Undergraduate Research Opportunity Programme in Science

A Mathematical Supplement To "The Sun in the Church: Cathedrals as Solar Observatories" by J.L. Heilbron

NG YOKE LENG

SUPERVISOR
ASSOC. PROF. HELMER ASLAKSEN

Department of Mathematics

National University of Singapore

Semester II, 2000/2001

INTRODUCTION

J.L. Heilbron's book "The Sun in the Church" addresses a basic problem: how is time measured? Since the time period of the Earth's orbit around the Sun is not neatly divisible into a whole number of days, it is hard to construct a calendar that will mark a moment in time back at the exact same point after the Earth makes a complete orbit around the Sun. In the book, Heilbron gives a comprehensive overview of the various approaches that have been proposed and implemented by the Catholic Church to fix the calendar. As the Church wanted to have a systematic way to determine when Easter should be celebrated, the calendar had to be made as accurate and reliable as possible. In consequence, the Church became deeply involved in improving the quality of observational data on which calendars were based. Huge cathedrals were considered to be ideal solar observatories: by making a hole in the ceiling and fixing a mark on the floor where the sun's shadow fell along a line laid out on the cathedral floor, exact measurements could be made regarding the position of the sun. The combination of the hole and line was known as the "meridiana".

Many historical and technical facts on how the cathedrals came to be used as an instrument of measurement have been recorded in this book. Heilbron has succeeded in providing the readers with a most enriching and interesting experience as they read the historical account. Nonetheless, most readers would probably find it hard to appreciate the technical sections of this book because these sections call for a certain level of understanding in some mathematical concepts. This paper has been specially written as a mathematical supplement to enable readers to have a clearer comprehension of how certain conclusions have been drawn or how certain values have been obtained. Due to the sheer volume of this book, the supplement provides explanations for only a few sections of it; similar supplements for the rest of the book are expected to appear in the near future.

The structure of this supplement is broadly categorized into two parts. Firstly, a collection of preliminary information used in later explanations is given. This includes some mathematical tools and the models created by Hipparchus, Ptolemy and Kepler to explain the motions of the Sun and the planets. Secondly, detailed explanations of selected sections of Heilbron's book are provided. These concern the steps in calculation involved in working out certain values and the comparison of models. The use of a meridiana to justify the superiority of one model over another is also discussed.

1. Preliminaries

1.1 Background of various mathematicians mentioned

This section aims to provide more information on the prominent mathematicians mentioned in this paper so that readers may have a more defined impression of their names as they read this paper. The primary source for the contents of this section is:

http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/.

1.1.1 Hipparchus

Little is known of Hipparchus' life, but he is known to have been born in Nicaea in Bithynia. The town of Nicaea is now called Iznik and is situated in north-western Turkey. Despite being a mathematician and astronomer of great importance, Hipparchus' work has not been well documented; only one work of his has survived, namely *Commentary on Aratus and Eudoxus*. Most of his discoveries have been made known to us only through Ptolemy's *Almagest*. His contributions to the field of mathematics include producing a trigonometric table of chords, introducing the division of a circle into 360° into Greece, calculating the length of the year within $6.5'$ and discovering precession of the equinoxes.

1.1.2 Claudius Ptolemy

One of the most influential Greek astronomers and geographers of his time, Ptolemy propounded the geocentric theory in a form that prevailed for 1400 years. We do not know much about his life but most of his major works have survived. His earliest and most important piece is the *Almagest* which gives in detail the mathematical theory of the motions of the Sun, Moon and planets. This thirteen-book treatise was not superseded until a century after another mathematician, Copernicus, presented his heliocentric theory in the *De revolutionibus* of 1543.

1.1.3 Johannes Kepler

Born in the small town of Weil der Stadt in Swabia and moved to nearby Leonberg with his parents in 1576, Kepler is now chiefly remembered for discovering the three laws of planetary motion that bear his name published in 1609 and 1619. As much of his correspondence survived, we know a significant amount of Kepler's life and character. He did important work in optics, discovered two new regular polyhedra, gave the first mathematical treatment of close packing of equal spheres which led to an explanation of the shape of the cells of a honeycomb, gave the first proof of how logarithms worked and devised a method of finding the volumes of solids of revolution that can be seen as contributing to the development of calculus. Moreover, he calculated the most exact astronomical tables then known, whose continued accuracy did much to establish the truth of heliocentric astronomy.

1.1.4 Giovanni Cassini

Cassini studied at the Jesuit college in Genoa and then at the abbey of San Fructuoso. He observed at Panzano Observatory between 1648 and 1669 and in 1650, became professor of astronomy at the University of Bologna. Aside from an interest in astronomy, Cassini was also an expert in hydraulics and engineering. He was later employed by the Pope to oversee fortifications and river management on the river Po. In 1669, the senate of Bologna granted him approval to visit Paris under the invitation of Louis XIV, thinking that the trip would be a short one; yet he never returned to Italy. He became director of the Paris Observatory in 1671 and then switched to French citizenship two years later.

1.2 Some Mathematical Tools

1.2.1 Ellipses

Many mathematical textbooks contain comprehensive accounts of what an ellipse is and what its properties are. A recommended reference text is "Basic Calculus: from Archimedes to Newton to its Role in Science" written by Alexander J Hahn, and in particular, pages 54 – 55 and 90 – 94.

Here, some of the properties have been selected and stated below for easy reference.

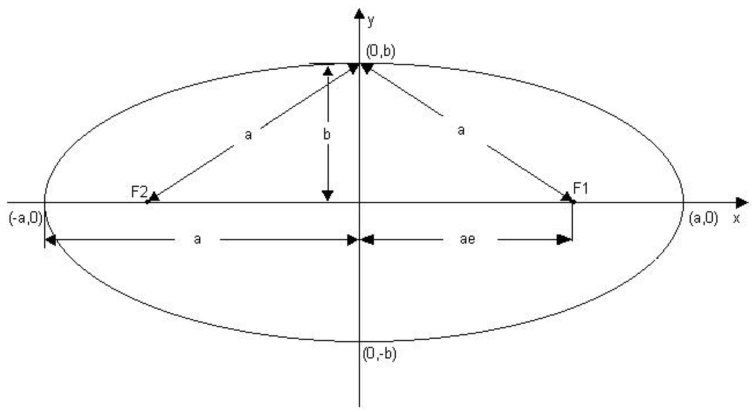


Figure 1

With reference to Figure 1, the **standard equation of the ellipse** is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Eccentricity, e, is defined as the ratio of the distance between the centre of the ellipse and one of the foci to the semimajor axis, or

$$e=\frac{OF_1}{a}=\frac{OF_2}{a}.$$

Then, $$OF_1=OF_2=ae.$$

Since the point $B=(0,b)$ is on the ellipse, $2BF_1=BF_1+BF_2=2a,$ and hence $BF_1=a.$ Similarly, $BF_2=a.$ By Pythagoras' Theorem,

$$a^2=b^2+(ae)^2$$
$$a^2=b^2+a^2e^2$$
$$b^2=a^2-a^2e^2$$
$$b^2=a^2(1-e^2).$$

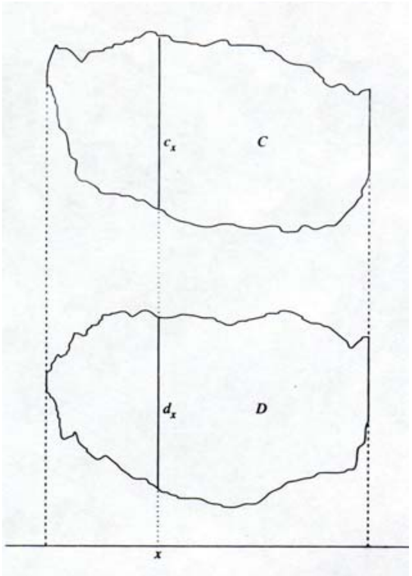


Figure 2

Referring to Figure 2, **Cavalieri's Principle** states that if $d_x = kc_x$ for all x and for a fixed positive number k, then D = kC.

Now, consider simultaneously the graph of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and that of the circle $x^2+y^2=a^2$, as shown in Figure 3.

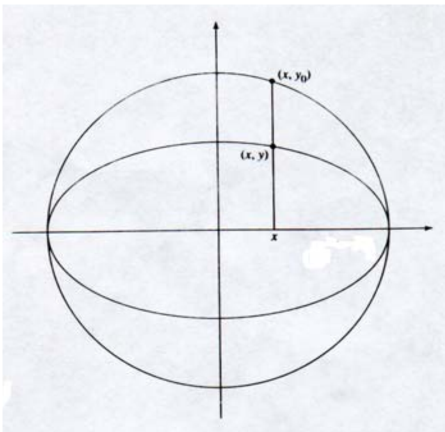


Figure 3

Let x satisfy $-a \le x \le a$ and, let (x, y) and (x, y_0) be the indicated points on the ellipse and circle, respectively. Since (x, y_0) satisfies $x^2 + y^2 = a^2$ and $y_0 \ge 0$, it follows that $y_0 = \sqrt{a^2 - x^2}$.

Since (x, y) is on the ellipse,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$

$$y^2 = \frac{b^2a^2 - b^2x^2}{a^2} = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$

$$y = \frac{b}{a}\sqrt{a^2 - x^2} = \frac{b}{a}y_0.$$

The above relation is frequently used in later calculations. In addition, if we suppose that the upper semicircles and the upper part of the ellipse are separated as shown in Fig. 4(a), we would then have demonstrated that $d_x = kc_x$ for all x.

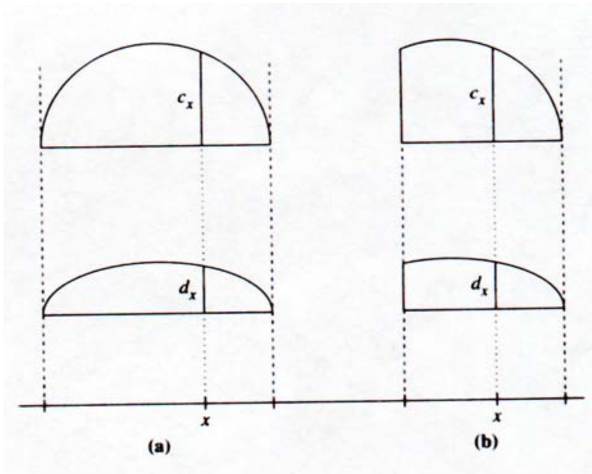


Figure 4

Since the area of a semicircle of radius a is $\frac{1}{2}\pi a^2$, it follows by Cavelieri's principle that the area of the upper half of the ellipse is equal to $\frac{b}{a}\left(\frac{1}{2}\pi a^2\right) = \frac{1}{2}\pi ab$. Therefore, the full ellipse with semimajor axis a and semiminor axis b has area πab. Note that Cavalieri's principle also applies to Figure 4(b). In particular, the area of the elliptical section has area $\frac{b}{a}$ times that of the semicircular section.

1.2.2 Small-angle approximations

Some angles dealt with in astronomy are small and may be approximated with little error caused. Consider θ being a small angle.

The Maclaurin series or Taylor series about the origin for $\sin\theta$, $\cos\theta$ and $\tan\theta$ up to terms in θ^2 are recorded below.

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1-\frac{\theta^2}{2}$$

$$\tan\theta \approx \theta .$$

1.2.3 Binomial Theorem

The Binomial Theorem states that for every pair of real numbers *x* and *y* and every natural number *n*,

$$(x+y)^n = x^n + nx^{n-1}y + \ldots + \frac{n(n-1)\ldots(n-k+1)}{k!}x^{n-k}y^k + \ldots + nxy^{n-1} + y^n .$$

1.3 Frames of reference

In this section, we investigate how the apparent motion between Sun and Earth changes when we shift from a **heliocentric**, or sun-in-the-centre, frame of reference to a **geocentric**, or earth-in-the-centre, frame of reference. A similar discussion is provided in the later part of this section concerning the Sun, the Earth and the planets. The former is based on page 118 of "The Exact Sciences in Antiquity" by O. Neugebauer while the latter is based on page 1010 of James Evans' article entitled "The division of the Martian eccentricity from Hipparchos to Kepler: A history of the approximations to Kepler motion".

Consider a system with only Sun S and Earth E. We know that the Earth revolves around the Sun and completes its orbit in a year. See Figure 5(a).

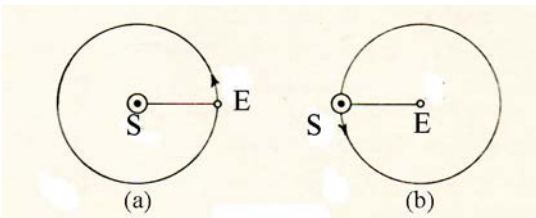


Figure 5

By arresting the motion of the Earth, we would also observe the Sun to revolve around Earth on a circular path but in the opposite direction. This is shown in Figure 5(b). The apparent path of the Sun around the Earth is called the **ecliptic**.

