
    $N$ March.
```

The same source had led to another one, [4], which included steps that allowed the algorithm to work even in AD 1582 or any year before. The more general algorithm is included below, with the "comment" omitted.

```
procedure Easter (year, month, day); value year; integer year, month, day;
comment (omitted)
begin integer golden number, century, Gregorian correction, Clavian
  correction, extra days, epact;
integer procedure mod (a, b); value a, b; integer a, b; mod := a – b × (a ÷ b);
golden number:= mod (Y, 19) + 1; if year ≤ 1582 then go to Julian;
Gregorian: century := year ÷ 100 + 1;
  Gregorian correction := (3 × century) ÷ 4 – 12;
  Clavian correction := (century – 16 – (century – 18) ÷ 25) ÷ 3;
  extra days := (5 × year) ÷ 4 – Gregorian correction – 10;
  epact := mod (11 × golden number + 20 + Clavian correction – Gregorian
    correction, 30);
```

```
  if epact ≤ 0, then epact := epact + 30;
  if (epact = 25 ∧ golden number > 11) ∨ epact = 24 then epact := epact + 1;
  go to ending routine;
Julian: extra days := (5 × year) ÷ 4; epact := mod (11 × golden number – 4,
  30) + 1;
ending routine: day := 44 – epact; if day < 21 then day := day + 30;
  day := day + 7 – mod (extra days + 7, 7);
 if day > 31 then begin month := 4; day := day – 31 end
  else month := 3 end Easter
```

In this second algorithm, the steps, no doubt are expressed differently from those in the first algorithm. Nonetheless, they refer to identical entities. The line headed by "Julian" lists the steps for finding an Easter date of a year prior to the time of adoption of the Gregorian calendar.

6.1.5 Carter's algorithm

The algorithms brought up so far can serve for extended periods of time. However, it is possible to find algorithms which are valid only for several years. The following rules are found from [19]. Derived by Carter, this algorithm only applies to the period between AD 1900 and AD 2099. We note that in AD 2000, both the solar and lunar equations are not applied to the lunar tables. In turn, the golden numbers in the period between AD 1900 and AD 2099 are not shifted about in the lunar tables. This means that Clavius' adjustments are not needed in any algorithm used to determine the date of Easter Sunday for that period. Consequently, Carter's algorithm may be seen as shorter than the previously mentioned algorithms. His algorithm is as follows:

Calculate $D = 225 - 11\ (Y \bmod 19)$

If D is greater than 50 then subtract multiples of 30 until the resulting new value of D is less than 51.

If D is greater than 48 subtract 1 from it.

Calculate $E = (Y + \lfloor Y/4 \rfloor + D + 1) \bmod 7$.

Calculate $Q = D + 7 - E$.

If Q is less than 32 then Easter is in March. If Q is greater than 31 then $Q - 31$ is its date in April.

6.2 Final remark

Given that Easter is related to the date of the vernal equinox and the age of the moon, the date for Easter could have been determined astronomically, even at the time of the Julian calendar. However, this has not been done because the exact time of any astronomical event depends on the local time of the observer. Furthermore, from the institution of leap days in the Julian calendar, these caused the date assigned to the vernal equinox to oscillate around the actual event.

Quoting Johannes Kepler, "Easter is a feast, not a planet." Since the adoption of a notional sun and moon, the determination of Easter Sunday has been more mathematical than astronomical. The algorithms highlighted in the previous section demonstrate the convenience brought about by mathematics. Mathematical manipulation may make one algorithm seem vastly different from another. Yet in general, these algorithms share the same basic principles of calculation and the same set of tools. Significant differences in them could arise in their presentations if they have been based on different calendars or admit exceptions. Hence, in using any Easter algorithm, it is important to be clear about its underlying method, calendar and validity period.

Finally, in general, the date of Easter slips back by about eight days each year until it hops forward again. Its regularity follows the algorithms discussed previously. However we should note that even though the found algorithms are designed to date Easter correctly forever, they would still slip into error after a very long period of time. This is because the rules on which these algorithms are based do contain residual imperfections. Furthermore, the astronomical lengths of the month and day are slowly changing due to tidal friction.

Appendix A: Contents page of the second edition of the Book of Common Prayer (extracted from [10])

THE CONTENTS OF THIS BOOK.

