Bisection Of The Eccentricity

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**An academic exercise presented in partial fulfillment
for the degree of Bachelor of Science with Honours in Mathematics.**

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I would also like to express my sincere thanks to all my dear friends, especially Shumei, Yuan Qin, Lay Ling and Lydia for all their supports and encouragements.

It was a tough journey to travel. I am grateful that I did not have to travel the journey alone. To my beloved family members, especially my granny, I would like to say thank you very much!

SUMMARY

The phenomena of the sky aroused people's interest a long time ago. The Cro Magnon people made bone engravings 30,000 years ago, which may depict the phases of the Moon. These calendars are the oldest astronomical documents, 25,000 years older than writing.

Agriculture required a good knowledge of the seasons. Religious rituals and prognostication were based on the locations of the celestial bodies. Thus, time reckoning became more important and accurate as people learned to calculate the movements of celestial bodies in advance.

During the rapid development of seafaring, when voyages extended farther and farther from homeports, position determination presented a problem for which astronomy offered a practical solution. Solving these problems of navigation were the most important tasks of astronomy in the 17^{th} and 18^{th} centuries, when the first precise tables on the movements of the planets and on other celestial phenomena were published. The basis for these developments was the discovery of the laws governing the motions of the planets by Ptolemy, Copernicus, Tycho Brahe, Kepler, Galieo and Newton.

Astronomical research has changed man's view of the world from geocentric, anthropocentric conceptions to the modern view of a vast universe where the man and the Earth play an insignificant role. Astronomy has taught us the real scale of the nature surrounding us.

Thus, the first chapter of this paper aims to provide an introduction to certain aspects of astronomy. This includes the Greeks' conception of the Universe (the geocentric model), the meridian line, the heliocentric model and the models created by Ptolemy and Kepler to explain the motions of the Sun and the planets.

The book entitled: “ The Sun in the Church” written by J.L. Heilbron, addresses a basic problem, that is, how is the calendar date fixed? Since the time period of the Earth’s orbit around the Sun is not neatly divisible into a whole number of days, it is difficult to construct a calendar that will mark a moment in time back at the exact same point after the Earth makes a complete orbit around the Sun.

Heilbron gives a comprehensive overview of the various approaches that have been proposed and implemented by the Catholic Church to fix the calendar. The Church was committed to improving the quality of observational data on which calendars were based because they wanted to have a systematic way to determine when Easter should be celebrated. Consequently, many huge cathedrals were used to serve as solar observatories. Simply by making a hole in the ceiling and fixing a mark on the floor where the Sun’s shadow fell along the meridian line laid out on the cathedral floor, precise measurements could be made regarding the position of the Sun.

“The Sun in the Church” includes many historical and technical facts on how the cathedrals came to be used to serve as solar observatories. No doubt, most readers would find the historical accounts interesting and enriching, but they might find it hard to appreciate the technical sections of this book. This is so because these sections require the readers to have certain level of knowledge in mathematics.

Thus, chapter two of my paper has been written to provide a mathematical supplement to selected sections of the book entitled: “The Sun in the Church”, as well as to clarify the concept of “Bisection of Eccentricity”.

Lastly, an appendix of mathematical tools that will be used frequently in some of the calculations can be found in the last section of this paper.

Author's Contributions

My contributions to this project are as follows:

1) With the help of my supervisor, A/P Aslaksen, I have managed to identify that the diagram in Appendix B of Heilbron's book and Yoke Leng's project are incorrect. I have corrected the diagram and derived the mathematical steps involved in the calculations. In addition, I have explained the concepts in greater details.
2) Both Heilbron's argument on Page 105 of his book and Yoke Leng's explanation in section 2.2 of her project contained errors that I have corrected.
3) I have filled in the details in the arguments in the paper entitled: "The division of the Martian eccentricity from Hipparchos to Kepler: A history of the approximations to Kepler motion" written by James Evans.