Now, consider the motion of a superior planet, or one that is farther than the Earth with respect to the Sun, on a heliocentric theory.

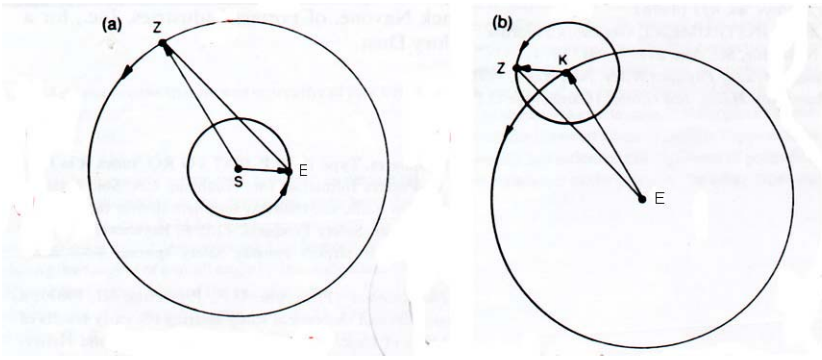


Figure 6

With reference to Figure 6(a), the Earth E executes an orbit about the stationary Sun S. In the course of a year, the position vector $\overrightarrow{SE}$ rotates anticlockwise about S. The angular speed of $\overrightarrow{SE}$ varies slightly in the course of the year, and so does the length of the vector. Similarly, a superior planet Z revolves around the Sun in a larger orbit. Vector $\overrightarrow{SZ}$ varies at its own slightly different rate. At any instant, the line of sight from the Earth to the planet coincides with $\overrightarrow{EZ}$ which is also equal to $-\overrightarrow{SE}+\overrightarrow{SZ}$. Since these vectors may be added in either order, $\overrightarrow{EZ}$ is also equivalent to $\overrightarrow{SZ}+\left(-\overrightarrow{SE}\right)$. The new form of the addition is shown in Figure 6(b).

Here we begin at the Earth E. A vector equal to $\overrightarrow{SZ}$ is drawn with its tail at E; let the head of this vector be called K. Then $\overrightarrow{EK}$ rotates in Figure 6(b) at the same rate as $\overrightarrow{SZ}$ rotates in Figure 6(a). At K, place the tail of a second vector, equal to $-\overrightarrow{SE}$, with its head at planet Z. This series of steps brings about a transformation in a superior planet from a heliocentric model (Figure 6(a)) to a geocentric model (Figure 6(b)). Point K serves as the centre of a small circle, or an **epicycle**, upon which Z revolves while K itself moves on a large carrying circle, or **deferent**, about the Earth E. The equivalence of Figures 6(a) and 6(b) is a consequence of the commutative property of vector addition. Hence, in the case of a superior planet, the epicycle corresponds to the orbit of the Earth about the Sun, and the deferent, to the heliocentric orbit of the planet itself.

In the case of an inferior planet, or one that is between the Sun and the Earth in the planetary system, the transformation is similar to that as explained for superior planets, but with adjustments made to the vectors and vector addition.

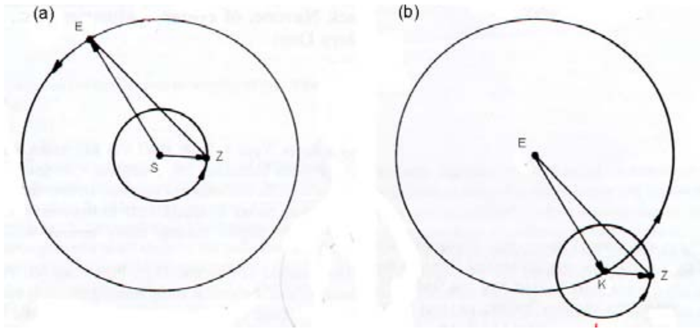


Figure 7

With reference to figures 7(a) and 7(b), S, E and Z retain their original representations. Notice in Figure 7(a), however, that Z is on the inner orbit and vector $\overrightarrow{EZ}$ is equal to $-\overrightarrow{SE}+\overrightarrow{SZ}$. We draw a vector equivalent to $-\overrightarrow{SE}$ with its tail at E and its head at a new point called K on the deferent. At K, we add a vector equal to $\overrightarrow{SZ}$ and call it $\overrightarrow{KZ}$. Once again, Figures 7(a) and 7(b) are equivalent to each other because of the commutative property of vector addition. Thus, for an inferior planet, the epicycle corresponds to its heliocentric orbit whilst the deferent corresponds to the orbit of the Earth about the Sun.

As a remark, the above discussion has assumed circular orbits for Earth and planets about Sun and hence, results in circular deferents and epicycles. Yet if Keplerian motion (elaborated on in Section 1.6), in particular, Kepler's First Law, is taken into account, then the same discussion would involve orbits that are elliptical, elliptical deferents and epi-ellipses.

1.4 Hipparchus' model of the motion of the Sun

Greek astronomers believe that all orbits of luminaries and planets treated in astronomy should be circles or components of circles. The simplest manner to represent the apparent motion of the Sun S as observed from the

Earth E would be a circle in the plane of the ecliptic, centered on the Earth E. This may be seen in Figure 8.

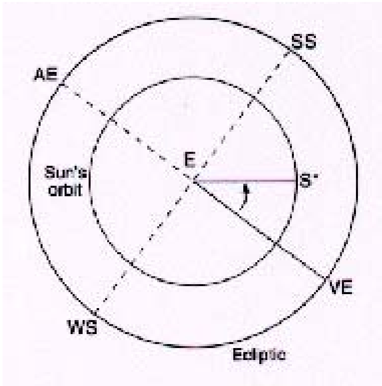


Figure 8

The solstices and equinoxes are located 90° apart as seen from the Earth E, and if we take y as the number of days in a year, we would expect the interval between the seasons to be exactly of length $\frac{y}{4}$ days. However, the observed facts show otherwise: the seasons are not equal. In particular, Hipparchus found the Sun to move 90° in the ecliptic plane from Spring equinox to Summer solstice in 94½ days and 90° from Summer solstice to Autumn equinox in 92½ days. Modern values for the lengths of the seasons are as follows: Spring – 92 days, 18 hours, 20 minutes or 92.764 days; Summer – 93 days, 15 hours, 31 minutes or 93.647 days; Autumn – 89 days, 20 hours, 4 minutes or 89.836 days; and Winter – 88 days, 23 hours, 56 minutes or 88.997 days. To explain this phenomenon, ancient astronomers would rather regard the inequalities in the speed of the Sun as an optical illusion than rule that the Sun does not move at uniform angular speed. For more information on how the illusion works, please refer to pages 104 – 105 in Heilbron's book. The essence of their solution was to displace the Earth from the centre of the Sun's orbit.

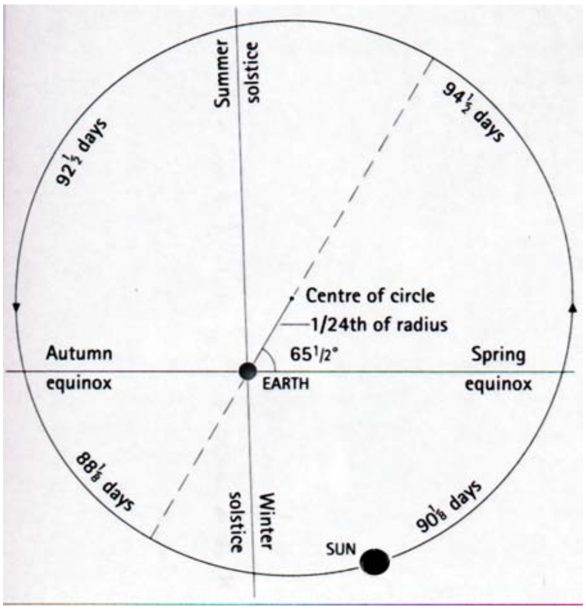


Figure 9

Much of the information in the following paragraph concerning Hipparchus' simple model of the orbit of the Sun is quoted from page 41 of "The Cambridge Illustrated of Astronomy" by Michael Hoskin.

In Figure 9, Hipparchus' solar model is given. The Earth is stationary at E, while the Sun S moves around the circle at a uniform angular speed about the centre of the circle C. The Sun's circle is thus said to be eccentric to the Earth. To generate the longer intervals between solstice and equinox, the Earth had to be removed from the centre in the opposite direction so that the corresponding arcs as seen from the Earth would each be more than $\frac{1}{4}$ of the circle, and it would take longer than $\frac{y}{4}$ days to traverse them. Hipparchus' calculations showed that the distance between the Earth and centre has to be 1/24 of the radius of the circle and that the line from the Earth to centre had to make an angle of 65½ ° with the Spring equinox.

At this point, I would like to highlight some of the terminology used in Heilbron's book as well as this paper.

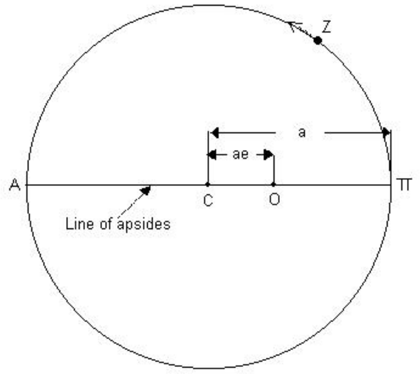


Figure 10

With reference to Figure 10, suppose *a* is the radius of the circle centered at C and Z is a body moving on the circle. The **eccentricity** *e* is defined as the ratio of the distance between O and C to radius *a*, that is,

$$e = \frac{OC}{a}.$$

This is similar to the definition of eccentricity for an ellipse. In Hipparchus' model for the Sun's orbit mentioned earlier, the figure "1/24" is in fact the eccentricity. The distance of separation *OC* is thus equivalent to *ae*.

Perigee Π occurs where Z is closest to O whilst **apogee** A occurs where Z is furthest from O.

The line joining the perigee and apogee is known as **the line of apsides**.

1.5 Ptolemy's solar and planetary models

Ptolemy's solar model is equivalent to that of Hipparchus'. In Heilbron's book, Ptolemy's method for obtaining the solar eccentricity and the angle between the line of apsides and the line joining the solstices ψ is illustrated and explained; the calculations involved are provided in Section 2.2 of this paper. When modern values for time differences between solstice and equinox are applied to Ptolemy's model, the value found for the angle ψ is fairly close to that found in the past. However the same could not be said for the value of eccentricity. In fact, the old value for solar eccentricity *e* exceeds the modern value by a factor of two. More discussion on this issue is provided later in this section.

To determine planetary positions accurately and conveniently, Ptolemy made use of a model that involved epicycles, deferents and equant points. I shall proceed to elaborate more on this model with reference to the same paper quoted earlier in Section 1.3 by James Evans, and another of his, entitled "On the function and the probable origin of Ptolemy's equant".

As indicated in Section 1.3, when we shift the point of view of a planet from the Sun to the Earth, the planet moves on an epicycle, with its centre carried on a deferent about the Earth. When seen from Earth, the planet appears to display periodic retrograde motion, that is, it seems to move backwards sometimes. Apollonios of Perge used the deferent-and-epicycle theory to provide an explanation for retrograde motion in the zero-eccentricity planetary model, or the model in which the centre of the deferent coincides with Earth.

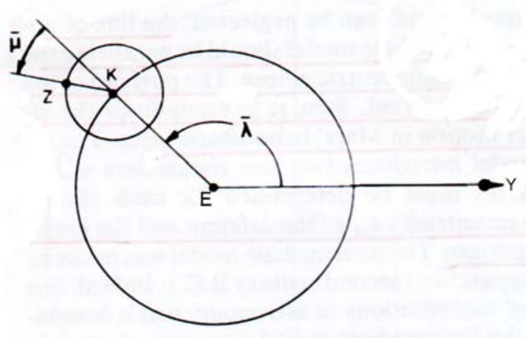


Figure 11

In Figure 11, the planet Z moves uniformly on an epicycle with centre K. K also moves uniformly around a deferent centred at Earth E. The fixed reference line Oϒ points to the vernal equinoctial point. Thus angles $\bar{\lambda}$ and $\bar{\mu}$ increase uniformly with time. If $\bar{\mu}$ increases at a much higher rate than $\bar{\lambda}$, then the planet would appear to display retrograde motion. In other words, in the ancient planetary theory, the epicycle is the mechanism that produces retrograde motion.

However, even though the model accounts for the retrogradations qualitatively, it fails to do so quantitatively. By the model, the retrograde loops produced would have the same size and shape, and be uniformly spaced around the ecliptic, as shown in Figure 12.