i. A PREFACE.
ii. Of ceremonies, why some be abolished and some retayned.
iii. The ordre howe the Psalter is appointed to be read.
iv. The Table for the order of the Psalmes to be sayd at Mornyng and Evening prayer.
v. The order how the rest of holy Scripture is appointed to be read.
vi. Propre Psalmes and Lessons at Morning and Evening Praier, for certayne feastes and dayes.
vii. An Almanack.
viii. The Table and Kalendar for Psalmes and Lessons, with necessarie Rules apperteynyng to the same.
ix. The order for Mornyng Prayer and Eveninge Praier throughout the yere.
x. The Letanie.
xi. The Collectes, Epistles, and Gospels, to be used at the ministracion of the holy Communion, throughout the yere
xii. The order of the ministracion of the holy Communion.
xiii. Baptisme both publique and private.
xiv. Confirmacion, where also is a Catechisme for children.
xv. Matrimonie.
xvi. Visitacion of the sicke.
xvii. The Communion of the sicke.
xviii. Burial.
xix. The thanksgiving of women after childe-birth.
xx. A Comminacion against sinners, with certain praiers to be used divers tymes in the yere.
xxi. The fourme and maner of makyng and consecrating of Bischoppes, Priestes, and Deacons.

Appendix B (Extracted from [10])

General Tables for Finding the Dominical or Sunday Letter,
And the places of the Golden Numbers in the Calendar

Table I

6	5	4	3	2	1	0
B	**C**	**D**	**E**	**F**	**G**	**A**
				1600	1700	1800
1900 2000	2100	2200	2300 2400	2500	2600	2700 2800
2900	3000	3100 3200	3300	3400	3500 3600	3700
3800	3900 4000	4100	4200	4300 4400	4500	4600
4700 4800	4900	5000	5100 5200	5300	5400	5500 5600
5700	5800	5900 6000	6100	6200	6300 6400	6500
6600	6700 6800	6900	7000	7100 7200	7300	7400
7500 7600	7700	7800	7900 8000	8100	8200	8300 8400
8500	&c.					

TO find the Dominical or Sunday Letter for any given Year of our Lord, add to the year its fourth part, omitting fractions, and also the number, which in Table I. standeth at the top of the column, wherein the number of hundreds contained in that given year is found: Divide the sum by 7, and if there is no remainder, then A is the Sunday Letter; but if any number remaineth, then the Letter, which standeth under that number at the top of the Table, is the Sunday Letter.

Table II

I.	II. Years of Our Lord	III.	I.	II. Years of Our Lord	III.
B	1600	0	B	5200	15
	1700	1		5300	16
	1800	1		5400	17
	1900	2		5500	17
B	2000	2	B	5600	17
	2100	2		5700	18
	2200	3		5800	18
	2300	4		5900	19
B	2400	3	B	6000	19
	2500	4		6100	19
	2600	5		6200	20
	2700	5		6300	21
B	2800	5	B	6400	20
	2900	6		6500	21
	3000	6		6600	22
	3100	7		6700	23
B	3200	7	B	6800	22
	3300	7		6900	23
	3400	8		7000	24
	3500	9		7100	24
B	3600	8	B	7200	24
	3700	9		7300	25
	3800	10		7400	25
	3900	10		7500	26
B	4000	10	B	7600	26
	4100	11		7700	26
	4200	12		7800	27
	4300	12		7900	28
B	4400	12	B	8000	27
	4500	13		8100	28
	4600	13		8200	29
	4700	14		8300	29
B	4800	14	B	8400	29
	4900	14		8500	0
	5000	15		&c.	
	5100	16			

TO find the Month and Days of the Month to which the Golden Numbers ought to be prefixed in the Calendar, in any given Year of our Lord, consisting of entire hundred years, and in all the intermediate years betwixt that and the next hundredth year following, look in the second column of Table II. for the given year consisting of entire hundreds, and note the number or cypher which stands against it in the third column; then, in Table III. look for the same number in the column under any given Golden Number, which when you have found, guide your eye side-ways to the left hand, and in the first column you will find the Month and Day to which that Golden Number ought to be prefixed in the Calendar, during that period of one hundred years.

The Letter B prefixed to certain hundredth years in Table II. denotes those years which are still to be accounted Bissextile or Leap-Years in the New Calendar; whereas all the other hundredth years are to be accounted only common years

Table III

Paschal Full Moon.	Sunday Letters.	The Golden Numbers.																		