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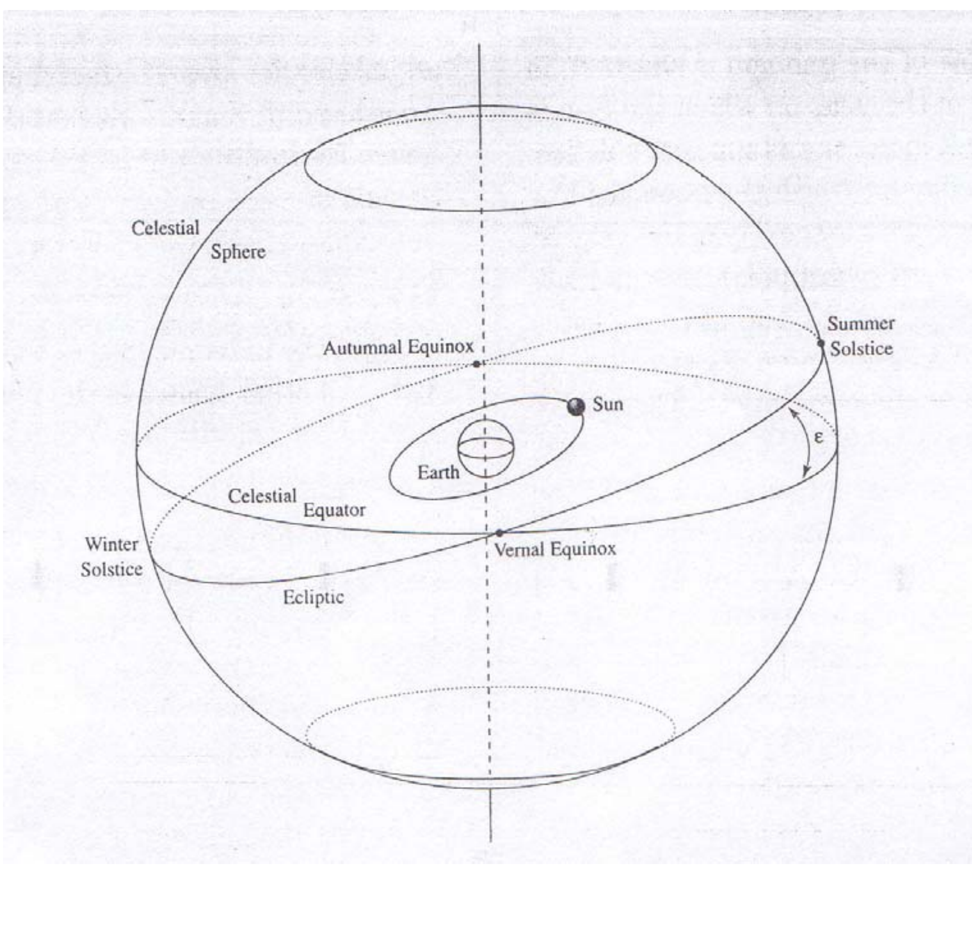
CHAPTER ONE

1.1 GREEKS' CONCEPTION OF THE UNIVERSE (GEOCENTRIC MODEL)

The Greek philosophers perceived the basis structure of the universe to be like this: The Earth is spherical in shape, it is fixed, and it lies at the center of the universe. All other planets are surrounded by the celestial sphere of fixed stars, which rotates about its axis once a day. Besides being carried along by the daily rotation of the sphere of fixed stars, the planets also have their own independent motion.

The celestial equator is the projection of the Earth's equator against the celestial sphere. The ecliptic is the projection of the path of the Sun against the celestial sphere. The two points where the ecliptic and celestial equator intersect are the vernal and autumnal equinox points. The point on the ecliptic where the sun reaches its highest position above the plane of the celestial equator is the summer solstice position. The point of lowest position below this plane is the winter solstice position.



Imagine if we can cut the celestial sphere in two, the cutting edges will be circles. If we cut near the surface we will get a small circle, but if we cut through the center of the sphere, we will get a circle of maximal length. Such a circle is called a great circle. This great circle is called the horizon. A circle on a sphere that is not a great circle is called a small circle. The distance of a heavenly body from the horizon is called its altitude.

When describing terrestrial position, we use latitude and longitude. Latitude is measured from 0 to 90 degrees north or south of the equator. Hence, places at the equator will have a latitude of 0 degree. The semi-great circles that run between the poles are called meridians. The meridian that runs through the Royal Greenwich Observatory outside London is called the prime meridian. The longitude of a place is determined by measuring the angular separation along the equator between the prime meridian and the meridian running through the point. Longitude is measured 180 degrees east or west of the Greenwich meridian.

The term co-latitude of a place refers to 90 degree minus the latitude of the place.