Figure 12

Yet this is overly simplified and does not occur in reality. The precise distance between one retrogradation and the next is quite variable, and thus the retrograde arcs are not equally spaced around the zodiac. There is no way for the uniformly spaced retrograde loops of the model to reproduce the unevenly spaced retrograde arcs of the planet itself.

A step taken to improve this zero-eccentricity model was to displace the Earth E from the centre of the deferent, C, slightly.

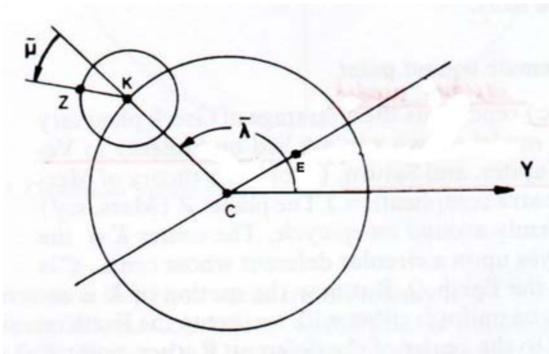


Figure 13

In Figure 13, planet Z moves uniformly on the epicycle and the centre of the epicycle, K, moves around the deferent at uniform speed. However, C no longer coincides with E, as in the "zero-eccentricity model". This is known as the "intermediate model". The intermediate model provided just the kind of variable spacing between two retrogradations of the planet yet it was still unable to fit the width of the retrograde loops perfectly. To tackle the problem, Ptolemy added a third device called the **equant point**, defined as a point about which the angular velocity of a body on its orbit is constant. Figure 14 shows Ptolemy's planetary model.

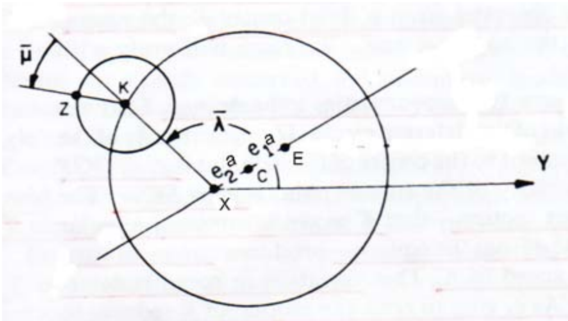


Figure 14

Here, Z, K and E have their usual meanings. However K is now assumed not to move uniform either with respect to the Earth E or the centre of the deferent, C. Instead, uniform motion of K is observed at X which is the mirror image of the Earth E, and is what Ptolemy named as the "equant point".

As a result of these two points X and E, there are now two eccentricities to be defined. Take the radius of the deferent to be a, the eccentricity of the Earth E with respect to centre C to be e_1 and the eccentricity of the equant point with respect to centre C to be e_2. Then,

$$e_1 = EC / a,$$
$$e_2 = XC / a.$$

The rule of equant motion – that K moves at constant angular speed as viewed from the equant – produces a physical variation in the speed of K. This variation in speed is determined by e_2 and if e_2 goes to zero, the motion of K reduces to uniform circular motion. Even if e_2 were zero, the eccentricity e_1 would cause the motion of K to appear non-uniform from the Earth E due to the same optical illusion mentioned in Section 1.4. The sum $e_1 + e_2$ is called the "total eccentricity". Ptolemy always puts $e_1 = e_2$ and such a situation has come to be referred to as the "bisection of eccentricity". This notion would once again be brought up in Section 2.3 where we make comparisons across Hipparchus', Ptolemy's and Kepler's solar and planetary models.

1.6 Kepler's laws

After a tedious and difficult research process, Kepler discovered three laws that could describe how the planets move with reference to the Sun with more precision as compared to Copernicus' and Ptolemy's planetary models.

The following information is based on pages 98 – 99 and 111 – 114 of “Text-Book on Spherical Astronomy” written by W.M. Smart.

Kepler’s First Law states that the path, or orbit, of a planet around the Sun is an ellipse, the position of the Sun being at a focus of the ellipse.

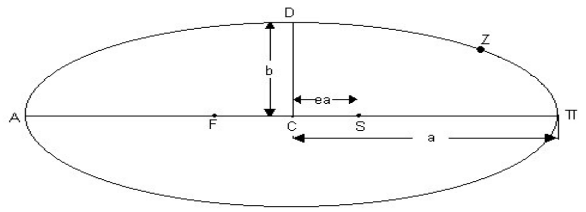


Figure 15

Figure 15 above shows an ellipse of which S and F are the two foci, C is the centre (midway between S and F) and ΠA is the major axis. The Sun will be supposed to be at S and the planet to move anticlockwise around the ellipse. At Π, the planet is at perigee, while at A, it is at apogee. ΠC is the semimajor axis; its length is given by a. DC is the semiminor axis; its length is denoted by b. The eccentricity e is given by the ratio SC : SA. Just as highlighted in Section 1.2.1, $b^2 = a^2(1-e^2)$. The perigee distance SΠ is $a(1-e)$ and apogee distance SA is $a(1+e)$.

Let ρ denote the distance of the planet Z from the Sun and the angle θ be the planet’s angular distance from perigee, or the true anomaly. The equation of its elliptical orbit is known to be

$$\rho = \frac{b^2/a}{1+e\cos\theta} = \frac{a(1-e^2)}{1+e\cos\theta}$$

The time required for the planet to describe its orbit is called the **period**, denoted by T. In time T, the radius vector SZ sweeps out an angle of 2π and thus, the **mean angular velocity** of the planet, w, is $\frac{2\pi}{T}$.

Kepler’s Second Law states that the radius vector SZ (in Figure 16) sweeps out equal areas in equal times.

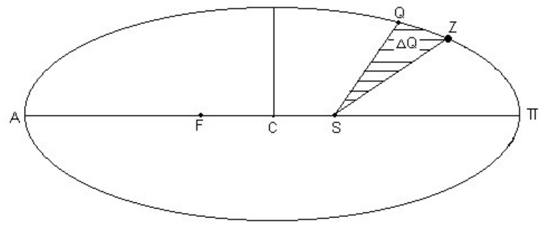


Figure 16

Let Z correspond to the planet's position at time t and Q its position at time $t+\Delta t$. Let $\rho+\Delta\rho$ denote the radius vector SQ and $\theta+\Delta\theta$ be $Q\hat{S}\Pi$. Hence, $Q\hat{S}Z=\Delta\theta$. If $\Delta\theta$ is sufficiently small, the arc ZQ may be regarded as a straight line and the area swept out in the infinitesimal interval Δt is simply the area of triangle QSZ which is equal to $\frac{1}{2}\rho\left(\rho+\Delta\rho\right)\sin\Delta\theta$, or with sufficient accuracy, $\frac{1}{2}\rho^2\Delta\theta$. The area velocity or the rate of description of area is the previous expression for area divided by Δt. As this rate is constant according to Kepler's second law, we can write,

$$h=\frac{1}{2}\rho^2\frac{d\theta}{dt} \qquad (1)$$

where h is a constant.

Now, the whole area of the ellipse is πab and this is described in the period T. Hence,

$$h=\frac{\pi ab}{T}$$

or,

$$h=\frac{\pi a^2(1-e^2)^{\frac{1}{2}}}{T} \qquad (2)$$

because $b^2=a^2(1-e^2)\Rightarrow b=a(1-e^2)^{\frac{1}{2}}$.

By (1) and (2), we have, $\frac{1}{2}\rho^2\frac{d\theta}{dt}=\frac{\pi a^2(1-e^2)^{\frac{1}{2}}}{T}$.

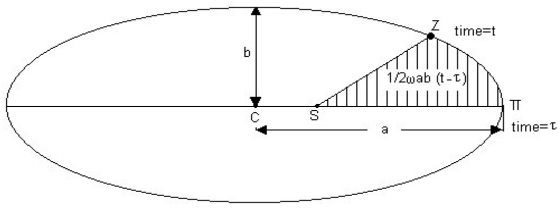


Figure17

Theoretically, if the values of the semimajor axis a, the eccentricity e, the time at which the planet passed through perigee τ and the orbital period T are known, Kepler's second law would enable us to determine the position of the planet in its orbit at any instant.

Referring to Figure 17, Z is the position of the planet at time t. In the interval (t - τ) the radius vector moving from SΠ to SZ sweeps out the shaded area SZΠ. By Kepler's Second Law,

Area SZΠ : Area of ellipse = t - τ : T.

Hence, Area SZΠ = $\dfrac{\pi ab(t-\tau)}{T}$,

Or, Area SZΠ = $\dfrac{1}{2}wab(t-\tau)$

where $w = \dfrac{2\pi}{T}$ and $b^2 = a^2(1-e^2)$.

This method may seem easy but in practice, it is inconvenient; the alternative is explained in detail in Section 2.6.

2. Clarifying selected sections of Heilbron's book

2.1 Notes to Appendix B: The Analemma in the Construction of San Petronio

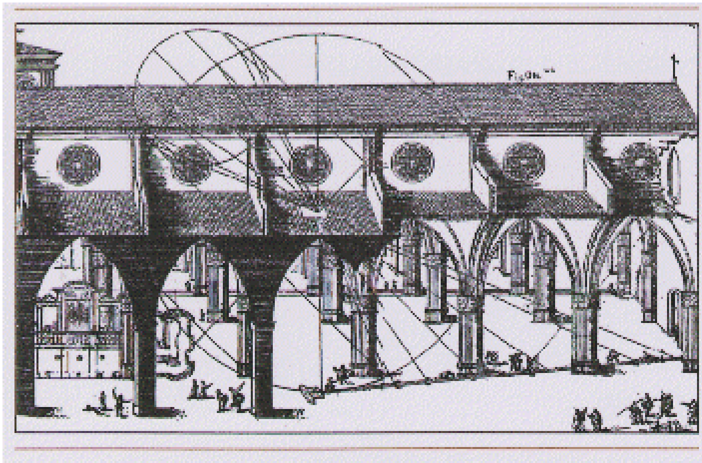


Figure 18

Figure 18 shows a group of people in the process of laying the meridiana at San Petronio. The circles imposed on the diagram make up the analemma used for a geometrical location of the zodiacal plaques along the meridiana. The geometry involved is mentioned in Appendix B of Heilbron's book and this section aims to provide an elaboration.

The vertical circles in Figure 18 have been redrawn in Figure 19 below.

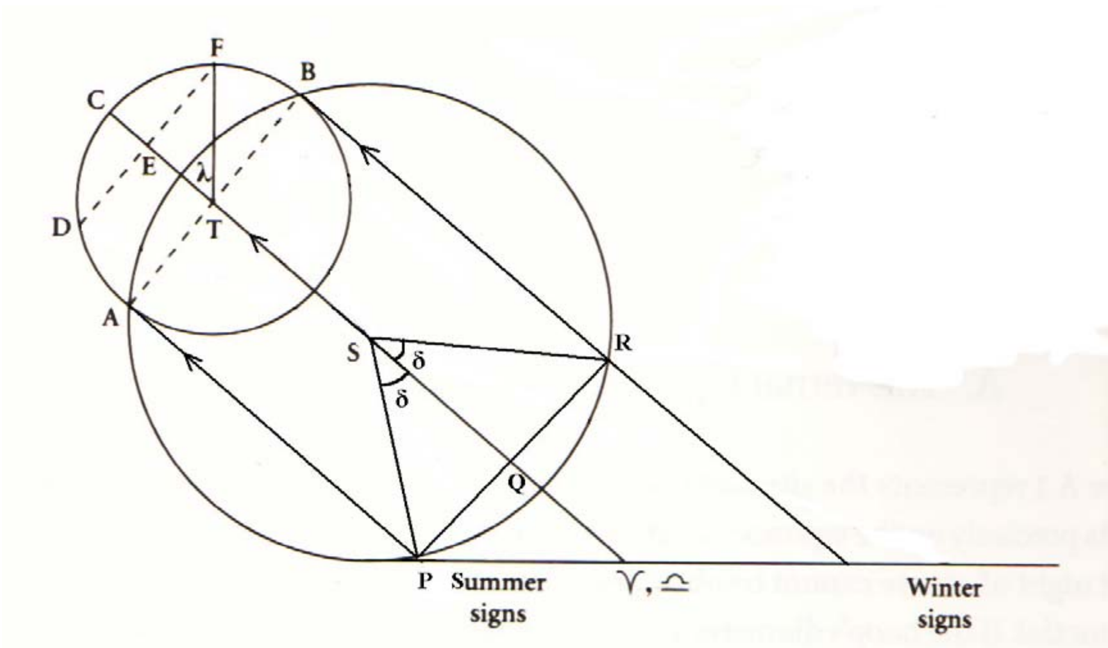


Figure 19

S indicates the centre of the gnomon, or an instrument that serves to indicate the time of day by casting its shadow upon a marked surface. In the case of

San Petronio, the marked surface would refer to the meridiana. The larger circle centred on S cuts the smaller circle centred on T at points A and B such that the chord AB of the bigger circle is identical to the diameter of the smaller circle. Let F be any point on the small circle and CTS be the noon ray at an equinox. In addition, $C\hat{T}F = \lambda$, $P\hat{S}Q = R\hat{S}Q = \delta$, and *r*, *R* are the radii of the smaller and bigger circles respectively.