		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Mar. 21	C	8	19	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26
Mar. 22	D	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27
Mar. 23	E	10	21	2	13	24	5	16	27	8	19	0	11	22	3	14	25	6	17	28
Mar. 24	F	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29
Mar. 25	G	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	0
Mar. 26	A	13	24	5	16	27	8	19	0	11	22	3	14	25	6	17	28	9	20	1
Mar. 27	B	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2
Mar. 28	C	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	0	11	22	3
Mar. 29	D	16	27	8	19	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4
Mar. 30	E	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5
Mar. 31	F	18	29	10	21	2	13	24	5	16	27	8	19	0	11	22	3	14	25	6
April 1	G	19	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7
April 2	A	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8
April 3	B	21	2	13	24	5	16	27	8	19	0	11	22	3	14	25	6	17	28	9
April 4	C	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10
April 5	D	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	0	11
April 6	E	24	5	16	27	8	19	0	11	22	3	14	25	6	17	28	9	20	1	12
April 7	F	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13
April 8	G	26	7	18	29	10	21	2	13	24	5	16	27	8	19	0	11	22	3	14
April 9	A	27	8	19	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15
April 10	B	28	9	20	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16
April 11	C	29	10	21	2	13	24	5	16	27	8	19	0	11	22	3	14	25	6	17
April 12	D	0	11	22	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18
April 13	E	1	12	23	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19
April 14	F	2	13	24	5	16	27	8	19	0	11	22	3	14	25	6	17	28	9	20
April 15	G	3	14	25	6	17	28	9	20	1	12	23	4	15	26	7	18	29	10	21
April 16	A	4	15	26	7	18	29	10	21	2	13	24	5	16	27	8	19	0	11	22
April 17	B	5	16	27	8	19	0	11	22	3	14	25	6	17	28	9	20	1	12	23
April 17	B												7	18	29	10	21	2	13	24
18	C	6	17	28	9	20	1	12	23	4	15	26								
April 18	C	7	18	29	10	21	2	13	24	5	16	27	8	19	0	11	22	3	14	25

Bibliography

Books and publications:

[1] Briggs, Asa (Editor-in-chief) *The Longman Encyclopedia* 1st Edition. Columbia University Press, 1989.

[2] Chambers' Encyclopaedia (new edition) Vol. II. London: George Newnes Limited, 1959.

[3] Heilbron, J. L. *The Sun in the Church: Cathedrals as Solar Observatories*. Cambridge: Harvard University Press, 1999.

[4] Knuth, D.E. "The Calculation of Easter" in *Communications of the ACM*, Vol. 5, No. 4, 1962, pg 209 – 210.

[5] Knuth, D.E. *Fundamental Algorithms*, 2nd Edition, in *The Art of Computer Programming*, Vol.1. Reading: Addison-Wesley Publishing Co., 1973, pg 155 – 156.

[6] O'Beirne, T.H. "Chapter 10 Ten Divisions Lead to Easter" in *Puzzles and Paradoxes*. London: Oxford University Press, 1965.

[7] Reingold, E.M. & Dershowitz, N. *Calendrical Calculations* – The Millenium Edition. Cambridge: Cambridge University Press, 2001.

[8] Richards, E.G. *Mapping Time: The Calendar and its History*. New York: Oxford University Press, 1998.

[9] Stewart, Ian. "Easter is a Quasicrystal" in *Scientific American* March 2001.

Websites:

[10] Anglican Resource Collection: The Book of Common Prayer, Nov 1998.
http://justus.anglican.org/resources/bcp/

[11] GM Arts Homepage: Easter dating, June 2000.
http://users.chariot.net.au/~gmarts/easter.htm

[12] Henk Reints. Easter date algorithms, 1999.
http://members.brabant.chello.nl/~h.reints/easter/index.htm

[13] Hill, Bonnie L. Calendar FAQ v1.6 – Part 1, October 2002.
http://www.webspace4me.net/~blhill/pages.aux/astrology/cal.FAQ.1.html

[14] Howell, Lynda M. The Book of Common Prayer.
http://www.eskimo.com/~lhowell/bcp1662/index.html

[15] Mallen, Ronald W. Astronomical Society of South Australia: Easter Dating Method, April 2002. http://www.assa.org.au/edm.html#Method

[16] Meyer, Peter. The Julian and Gregorian Calendars. http://serendipity.magnet.ch/hermetic/cal_stud/cal_art.htm

[17] Robert Harry van Gent. A Bibliography on Easter, the Computus and Easter Algorithms. April 2002. http://www.phys.uu.nl/~vgent/easter/easterbibliography.htm

[18] Ross, Kelley L. The Determination of Easter, on both the Julian and Gregorian Calendars, 1999. http://www.friesian.com/easter.htm

[19] Royal Observatory Greenwich: The Date of Easter. http://www.rog.nmm.ac.uk/

[20] The Catholic Encyclopaedia Online Edition: Dominical Letter, Dec 2002. http://www.newadvent.org/cathen/05109a.htm

[21] Tobin, Paul N. The Rejection of Pascal's Wager: The Last Supper, 2000. http://www.geocities.com/paulntobin/lastsupper.html

[22] Tondering, Claus. Frequently Asked Questions about Calendars Version 2.4, 28 October 2001. http://www.tondering.dk/claus/calendar24.html

[23] US Naval Observatory http://aa.usno.navy.mil/