There are several ways to describe position on the celestial sphere. Celestial longitude is the distance in degrees, reckoned from the vernal equinox, on the ecliptic, to a circle at right angles to the ecliptic passing through the heavenly body. On the other hand, declination is the angular distance of any heavenly body from the celestial equator, either northward or southward.

1.2 THE MERIDIAN LINE

Meridians are semi-great circles running along the earth's surface from the North to the South Pole. Our own meridian is nothing more than a north-south line running through the particular spot where we are situated.

Meridian line can be found by using the "method of equal altitudes." This involves marking the direction of the sun at some time in the morning, and marking it again in the afternoon when the sun has the same altitude. For instance, we can drive a nail perpendicularly into a board as at A in figure1, and carefully level the board on a flat horizontal ground .We mark the position of the tip of the nail's shadow at B in the morning, and again at C when the sun has the same altitude again that afternoon.

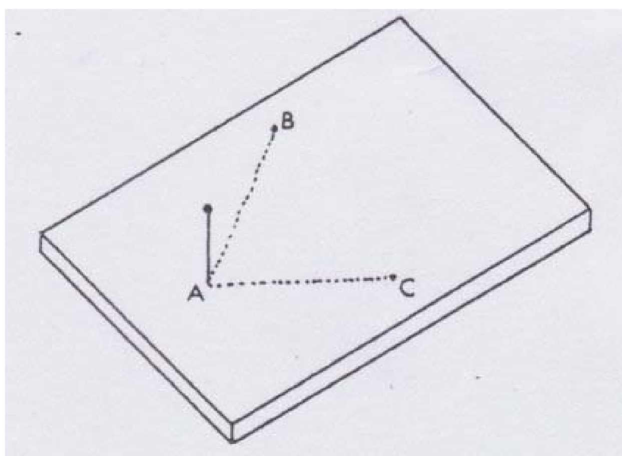


Figure 1

To find out when the altitudes are equal, we can draw a circular arc with a radius of AB and centered at A, the foot of the nail, as shown in figure 2, and when the tip of the shadow crosses the same arc, as at C, we mark the point again. Then the lines AB and AC are drawn, and the angle BAC is bisected by the line AN, which is the meridian line.

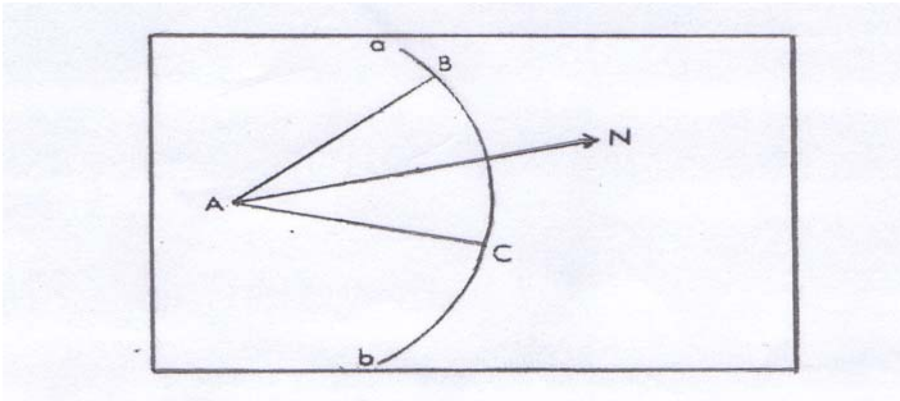


Figure 2

It has been observed that when the shadow of the Sun has the shortest length on the meridian, it is at its highest position in the sky. This moment is noon of the day. Accordingly, the summer solstice is the day on which the noon shadow is the shortest and winter solstice is the day on which the noon shadow is the longest. This implies that summer solstice is the day of longest daylight and winter solstice is the day of shortest daylight. There are two days on which day and night have the same lengths. The one occurring before summer solstice is the vernal equinox and the other occurring after is the autumnal equinox. Accordingly, the seasons of year are as follows: Spring is the period from vernal equinox to summer solstice; summer runs from summer solstice to autumnal equinox; autumn from autumnal equinox to winter solstice, lastly winter from winter solstice to vernal equinox. The above mentioned corresponds to only places situated at the northern hemisphere and does not apply to places at the southern hemisphere and the tropics.