If DF is parallel to AB, in triangle EFT, $\frac{EF}{r} = \sin\lambda \Rightarrow EF = r\sin\lambda$. Since $EF = QR$, and in triangle RSQ, $\frac{QR}{R} = \sin\delta \Rightarrow QR = R\sin\delta$. Then, $EF = R\sin\delta$. Therefore,

$$R\sin\delta = r\sin\lambda$$

$$\sin\delta = \frac{r}{R}\sin\lambda$$

Taking $\sin\varepsilon = \frac{r}{R}$, we have

$$\sin\delta = \sin\varepsilon\sin\lambda. \quad (1)$$

Now, refer to Figure 20.

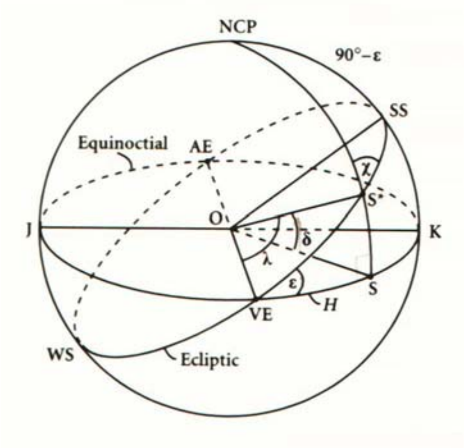


Figure 20

The above figure gives the earlier-mentioned angles λ and δ on the celestial sphere. S* marks the true Sun while S the projection of the true Sun on the equinoctial. δ is the Sun's declination and ε the obliquity of the ecliptic. Let the radius of the celestial sphere be K.

Since triangles OSS*, VE.SS* and O.VE.S* are approximately right-angled triangles, we have

$$\sin\delta = \frac{SS*}{K} \Rightarrow SS* = K\sin\delta \text{ and } \sin\varepsilon = \frac{SS*}{S.VE} \Rightarrow SS* = S.VE\sin\varepsilon.$$

Hence, $K\sin\delta = S*VE\sin\varepsilon$

$$\sin\delta = \frac{S*VE}{K}\sin\varepsilon \qquad (2)$$

Comparing (1) and (2), $\sin\lambda = \frac{S*VE}{K}$. Taking λ small, we have

$$\lambda \approx \frac{S*VE}{K} \Rightarrow S*VE \approx K\lambda.$$

Kλ gives the ecliptic longitude and since K is constant, it is sufficient to mark the point where the noon ray falls on the meridiana at an equinox, and then by increasing λ in steps of 30°, the rest of the zodiacal plaques could be positioned accordingly.

2.2 Notes to Heilbron, pg 105: Ptolemy's solar eccentricity

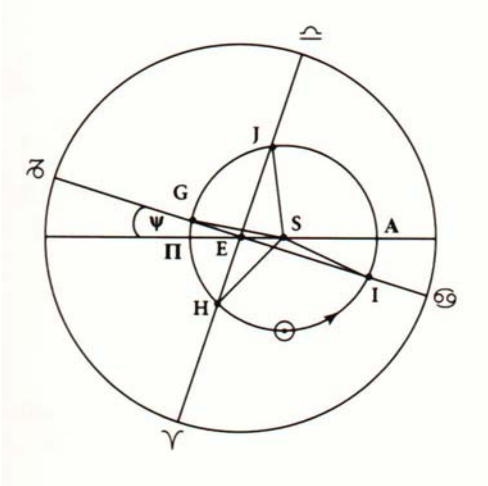


Figure 21

As highlighted in Section 1.5, Ptolemy made use of Hipparchus' model for the Sun's motion around the earth and calculated values for the solar eccentricity, e, and the angle ψ between the line of apsides and the line joining the solstices. Figure 21 shows the model that Ptolemy used to work out the aforesaid values and this section aims to provide a detailed set of working to obtain the result quoted at the end of the page: $e = 0.0334$, $\psi = 12^{\circ}58'$, by applying modern seasonal lengths.

Since fall is longer than spring, EA must point somewhere between the summer solstice and autumnal equinox. By doing so, when viewed from the earth, the sun's orbit between points I and J would be longer than $\frac{1}{4}$ arc of the entire path. Then consequently, EΠ points between the winter solstice and the spring equinox. Taking the Julian value for the number of days in a year (denoted by y), the mean angular motion of the sun, w, is equal to $360^{\circ}/y$ per day, or

$$w = 360^{\circ}/365.25 \approx 0.9856^{\circ} \text{ per day}$$

Given that the differences in time between the sun's arrival at G, H, I and J on its orbit, when, viewed from the earth E, as it appears at the first points of Capricorn, Aries, Cancer and Libra, respectively are $89^{d}0^{h}$, $92^{d}18^{h}$, $93^{d}15^{h}$, $89^{d}20^{h}$, and, we work out the following angles:

$$G\hat{S}H = (89)w = (89)\frac{360^\circ}{365.35} \approx 87.7184^\circ$$

$$H\hat{S}I = \left(92\frac{18}{24}\right)w = \left(92\frac{18}{24}\right)\frac{360^\circ}{365.35} \approx 91.4144^\circ$$

$$J\hat{S}I = \left(93\frac{15}{24}\right)w = \left(93\frac{15}{24}\right)\frac{360^\circ}{365.35} \approx 92.2768^\circ$$

$$G\hat{S}J = \left(89\frac{20}{24}\right)w = \left(89\frac{20}{24}\right)\frac{360^\circ}{365.35} \approx 88.5397^\circ$$

These angles correspond to the seasons Winter, Spring, Summer and Autumn respectively.

As a remark, in Heilbron's book, the order of listing of intervals between Capricorn, Aries, Cancer and Libra did not correspond directly with that of the angles that followed. Had readers overlooked the mis-correspondence, and proceeded with the calculations for e and ψ, errors would have been unavoidable.

Now we continue with the working steps to obtain values for e and ψ. Consider triangle JSH. Since it is an isosceles triangle, the perpendicular, of length y, when dropped from point S to the base JH, would bisect the angle JSH. In addition, take $JS = HS = a$ where a is the radius of circle centred at S. Then,

$$\cos\frac{G\hat{S}J + G\hat{S}H}{2} = \frac{y}{a}$$

Similarly, in isosceles triangle GSI, taking x as the length of the perpendicular dropped from point S to the base GI, we have,

$$\cos\frac{G\hat{S}H + H\hat{S}I}{2} = \frac{x}{a}$$

Due to the fact that $\tan\psi = \frac{x}{y} = \frac{x/a}{y/a}$, that is,

$$\tan\psi = \frac{\cos\frac{G\hat{S}H + H\hat{S}I}{2}}{\cos\frac{G\hat{S}J + G\hat{S}H}{2}}$$

Hence, $$\psi = \tan^{-1}\left(\frac{\cos\frac{G\hat{S}H + H\hat{S}I}{2}}{\cos\frac{G\hat{S}J + G\hat{S}H}{2}}\right)$$

By substituting the exact values of the angles into the above equation, we obtain the value of ψ to be approximately $12^\circ 59'$, almost equivalent to the value quoted in Heilbron's book.

To work out the value of eccentricity, *e*, we note that $e = \frac{ES}{a}$. But $\sin\psi = \frac{x}{ES} \Rightarrow ES = \frac{x}{\sin\psi}$. Therefore, $e = \frac{\frac{x}{\sin\psi}}{a} = \frac{\frac{x}{a}}{\sin\psi}$. Using the result from above, we have,

$$e = \frac{\cos\frac{G\hat{S}H + H\hat{S}I}{2}}{\sin\psi}$$

After substituting the relevant values into the above relation, we obtain the value of *e* to be around 0.0335, which is also close to that quoted in the book.

However, Ptolemy's value of solar eccentricity which is 0.0334 exceeded that found by Kepler which is 0.0167. To account for this factor-of-two difference, we have to look into the way in which Ptolemy and Kepler measured the separation between the centre of the sun's orbit and the earth. Figure 22 provides a visual aid to this explanation.

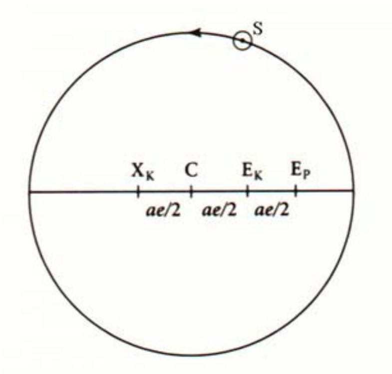


Figure 22

C represents the centre of the sun's orbit, E_P denotes where Ptolemy positions the earth, E_K denotes where Kepler positions the earth and X_K marks Kepler's equant point. Consider radius of the sun's orbit to be *a*, then the intervals between X_K, C, E_K and E_P would each be $\frac{ae}{2}$. Ptolemy calculated eccentricity as $\frac{CE_P}{a} = \frac{ae/2 + ae/2}{a} = e$ whereas Kepler calculated the same quantity as $\frac{CE_K}{a} = \frac{ae/2}{a} = \frac{e}{2}$. Thus, the factor-of-two difference is produced.

Even though we know now that Kepler's model is the accurate one, it would be interesting to find out how Ptolemy's model had provided such a close approximate to Kepler's model. Since both models gave near similar observations, mathematicians and astronomers alike had to struggle with deciding which of the models was the correct one until the development of the meridiana. In fact, the use of the meridiana to make measurements that would allow substantial conclusions to be drawn from them is another area worth exploring into. In the following sections of the paper, the discussion would focus on the comparison of models and the use of the meridiana.

2.3 Comparison of models

In this paper, three sets of solar and planetary models have been brought into discussion, namely, by Hipparchus, Ptolemy and Kepler. Ptolemy's equant theory for the planets is an amazingly close approximate to Kepler's planetary model. In this section, we aim to find out mathematically how this has been possible. Prior to that, we would derive the equations defining the position of a body in its orbit in each of the three models. The defining parameters of the body's position are the true anomaly and the radius vector. In finding out the equations of these parameters, we would also make use of the eccentricity. Information on which this section is based mostly on Evans' article entitled "The division of the Martian eccentricity from Hipparchos to Kepler: A history of the approximations to Kepler motion" and shall be complemented with brief notes from other books as recorded in the reference section.

As a remark, in Heilbron's book, the value of eccentricity in Ptolemy's planetary theory is e whereas that in Kepler's is $\frac{e}{2}$. However, in most mathematical textbooks, the former is usually taken as 2e while the latter as $2e$. For the rest of the paper, we wish to identify with Heilbron's notation. Hence, the form of any equation involving eccentricity will be maintained as that found in general textbooks but the eccentricity values will be altered to correspond with Heilbron's and written in parentheses without expansion.

As described in Section 1.6, the equation for the elliptical orbit is

$$\rho = \frac{a\left[1-\left(\frac{e}{2}\right)^2\right]}{1+\frac{e}{2}\cos\theta} \tag{1}$$

where ρ is the planet's distance from the Sun, θ is the true anomaly and $\frac{e}{2}$ is the eccentricity.

By means of the binomial theorem, the above equation may be expanded as the following:

$$\begin{aligned}\rho(\theta) &= a\left[1-\left(\frac{e}{2}\right)^2\right]\left[1+\frac{e}{2}\cos\theta\right]^{-1} \\ &= a\left[1-\left(\frac{e}{2}\right)^2\right]\left[1-\frac{e}{2}\cos\theta+\left(\frac{e}{2}\right)^2\cos^2\theta-\ldots\right] \\ &= a\left[1-\left(\frac{e}{2}\right)^2\right]\left[1-\frac{e}{2}\cos\theta+\left(\frac{e}{2}\right)^2(1-\sin^2\theta)-\ldots\right] \\ &= a\left[1-\left(\frac{e}{2}\right)^2\right]\left[1-\frac{e}{2}\cos\theta+\left(\frac{e}{2}\right)^2-\left(\frac{e}{2}\right)^2\sin^2\theta-\ldots\right]\end{aligned}$$

Hence, $$\rho(\theta) = a\left[1-\frac{e}{2}\cos\theta-\left(\frac{e}{2}\right)^2\sin^2\theta+\ldots\right] \qquad (2)$$

Also derived in section 1.6 is the condition of constant areal velocity, that is,

$$\frac{1}{2}\rho^2\frac{d\theta}{dt} = \frac{\pi a^2\left[1-\left(\frac{e}{2}\right)^2\right]^{\frac{1}{2}}}{T}.$$

When (1) is substituted into the above, and the resulting differential equation for $\theta(t)$ is expanded and integrated through order e^2, we obtain

$$\theta(t) = wt + 2\left(\frac{e}{2}\right)\sin wt + \frac{5}{4}\left(\frac{e}{2}\right)^2\sin 2wt, \qquad (3)$$

where w is the angular velocity.

Equations (2) and (3) are the defining equations for a body on the Keplerian model. We next derive similar equations for a body on an eccentric orbit with equant. Figure 23 is the reference diagram for the derivation steps that follow.

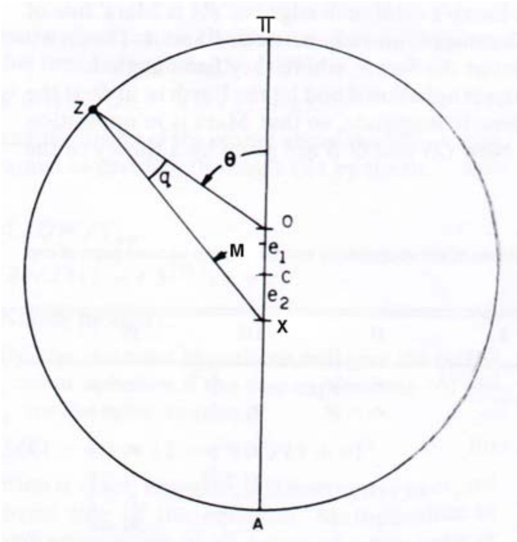


Figure 23

The equation of the orbit in polar coordinates for Z is that of a circle eccentric to C, which is,

$$\rho(\theta) = -e_1 \cos\theta + (1 - e_1^{\,2} \sin^2\theta)^{\frac{1}{2}}$$

Hence, by binomial theorem,

$$\rho(\theta) \approx 1 - e_1 \cos\theta - \frac{1}{2} e_1^{\,2} \sin^2\theta \qquad (4)$$

Since mean anomaly M is equivalent to wt and $q = \theta - M \Rightarrow \theta = M + q$, therefore, $\theta = wt + q$.

Applying Sine Rule to triangle OZX, we have

$$\frac{OX}{\sin q} = \frac{OZ}{\sin M} \Rightarrow \frac{e_1 + e_2}{\sin q} = \frac{\rho}{\sin wt} \Rightarrow \sin q = \frac{(e_1 + e_2)\sin wt}{\rho}.$$

This gives $q = \sin^{-1}\left[\frac{(e_1 + e_2)\sin wt}{\rho}\right]$.

Hence,

$$\theta(t) = wt + \sin^{-1}\left[\frac{(e_1 + e_2)\sin wt}{\rho}\right].$$

By substituting the expression of ρ for an eccentric with equant model into this equation, and expanding to second order in e, we get

$$\theta(t) = wt + (e_1 + e_2)\sin wt + \frac{1}{2} e_1 (e_1 + e_2)\sin 2wt \qquad (5)$$

For Hipparchus' solar theory, $e_1 = e$ and $e_2 = 0$. On Ptolemy's equant theory, he does a bisection of eccentricity by putting $e_1 = e_2$. Hence each is of value $\frac{e}{2}$. By substituting the two sets of eccentricity values into equations (3) and (4), we find the corresponding true anomaly and radius vector equations as shown below. In addition, equations (2) (include up to second order terms in *e* and set the semimajor axis as unity) and (3) have also been added into the table for comparison purposes.

Model	Eccentricities	True anomaly, $\theta(t)$	Radius vector, $\rho(t)$
Hipparchus	$e_1 = e$	$wt + 2\left(\frac{e}{2}\right)\sin wt + 2\left(\frac{e}{2}\right)^2 \sin 2wt$	$1 - 2\left(\frac{e}{2}\right)\cos\theta - 2\left(\frac{e}{2}\right)^2 \sin^2\theta$
Ptolemy	$e_1 = e$ and $e_2 = e$	$wt + 2\left(\frac{e}{2}\right)\sin wt + \left(\frac{e}{2}\right)^2 \sin 2wt$	$1 - \left(\frac{e}{2}\right)\cos\theta - \frac{1}{2}\left(\frac{e}{2}\right)^2 \sin^2\theta$
Kepler	$e_1 = e_2 = \frac{e}{2}$	$wt + 2\left(\frac{e}{2}\right)\sin wt + \frac{5}{4}\left(\frac{e}{2}\right)^2 \sin 2wt$	$1 - \left(\frac{e}{2}\right)\cos\theta - \left(\frac{e}{2}\right)^2 \sin^2\theta$

Seeing the data, suppose we ignore the second order terms, then the first discrepancy arises in the radius vector or Hipparchus' model. Consider the true anomalies of Ptolemy and Kepler. The difference between them is a mere $\frac{1}{4}\left(\frac{e}{2}\right)^2 \sin 2wt$ that amounts to a maximum of 8 min of arc. The derivation of this difference is given in Section 2.6. Considering the low level of precision in instruments used to make measurements of celestial bodies in ancient days, it would have been hard to recognize that this minute difference points towards an error in the model. It is no wonder that so much controversy had occurred amongst the astronomers then, in their desire to conclude which planetary theory was the true and exact one.

In fact, Ptolemy's and Kepler's models have approximated each other so well that up to first order terms in *e*, the empty focus of the Keplerian ellipse is indistinguishable form the equant point in Ptolemy's model. The working steps that follow provide the mathematics behind such an approximation.

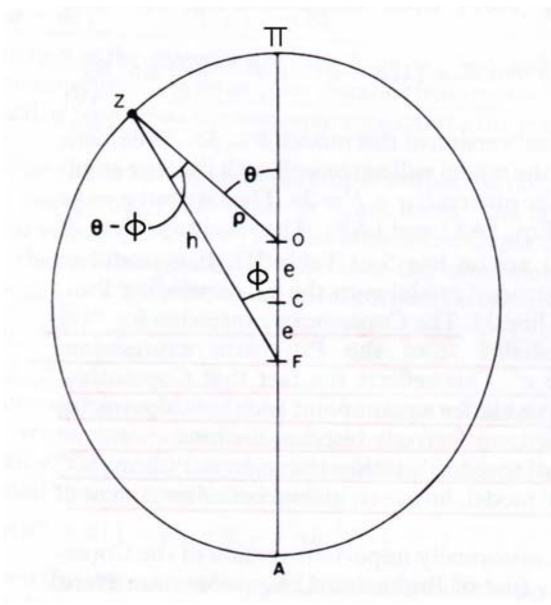


Figure 24

Referring to Figure 24, FZ makes an angle ϕ with the line of apsides AΠ; θ and ρ have their usual meanings. Take FZ = h. By properties of ellipses, FZ = AΠ - OZ. That is, $h = 2a - \rho$, or if the semimajor axis has been set to unity, $h = 2 - \rho$. In addition, $O\hat{Z}F = \theta - \phi$. We aim to find an expression for $\phi(t)$ to represent the motion of the planet as observed from the empty focus. Applying Sine Rule to triangle OZF, we have,

$$\frac{2\left(\frac{e}{2}\right)}{\sin(\theta - \phi)} = \frac{h}{\sin\left(\frac{\pi}{2} - \theta\right)} = \frac{h}{\sin\theta} \Rightarrow \sin(\theta - \phi) = \frac{2\left(\frac{e}{2}\right)\sin\theta}{h}$$

$$\Rightarrow \sin(\theta - \phi) = \frac{2\left(\frac{e}{2}\right)\sin\theta}{2 - \rho}$$

Upon substituting the Keplerian expressions for $\theta(t)$ and $\rho(t)$, which were found earlier in this section, and expanding, we yield the following:

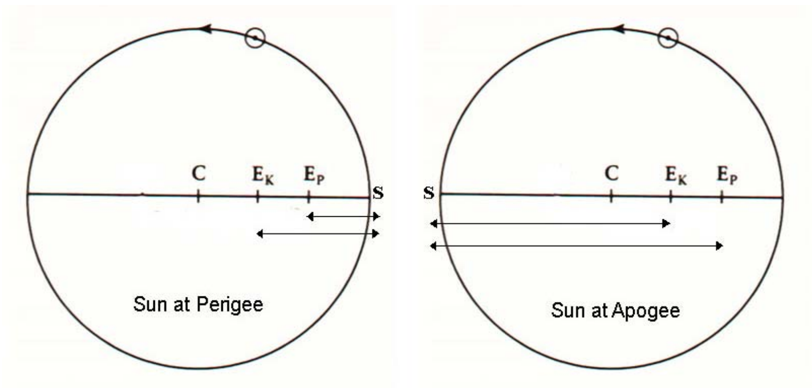
$$\phi(t) = wt + 0 + \frac{1}{4}\left(\frac{e}{2}\right)^2 \sin 2wt.$$

Therefore, if we only consider terms in e up to first order, $\phi(t)$ is equivalent to wt which in turn gives constant motion of the planet Z when observed at the empty focus F. This is why Kepler's empty focus is said to behave as Ptolemy's equant point, to order e.

2.4 Use of meridiana and Notes to Appendix C

There had been a great controversy between adherents of Ptolemy's traditional solar theory and proponents of Kepler's "bisection of the eccentricity" in his solar model. This motivated Cassini to find out which theory was the accurate one by using the meridiana at San Petronio; and he succeeded. The basis of and working steps to his conclusion are elaborated in this section.

As highlighted in Section 2.2, the modern value of the eccentricity is half of the ancient. Suppose in Ptolemy's model, the solar eccentricity is of value *e*, then the same quantity would be of value $\frac{e}{2}$ in Kepler's model.



(a) (b)

Figure 25

Referring to Figure 25(a), we see that at perigee, the distance between Sun and Earth is shorter on Ptolemy's model than that on Kepler's. From Figure 25(b), the opposite occurs: at apogee, the same distance is longer on Ptolemy's model than that on Kepler's. If the apsidal separation between Sun and Earth could be observed directly, then it would be possible to test out which of the two models is the correct one. Such a direct observation cannot be obtained but fortunately, a convenient substitute exists in the Sun's apparent diameter, which is inversely proportional to its separation from the Earth. Astronomers had found that the Sun's apparent diameter at mean distance ranges from $30'30''$ to $32'44''$. In addition, the difference between the Sun's diameters at the absides was found to be either $1'$ or $2'$. The table below lists the apogee distance, perigee distance and apsidal difference between Sun and Earth corresponding to Kepler's and Ptolemy's theories, or as Heilbron names them, equant theory, and pure eccentric or perspectival theory respectively.

Theory by	Apogee distance	Perigee distance	Absidal Difference
Kepler	$a\left(1+\frac{e}{2}\right)$	$a\left(1-\frac{e}{2}\right)$	ae
Ptolemy	a(1 + e)	a(1 – e)	2ae

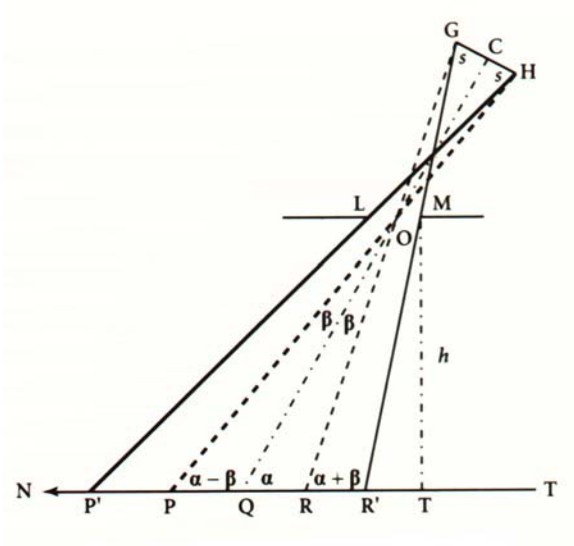


Figure 26

In Figure 26, GH is of length 2s, OC refers to the separation between the Sun and the Earth at the apsides. Let GO = HO = d. Since triangles GOC and HOC are right-angled, by Pythagoras' Theorem, d is equivalent to $\sqrt{s^2+OC^2}$. Finally, β refers to half the apparent apsidal diameter of the Sun. Since β is small, it may be approximated by sinβ. In the calculations that follow, the subscripts Π and A, when attached to a certain quantity, mark the same quantity as that measured at perigee and apogee respectively.

On the perspectival theory, $\beta_\Pi \approx \sin\beta_\Pi = \dfrac{s}{\sqrt{s^2+OC_\Pi{}^2}}$. Since $OC_\Pi = a(1-e)$ and s^2 is negligibly small, $\beta_\Pi = \dfrac{s}{\sqrt{s^2+a^2(1-e)^2}} \approx \dfrac{s}{a(1-e)}$. By similar derivation, and using $OC_A = a(1+e)$, we obtain $\beta_A = \dfrac{s}{a(1+e)}$.

The difference between the apsidal diameters is

$$\begin{aligned}2\beta_\Pi - 2\beta_A &= 2(\beta_\Pi - \beta_A)\\ &= 2\left(\frac{s}{a(1-e)} - \frac{s}{a(1+e)}\right)\\ &= \frac{2s}{a}\left(\frac{1}{1-e} - \frac{1}{1+e}\right),\end{aligned}$$

and considering up to terms in e only,

$$\begin{aligned}&\approx \frac{2s}{a}[(1+e)-(1-e)]\\ &= \frac{4se}{a} = 2e\sigma\end{aligned}$$

where $\sigma = \dfrac{2s}{a}$.

Similarly, for the equant theory, by using $OC_\Pi = a\left(1-\dfrac{e}{2}\right)$ and $OC_A = a\left(1+\dfrac{e}{2}\right)$, we get

$$\begin{aligned}2\beta_\Pi - 2\beta_A &= 2\left(\frac{s}{a\left(1-\frac{e}{2}\right)} - \frac{s}{a\left(1+\frac{e}{2}\right)}\right)\\ &= \frac{2s}{a}\left(\frac{1}{1-\frac{e}{2}} - \frac{1}{1+\frac{e}{2}}\right)\\ &\approx \frac{2s}{a}\left[\left(1+\frac{e}{2}\right)-\left(1-\frac{e}{2}\right)\right]\\ &= \frac{2se}{a} = e\sigma.\end{aligned}$$

According to Kepler, $e = 0.036$ whilst $\sigma = 30'$.

Hence, if the perspectival theory holds true for the Sun, the difference in the apparent diameters of the Sun at the apsides would be

$$\begin{aligned}2\beta_\Pi - 2\beta_A &= 2e\sigma\\ &= 2(0.036)(30')\\ &\approx 2',\end{aligned}$$

or if the equant theory holds true instead, the same quantity is

$$
\begin{aligned}
2\beta_{\Pi} - 2\beta_{A} &= e\sigma \\
&= (0.036)(30') \\
&\approx 1'.
\end{aligned}
$$

However, there was no consensus amongst the astronomers on which the correct value for the difference between the diameters at the apsides is. Cassini could not make any conclusion and thus turned towards using the Sun's image along the meridiana at San Petronio. This piece of instrument offered higher precision and Cassini was able to measure the quality of interest, $e\sigma$ by considering the length $e\sigma h$ in the diameter of the Sun's image along the meridiana, where h is the height of the gnomon.

Referring to Appendix C and Figure 26, α is the altitude of the Sun's centre and β is half the Sun's apparent size.

In triangle OPR, by Sine Rule, $\dfrac{PR}{\sin 2\beta} = \dfrac{OR}{\sin(\alpha-\beta)} \Rightarrow PR = \dfrac{OR\sin 2\beta}{\sin(\alpha-\beta)}$. Since $\sin(\alpha-\beta) = \dfrac{h}{OR}$, we have $OR = \dfrac{h}{\sin(\alpha+\beta)}$. Then,

$$PR = \frac{h\sin 2\beta}{\sin(\alpha+\beta)\sin(\alpha-\beta)}.$$

For 2β small, $\sin 2\beta \approx 2\beta$. Thus $PR = \dfrac{h(2\beta)}{\sin(\alpha)\sin(\alpha)} = \dfrac{h(2\beta)}{\sin^2\alpha}$. That is, the diameter PR of the central portion of the Sun's image along the meridian line in Figure 23 is $h\csc^2\alpha(2\beta)$.

Now, see Figure 27.

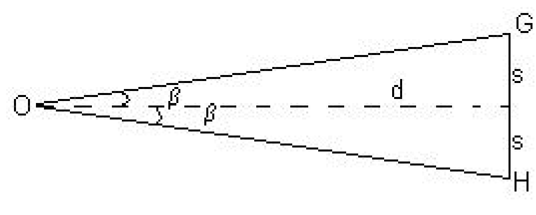


Figure 27

Due to the fact that $\sin\beta = \frac{s}{\sqrt{s^2+d^2}}$, and $\cos\beta = \frac{d}{\sqrt{s^2+d^2}}$, we obtain $\sin 2\beta = \frac{2sd}{s^2+d^2}$.

Taking 2β as a small angle and s small such that s^2 is negligibly small, the previous expression may be approximated by

$$2\beta = \frac{2sd}{d^2} = \frac{2s}{d} = \text{ sun's diameter/sun's distance.}$$

On the equant theory, $d = a\left(1 \pm \frac{e}{2}\right) \Rightarrow 2\beta = \frac{2s}{a\left(1 \pm \frac{e}{2}\right)} = \frac{\sigma}{1 \pm \frac{e}{2}}$.

On the perspectival theory, $d = a(1 \pm e) \Rightarrow 2\beta = \frac{2s}{a(1 \pm e)} = \frac{\sigma}{1 \pm e}$.

The corresponding expressions for length PR in the two models are thus, $h\csc^2\alpha\left(\frac{\sigma}{1 \pm \frac{e}{2}}\right)$ and $h\csc^2\alpha\left(\frac{\sigma}{1 \pm e}\right)$.

Given the values of α at winter solstice WS, and summer solstice SS are 22° and 69°, we may calculate the difference ΔI between the diameters of the image at WS and SS which occur close to the apsides.

We note that $\Delta I = PR_{WS} - PR_{SS}$.

Therefore, in the equant theory, $\Delta I_K = h\sigma\left(\frac{\csc^2 22^\circ}{1-\frac{e}{2}} - \frac{\csc^2 69^\circ}{1+\frac{e}{2}}\right)$.

Or, $$\Delta I_K \approx h\sigma\left[7\left(1+\frac{e}{2}\right) - \left(1-\frac{e}{2}\right)\right] = h\sigma(6+4e).$$

In the perspectival theory, $\Delta I_P = h\sigma\left(\frac{\csc^2 22^\circ}{1-e} - \frac{\csc^2 69^\circ}{1+e}\right)$

which gives $$\Delta I_P \approx h\sigma\left[7(1+e) - (1-e)\right] = h\sigma(6+6e).$$

Take $\sigma = 30'$, $e = 0.036$, and $h = 27.1$ m. This gives

$\sigma he = \left[(30)\frac{\pi}{(180)(60)}(27.1) \right] \approx 8.5$ mm. In order for a decisive confirmation of one theory over the other, Cassini had to measure σhe to within 8.5mm; otherwise, a finding of $\Delta I = \sigma h(6+6e)$ would have decided nothing.

Later on, Cassini wanted to find a way to make measurements not only when the Sun is at perigee or apogee with respect to the Earth. He found that the relationship between the change in apparent diameter, $\Delta\rho$, of the Sun in any time interval, Δt, and the change in the Sun's apparent position, $\Delta(\theta - M)$ in the same time interval, is twice as big with a whole eccentricity as compared to a bisected one. Here, θ is the Sun's angular distance from perigee, or the true anomaly; M is the displacement measured from the centre of the orbit C, or the mean anomaly. At time t, M is equivalent to wt or $\frac{2\pi}{T}.t$, where T is the period of the Sun's orbit.

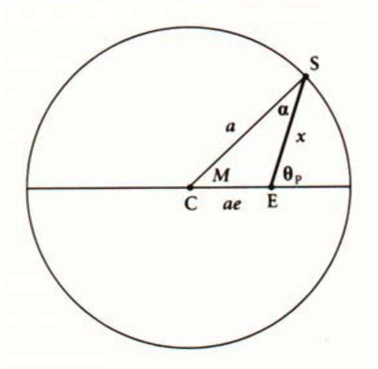


Figure 28

Figure 28 illustrates Cassini's method being applied on Ptolemy's model. Applying Sine Rule to triangle CSE, we have

$$\frac{ae}{\sin\alpha} = \frac{x}{\sin M} \Rightarrow \sin\alpha = \frac{ae\sin M}{x} \qquad (1)$$

In addition, by applying Cosine Rule to triangle CSE, we get

$$x^2 = a^2 + (ae)^2 - 2a^2 e\cos M = a^2(1 + e^2 - 2e\cos M).$$

Hence, $x = a\sqrt{1 + e^2 - 2e\cos M}$.

From the expression above, we obtain,

$$\frac{1}{x} = \frac{1}{a}(1+e^2-2e\cos M)^{-\frac{1}{2}}$$
$$\approx \frac{1}{a}(1+e^2+e\cos M)$$

and considering up to terms in e only,

$$= \frac{1}{a}(1-e\cos M). \qquad (2)$$

Therefore, by (1) and (2), we get

$$\sin\alpha \approx ae\sin M.\frac{1}{a}(1+e\cos M) = e\sin M(1+e\cos M).$$

Since α is small, $\sin\alpha=\alpha$. That is,

$$\alpha = e\sin M(1+e\cos M).$$

Due to the fact that $\theta_P = M+\alpha$, we have $\Delta\theta_P = \Delta M + \Delta\alpha$.

Thus, by differentiation and up to terms in e,

$$\Delta(\theta_P - M) = \Delta\alpha$$
$$= \Delta[e\sin M(1+e\cos M)]$$
$$= e\cos M.\Delta M$$

Recall that 2β = Sun's diameter / Sun's distance.

Therefore, the change in apparent diameter in Δt is

$$\Delta\rho_P = \frac{2s}{\Delta x}$$
$$= \frac{2sa}{a\Delta x}$$
$$= \frac{a\sigma}{\Delta x}$$
$$= a\sigma.\Delta\left(\frac{1}{x}\right)$$
$$= a\sigma.\Delta\left[\frac{1}{a}(1+e\cos M)\right]$$

$$= \frac{2sa}{a\Delta x}$$

$$= \frac{a\sigma}{\Delta x}$$

$$= a\sigma.\Delta\left(\frac{1}{x}\right)$$

$$= a\sigma.\Delta\left[\frac{1}{a}(1+e\cos M)\right]$$

$$= \sigma.-e\sin M.\Delta M \qquad \text{(by differentiation)}$$

$$= -e\sigma\sin M.\Delta M$$

Hence, in interval Δt, the ratio of the Δ in apparent diameter of sun to the Δ in inequality in sun's apparent position is

$$\frac{\Delta\rho_P}{\Delta(\theta_P - M)} = \frac{-e\sigma\sin M.\Delta M}{e\cos M.\Delta M} = -\sigma\tan M \qquad (3)$$

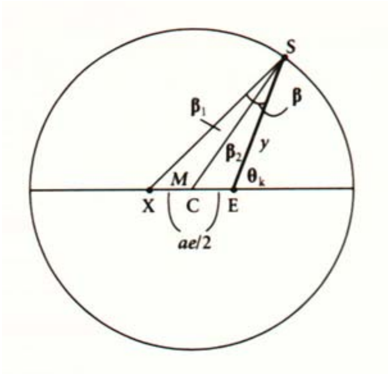


Figure 29

Figure 29 illustrates Cassini's method being applied to Kepler's model. Applying Sine Rule to triangle XSC, we have

$$\frac{\frac{ae}{2}}{\sin\beta_1} = \frac{a}{\sin M} \Rightarrow \sin\beta_1 = \frac{\frac{ae}{2}\sin M}{a} = \frac{e}{2}\sin M.$$

In addition, with the same rule applied to triangle ESC, we have

$$\frac{\frac{ae}{2}}{\sin\beta_2} = \frac{a}{\sin(\pi-\theta_K)} \Rightarrow \sin\beta_2 = \frac{\frac{ae}{2}\sin(\pi-\theta_K)}{a} = \frac{e}{2}\sin\theta_K = \frac{e}{2}\sin(M+\beta)$$

where $\theta_K = M + \beta$.

Consider β_1 and β_2 to be small angles then each of them may be approximated by $\sin\beta_1$ and $\sin\beta_2$ respectively.

Since $\beta = \beta_1 + \beta_2$,

$$\begin{aligned}\beta &\approx \sin\beta_1 + \sin\beta_2 \\ &= \frac{e}{2}\sin M + \frac{e}{2}\sin(M+\beta) \\ &= \frac{e}{2}\sin M + \frac{e}{2}(\sin M\cos\beta + \cos M\sin\beta) \\ &= \frac{e}{2}\sin M(1+\cos\beta) + \frac{e}{2}\cos M\sin\beta.\end{aligned}$$

β is small and thus, by small-angle approximations,

$$\begin{aligned}\beta &\approx \frac{e}{2}\sin M\left(1+1-\frac{\beta^2}{2}\right) + \frac{e}{2}(\cos M)\beta \\ &= e\sin M + \left(\frac{e}{2}\cos M\right)\beta\end{aligned}$$

$$\left(1-\frac{e}{2}\cos M\right)\beta = e\sin M$$

$$\begin{aligned}\beta &= \frac{e\sin M}{1-\frac{e}{2}\cos M} \\ &= e\sin M\left(1-\frac{e}{2}\cos M\right)^{-1} \\ &= e\sin M\left(1+\frac{e}{2}\cos M\right).\end{aligned}$$

By applying Sine Rule to triangle XES, we have $\frac{y}{\sin M} = \frac{ae}{\sin\beta}$.

Hence, when β is small,

$$\begin{aligned}\frac{1}{y} &= \frac{\sin\beta}{ae\sin M} \\ &\approx \frac{\beta}{ae\sin M} \\ &= \frac{e\sin M\left(1+\frac{e}{2}\cos M\right)}{ae\sin M} \\ &= \frac{1}{a}\left(1+\frac{e}{2}\cos M\right).\end{aligned}$$

Similar to earlier working, since $\theta_K = M + \beta$, we have $\Delta\theta_K = \Delta M + \Delta\beta$.

Thus,

$$\begin{aligned}\Delta(\theta_K - M) &= \Delta\beta \\ &= \Delta\left[e\sin M\left(1+\frac{e}{2}\cos M\right)\right] \\ &= e\cos M.\Delta M\end{aligned}$$

by differentiation up to terms in *e*.

Therefore, the change in apparent diameter in Δt is

$$\begin{aligned}\Delta\rho_K &= \frac{2s}{\Delta y} \\ &= \frac{2sa}{a\Delta y} \\ &= \frac{a\sigma}{\Delta y} \\ &= a\sigma.\Delta\left(\frac{1}{y}\right) \\ &= a\sigma.\Delta\left[\frac{1}{a}\left(1+\frac{e}{2}\cos M\right)\right] \\ &= \sigma.-\frac{e}{2}\sin M.\Delta M \qquad \text{(by differentiation)} \\ &= -\frac{e}{2}\sigma\sin M.\Delta M\end{aligned}$$

Hence in interval Δt, the ratio of the Δ in apparent diameter of sun to the Δ in inequality in sun's apparent position is

$$\frac{\Delta\rho_K}{\Delta(\theta_K - M)} = \frac{-\frac{e}{2}\sigma\sin M.\Delta M}{e\cos M.\Delta M} = -\frac{\sigma}{2}\tan M. \qquad (4)$$

Comparing (3) and (4), we observe that the latter is half the result of the former. Using the meridiana, Cassini made measurements that confirmed that the relationship between $\Delta(\theta - M)$ and $\Delta\rho$ was indeed that expected for the theory of the bisected eccentricity. Thus, Kepler's theory is confirmed as the correct model.

2.5 Bisection of eccentricity

The notion of "bisection of eccentricity" has been briefly mentioned in Sections 1.5 and 2.3. Here, we provide a more detailed discussion of this concept and aim towards clarifying the double meaning it holds. Most of the content presented in this section is not new but having the information on bisection of eccentricity grouped in one place should boost our understanding of it significantly.

We recall that on Ptolemy's planetary model, he had added a third device called the "equant point" to the intermediate deferent-epicycle model such that the width of the retrograde loops could fit exactly onto it. The angular velocity of a body on orbit, when observed from the equant point, is constant. Figure 14 is repeated below for easy reference.

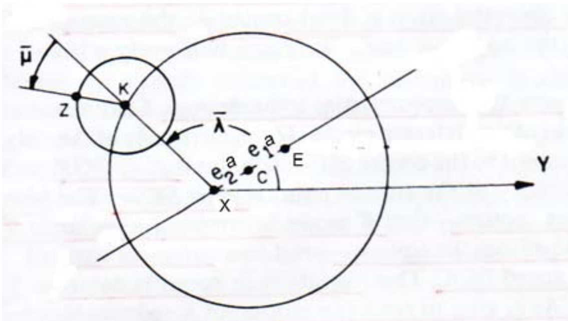


Figure 14

As a result of the addition, two eccentricities were defined. Taking the radius of the deferent to be a, e_1 represented the eccentricity of the Earth E with respect to centre C and e_2 represented the eccentricity of the equant point X with respect to C. That is,

$$e_1 = EC / a,$$
$$e_2 = XC / a.$$

The total eccentricity refers to the sum of e_1 and e_2. In this case, by "bisection of eccentricity", it refers to the situation where Ptolemy places the equant point on the mirror image of E along the line of apsides such that $e_1 = e_2$ is obtained.

The other meaning of "bisection of eccentricity" arises when we turn towards comparing Kepler's eccentricity value with Ptolemy's (or Hipparchu's) value. The former is half the value of the latter because of the difference in the way Kepler and Ptolemy had measured the separation between Sun and Earth on their respective solar models. A similar explanation to that in Section 1.5, together with figure 19, is given below.

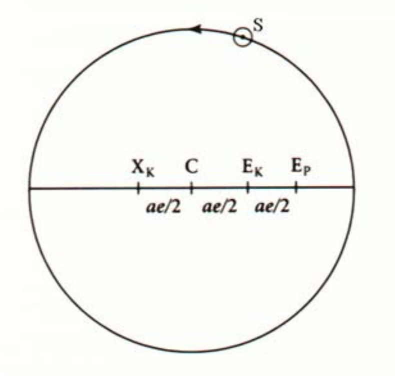


Figure 19

C represents the centre of the sun's orbit, E_P denotes where Ptolemy positions the earth, E_K denotes where Kepler positions the earth and X_K marks Kepler's equant point. Consider radius of the sun's orbit to be *a*, then the intervals between X_K, C, E_K and E_P would each be $\frac{ae}{2}$.

Both Ptolemy and Kepler had calculated the solar eccentricity as the ratio of the separation between Earth and centre of Sun's orbit to the orbital radius of the Sun. However, with reference to Figure 19, we see that Kepler would have measured that particular separation as $CE_K = \frac{ae}{2}$ whereas Ptolemy would have measured it as $CE_P = \frac{ae}{2} + \frac{ae}{2} = ae.$ The resulting value of eccentricity found by Kepler is thus equivalent to the bisected value of Ptolemy's eccentricity.

In summary, we see that "bisection of eccentricity" either refers to Ptolemy splitting the total eccentricity exactly into two on his planetary theory, or refers to Kepler dividing the solar eccentricity as defined in Ptolemy's (or Hipparchus') solar theory. In Heilbron's book, the discussion has focused on the second definition.

2.6 Notes to Heilbron Pages 114 to 117 and Appendices D and E

Having discovered the model of planets traveling on elliptical orbits, Kepler faced the problem of finding a simple geometrical method of deducing the elements of such orbits, which include the eccentricity e and the direction of the line of apsides ψ, from observations. Instead, he could only obtain these values by trial and error. Once they have been found, Kepler wanted to determine geometrically from his area law the position of planets. This is the focus of discussion on pages 114 and 115 of Heilbron's book which shall be elaborated on here, with reference to W.M.Smart's "Textbook in Spherical Astronomy".

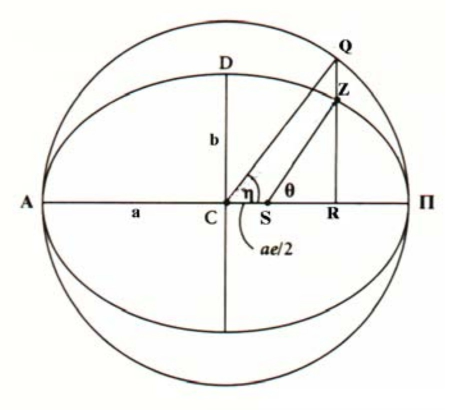


Figure 30

Let the radius SZ make the angle θ with SΠ; θ is called the true anomaly. Let a circle be described on the major axis AΠ as diameter; its radius is thus a. Let RZ, the perpendicular from Z to AΠ, be produced to meet this circle at Q. Then angle QCΠ is called the eccentric anomaly, denoted by η. By properties of ellipses,

$$RZ : RQ = b : a \tag{1}$$

where a is the semi-major axis CΠ.

Since $RZ = r\sin\theta$ and $RQ = CQ\sin\eta = a\sin\eta$, then by (1),

$$\frac{r\sin\theta}{a\sin\eta} = \frac{b}{a} \Rightarrow r\sin\theta = b\sin\eta. \tag{2}$$

Since $SR = r\cos\theta$ and also, $SR = CR - CS = a\cos\eta - \frac{ae}{2}$, therefore

$$r\cos\theta = a\cos\eta - \frac{ae}{2} = a\left(\cos\eta - \frac{e}{2}\right). \tag{3}$$

$(2)^2 + (3)^2$ gives

$$r^2 = b^2 \sin^2 \eta + a^2 \left(\cos\eta - \frac{e}{2} \right)^2 .$$

Using the relation $b^2 = a^2 \left[1 - \left(\frac{e}{2} \right)^2 \right]$, and after a little reduction, we obtain

$$r = a\left[1 - \frac{e}{2}\cos\eta \right]. \quad (4)$$

Since $r\cos\theta = r\left(1 - 2\sin^2 \frac{\theta}{2} \right)$, we have $2r\sin^2 \frac{\theta}{2} = r - r\cos\theta$. Upon substituting terms on the right hand side of the equation with (3) and (4) and manipulation, we obtain

$$2r\sin^2 \frac{\theta}{2} = a\left(1 + \frac{e}{2} \right)(1 - \cos\eta). \quad (5)$$

Similarly,

$$2r\cos^2 \frac{\theta}{2} = a\left(1 - \frac{e}{2} \right)(1 + \cos\eta). \quad (6)$$

Divide (5) by (6) and taking the square root, we get

$$\tan\frac{\theta}{2} = \left(\frac{1 + \frac{e}{2}}{1 - \frac{e}{2}} \right)^{\frac{1}{2}} \tan\frac{\eta}{2}. \quad (7)$$

Equations (4) and (7) therefore express the radius vector r and the true anomaly θ in terms of the eccentric anomaly η. To obtain η, we turn towards solving Kepler's Equation.

Recall that w denotes mean angular velocity and the product $w(t - \tau)$ represents the angle described in an interval $(t - \tau)$ by a radius vector rotating about S with constant angular velocity w. We define the mean anomaly, denoted by M, such that

$$M = w(t - \tau).$$

In Figure 30, the area SZΠ is thus given by

$$\text{Area SZ}\Pi = \frac{1}{2} abM. \quad (8)$$

To express the area in terms of the eccentric anomaly η, we consider area SZΠ as the sum of area ZSR and area RZΠ. Take first the area of triangle ZSR. Its area is $\frac{1}{2}SR.RZ$.

Since $SR = CR - CS = a\cos\eta - \frac{ae}{2}$ and $RZ = \frac{b}{a}RQ = \frac{b}{a}\left(a\sin\eta\right) = b\sin\eta$, area ZSR = $\frac{1}{2}ab\sin\eta(\cos\eta - \frac{e}{2})$. Next consider area RZΠ. By application of Cavalieri's Principle to ellipses, we know that area RZΠ is equal to $\frac{b}{a}$ times of area QRΠ. But area QRΠ is area of sector CQΠ minus the area of triangle QCR, where angle QCR is η, area CQΠ is $\frac{1}{2}a^2\eta$ and area QCR is $\frac{1}{2}.a\cos\eta.a\sin\eta$ or $\frac{1}{2}a^2\sin\eta\cos\eta$. Hence,

$$\text{Area RZ}\Pi = \frac{b}{a}\left(\frac{1}{2}a^2\eta - \frac{1}{2}a^2\sin\eta\cos\eta\right) = \frac{1}{2}ab(\eta - \sin\eta\cos\eta).$$

Then adding areas ZSR and RZΠ gives

$$\text{Area SZ}\Pi = \frac{1}{2}ab(\eta - \sin\eta). \tag{9}$$

By equations (8) and (9), we then have

$$\eta - \frac{e}{2}\sin\eta = M \equiv w(t - \tau).$$

This is Kepler's Equation which relates the eccentric anomaly η and the mean anomaly *M*. If *M* and *e* are known, it is then possible to determine the corresponding value of η. Thereafter, it is possible to find the value of the true anomaly θ by substituting the values of the eccentricity *e* and found value of eccentric anomaly η into equation (7); this then allows us to determine the position of planet Z at time *t*.

Nonetheless, just as indicated in Heilbron's book, even though the Kepler's Equation is simple to write down, it can be solved only by guesswork and successive approximations. I shall not attempt to demonstrate how this is done exactly but a good reference for the detailed steps is found in Smart's book, pages 117 – 119.

Seth Ward, professor of geometry at Oxford, had thought he found a geometrical method that could solve Kepler's problem simply and accurately.

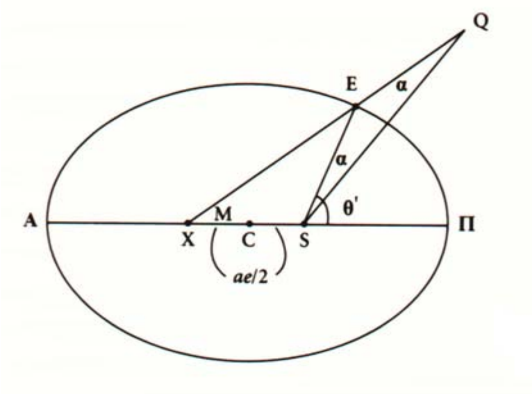


Figure 31

With reference to Figure 31, AΠ is the line of apsides, E and S are Earth and Sun respectively, X marks the equant point in the unoccupied focus of the ellipse, and XS = *ae*. Let the mean anomaly be *M* whilst the true anomaly be θ'. By extending XE to Q such that EQ = SE, and by a basic property of ellipses, we have XQ = AΠ = 2*a*. In triangle XSQ, by Sine Rule,

$$\frac{ae}{\sin\alpha} = \frac{XQ}{\sin(\pi - M - \alpha)} = \frac{2a}{\sin[\pi - (M + \alpha)]}$$

$$\frac{e}{\sin\alpha} = \frac{2}{\sin(M + \alpha)}$$

$$\sin\alpha = \frac{e}{2}\sin(M + \alpha).$$

Since $\alpha = \frac{\theta' - M}{2}$, we have

$$\sin\left(\frac{\theta' - M}{2}\right) = \frac{e}{2}\sin\left(M + \frac{\theta' - M}{2}\right) = \frac{e}{2}\sin\left(\frac{\theta' + M}{2}\right).$$

Then,
$$\frac{e}{2} = \frac{\sin\left(\frac{\theta' - M}{2}\right)}{\sin\left(\frac{\theta' + M}{2}\right)}.$$

Using the above expression, the following can be deduced,

$$\frac{1+\frac{e}{2}}{1-\frac{e}{2}}=\frac{1+\frac{\sin\left(\frac{\theta'-M}{2}\right)}{\sin\left(\frac{\theta'+M}{2}\right)}}{1-\frac{\sin\left(\frac{\theta'-M}{2}\right)}{\sin\left(\frac{\theta'+M}{2}\right)}}=\frac{\sin\left(\frac{\theta'+M}{2}\right)+\sin\left(\frac{\theta'-M}{2}\right)}{\sin\left(\frac{\theta'+M}{2}\right)-\sin\left(\frac{\theta'-M}{2}\right)}$$

$$=\frac{2\sin\frac{\theta'}{2}\cos\frac{M}{2}}{2\cos\frac{\theta'}{2}\sin\frac{M}{2}}.$$

$$=\frac{\tan\frac{\theta'}{2}}{\tan\frac{M}{2}}$$

This equation relates the true anomaly to the apsidal distances and mean anomaly. Hence, Ward thought he had managed to devise a geometrical method that is simple to apply and would give the value of the true anomaly directly. Unfortunately, Ward was wrong; his method did not give the exact value of the true anomaly. The following gives an explanation of why this occurred, with reference to Appendix E.

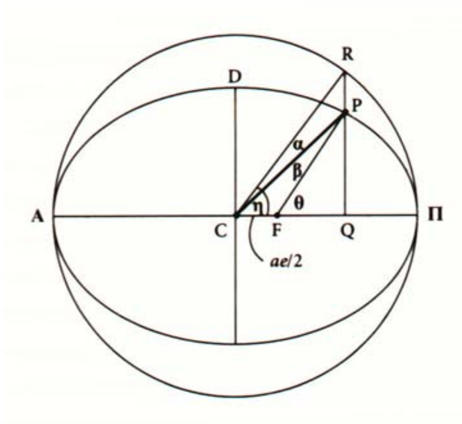


Figure 32

According to Figure 32, the true anomaly is $\theta = \eta - \alpha + \beta$.

In triangle CPR,

$$\frac{RP}{\sin\alpha}=\frac{a}{\sin C\hat{P}R}\Rightarrow \sin\alpha=\frac{RP}{a}\sin C\hat{P}R. \qquad (1)$$

By Cavalieri's Principle, $PQ=\frac{b}{a}RQ.$

However in triangle CRQ, $\sin\eta = \dfrac{RQ}{a} \Rightarrow RQ = a\sin\eta$.

Then $RP = RQ - PQ$

$$
\begin{aligned}
&= a\sin\eta - b\sin\eta \\
&= (a-b)\sin\eta \qquad (2)
\end{aligned}
$$

Since $C\hat{R}Q = 90^\circ - \eta$, $C\hat{P}R = 180^\circ - \alpha - C\hat{R}Q$

$$
\begin{aligned}
&= 180^\circ - \alpha - (90^\circ - \eta) \\
&= 90^\circ - (\alpha - \eta) \qquad (3)
\end{aligned}
$$

Substitute (2) and (3) in (1), we get

$$
\begin{aligned}
\sin\alpha &= \frac{a-b}{a}\sin\eta\sin[90^\circ - (\alpha - \eta)] \\
&= \frac{a-b}{a}\sin\eta\cos(\alpha - \eta) \\
&= \left(1 - \frac{b}{a}\right)\sin\eta\cos(\eta - \alpha) \\
&= \left(1 - \frac{b}{a}\right)\sin\eta\cos\eta, \qquad \text{where } \alpha \text{ is small.}
\end{aligned}
$$

By properties of ellipses, $b^2 = a^2\left[1 - \left(\dfrac{e}{2}\right)^2\right]$

$$\frac{b^2}{a^2} = 1 - \frac{e^2}{4}$$

$$\frac{b}{a} = \left(1 - \frac{e^2}{4}\right)^{\frac{1}{2}} \approx 1 - \frac{e^2}{8}$$

Hence, $\sin\alpha \approx \dfrac{e^2}{8}\sin\eta\cos\eta$. In addition, α is small. Thus, $\alpha \approx \dfrac{e^2}{8}\sin\eta\cos\eta$.

Similarly, in triangle CPF, by Sine Rule,

$$\frac{\frac{ae}{2}}{\sin\beta} = \frac{PF}{\sin(\eta - \alpha)} \Rightarrow \sin\beta = \frac{ae}{2PF}\sin(\eta - \alpha) = \frac{\frac{e}{2}\sin\eta}{1 - \frac{e}{2}\cos\eta}.$$

β is small and thus by approximation, $\beta = \dfrac{\frac{e}{2}\sin\eta}{1 - \frac{e}{2}\cos\eta}$.

Recall that $\theta = \eta - \alpha + \beta$.

Hence, $\theta = M + \frac{e}{2}\sin\eta - \frac{e^2}{8}\sin\eta\cos\eta + \frac{\frac{e}{2}\sin\eta}{1-\frac{e}{2}\cos\eta}$

$$\approx M + \frac{e}{2}\sin\eta - \frac{e^2}{8}\sin\eta\cos\eta + \frac{e}{2}\sin\eta\left(1+\frac{e}{2}\cos\eta\right)$$

To express $\sin\eta$ in terms of e and M, we have

$$\sin\eta = \sin\left(M + \frac{e}{2}\sin\eta\right)$$
$$= \sin\left[M + \frac{e}{2}\sin\left(M + \frac{e}{2}\sin\eta\right)\right]$$
$$= \sin\left[M + \frac{e}{2}\left(\sin M\cos\left(\frac{e}{2}\sin\eta\right) + \cos M\left(\frac{e}{2}\sin\eta\right)\right)\right]$$
$$\approx \sin\left[M + \frac{e}{2}\left(\sin M\left(1-\frac{e^2}{4}\sin\eta\right) + \cos M . \frac{e}{2}\sin\eta\right)\right]$$
$$= \sin\left(M + \frac{e}{2}\sin M\right) \qquad \text{(up to first order terms in e)}$$
$$= \sin M\cos\left(\frac{e}{2}\sin M\right) + \cos M\sin\left(\frac{e}{2}\sin M\right)$$
$$\approx \sin M\left(1-\frac{e^2}{4}\sin^2 M\right) + \cos M . \frac{e}{2}\sin M$$
$$= \sin M + \frac{e}{2}\sin M\cos M.$$

To express $\cos\eta$ in terms of e and M, we have

$$\cos\eta = \cos\left(M + \frac{e}{2}\sin\eta\right)$$
$$= \cos\left[M + \frac{e}{2}\left(\sin M + \frac{e}{2}\sin M\cos M\right)\right]$$
$$= \cos M\cos\left[\frac{e}{2}\left(\sin M + \frac{e}{2}\sin M\cos M\right)\right] - \sin M\sin\left[\frac{e}{2}\left(\sin M + \frac{e}{2}\sin M\cos M\right)\right]$$
$$\approx \cos M - \sin M . \frac{e}{2}\sin M \qquad \text{(up to first order terms in e)}$$
$$= \cos M - \frac{e}{2}\sin^2 M.$$

Thus,

$$\theta = M + \frac{e}{2}\left(\sin M + \frac{e}{2}\sin M \cos M \right) - \frac{e^2}{8}\left(\sin M + \frac{e}{2}\sin M \cos M \right)\left(\cos M - \frac{e}{2}\sin^2 M \right)$$

$$= \frac{e}{2}\left(\sin M + \frac{e}{2}\sin M \cos M \right)\left(1 + \frac{e}{2}\left(\cos M - \frac{e}{2}\sin^2 M \right)\right)$$

$$= M + \left(\frac{e}{2}\sin M + \frac{e^2}{8}\sin 2M \right) - \frac{e^2}{16}\sin 2M + \frac{e}{2}\sin M + \frac{e^2}{4}\sin 2M$$

$$= M + e\sin M + \frac{5}{16}e^2 \sin 2M.$$

Referring to Figure 27, in triangle XSQ, by Sine Rule,

$$\frac{ae}{\sin\alpha} = \frac{XQ}{\sin(180^\circ - M - \alpha)}$$

$$\frac{ae}{\sin\alpha} = \frac{2a}{\sin(180^\circ - (M + \alpha))}$$

$$\frac{e}{\sin\alpha} = \frac{2}{\sin(M + \alpha)}$$

$$\sin\alpha = \frac{e}{2}\sin(M + \alpha)$$

$$\sin\alpha = \frac{e}{2}\sin M \cos\alpha + \frac{e}{2}\cos M \sin\alpha$$

$$\tan\alpha = \frac{e}{2}\sin M + \frac{e}{2}\cos M \tan\alpha.$$

$$\tan\alpha\left(1 - \frac{e}{2}\cos M \right) = \frac{e}{2}\sin M$$

$$\therefore \tan\alpha = \frac{\frac{e}{2}\sin M}{1 - \frac{e}{2}\cos M}.$$

Consider α a small angle. Then by small angle approximations, $\tan\alpha \approx \alpha$.

Since $\theta' = M + 2\alpha$, we have

$$
\begin{aligned}
\theta' &\approx M + \frac{e \sin M}{1 - \frac{e}{2}\cos M} \\
&\approx M + e \sin M \left(1 + \frac{e}{2}\cos M\right) \\
&= M + e \sin M + \frac{e^2}{4}\sin 2M
\end{aligned}
$$

Thus, $\theta - \theta' = \frac{1}{16}e^2 \sin 2M$. In other words, there was a discrepancy of a maximum $\frac{1}{16}e^2 \sin 2M$ between Ward's found value for "true anomaly" and the actual true anomaly.

References:

1. Evans, J. *On the function and the probable origin of Ptolemy's equant*. Am. J. Phys. **52** (12), 1080 (1984)
2. Evans, J. *The division of the Martian eccentricity from Hipparchos to Kepler: A history of the approximations to Kepler motion*. Am. J. Phys. **56** (11), 1009 (1988)
3. Hahn, Alexander J. *Basic Calculus: from Archimedes to Newton to its role in science*. New York: Springer-Verlag, 1998.
4. Heilbron, J.L. *The Sun in the Church: Cathedrals as Solar Observatories*. Cambridge: Harvard University Press, 1999.
5. Hoskin, Michael. *The Cambridge Illustrated History of Astronomy*. Cambridge: Cambridge University Press, 1997.
6. Neugebauer, O. *The Exact Sciences in Antiquity*. Princeton: Princeton University Press, 1952.
7. Smart, W.M. *Textbook on Spherical Astronomy* (4^{th} Edition). Cambridge: Cambridge University Press, 1960.
8. O'Connor, J.J. and Robertson, E.F., *The MacTutor History of Mathematics Archive*, http://www-groups.dcs.st-andrews.ac.uk/~history/Mathematicians/, University of St Andrews Scotland, Mar 2001.