The astronomer-mathematician Hipparchus lived and worked from about 190-125 B.C. in Nicaea provided the following result from his shadow measurements:

Spring 94.5 days

Summer 92.5 days

Autumn 88.125days

Winter 90.125days

With reference to the diagram in page 4, observe that the vernal equinox, summer solstice, autumnal equinox, and winter solstice positions divide the ecliptic into four equal circular arcs. The fact that the lengths of the seasons are different implies that the sun will trace out these four arcs in different lengths of time. This however, stands in clear contradiction to the two fundamental principles of Greek philosophy:

1) The Earth is fixed and positioned at the center of the universe (geocentric hypothesis)
2) All other planets in the universe move with constant velocity in the circular paths.

To account for this phenomenon, the ancient astronomers would rather regard the inequalities in the speed of the sun as an optical illusion than to rule that the sun does not move at uniform angular speed. Thus, Claudius Ptolemy (about 100-165 A.D.), one of the most influential Greek astronomers of his time, came up with a new theory to account for this phenomenon. Ptolemy developed a theory that has the sun moving with constant velocity in a circular path around the Earth and at the same time explained the observed differences in the lengths of the seasons.

1.3 PTOLEMY'S SOLAR MODEL

In Ptolemy's solar model, the center of the Sun's orbit S (figure 3.11) no longer coincides with the center of the ecliptic E. He let their separation be ae, where a is the radius of the orbit and e is the eccentricity. The eccentricity, e, is defined as the ratio of the distance between E and S to radius a, that is,

$$e = ES/a,$$

which implies, ES= ae.

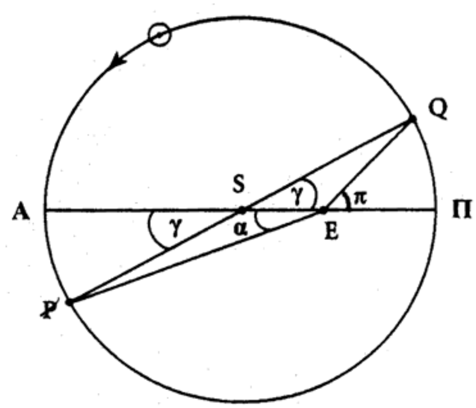


FIG. 3.11. Apparent speed of the sun around perigee and apogee (the apsides).

The diameter of the sun's orbit that runs through the center of the ecliptic (or of the earth) is called the "line of apsides." The point Π on this diameter closest to the earth is the "perigee" or perihelion and that furthest away; A is called the "apogee" or aphelion (figure 3.11).

With Ptolemy's solar model, if the sun, S, moves uniformly about its orbit, it will appear from the earth, E, to slow down around the apogee and speed up around the perigee.

With reference to figure 3.11, suppose now that the sun go from A to P through an angle $\gamma = \omega t$ in time t as seen from the center of the sun's orbit S; it will appear to move through the smaller angle $\alpha = \angle AEP$ as seen from the earth. To prove this, we first notice that γ is an external angle of triangle SEP, thus it will be equal to the sum of the opposite internal angles. In this case, $\angle\gamma = \angle\alpha + \angle SPE$. This clearly shows us that $\gamma > \alpha$. This implies that in time t, the sun will have moved in the ecliptic over a smaller angle than ωt. That is precisely the behaviour it displays around the summer solstice.

Similarly, at perigee if the sun ran through the same angle$\gamma = \omega t$ as seen from the center of its orbit, it will move through an $\angle\pi$ in the ecliptic. As shown in the previous case, evidently we have $\angle\pi > \gamma$. Thus at the perigee, the sun will appear to move faster than ω. That is how it behaves in midwinter.

Ptolemy's method for obtaining solar eccentricity and the angle between the line of apsides and the line joining the solstices ψ will be illustrated and explained in chapter 2, section 2.2 of this paper. When modern values for time differences between solstice and equinox are applied to Ptolemy's model, the value found for the angle ψ is fairly precise. Nevertheless, it was not true for the value of eccentricity. In fact, the old value for solar eccentricity, e, exceeds the modern value by a factor of two.

1.4 HELIOCENTRIC MODEL

Today, we know that the earth revolves around the sun in a plane called the ecliptic plane. The earth also rotates once a day about its north-south axis that is not perpendicular to the ecliptic plane but tilted 23.5 degree with respect to the celestial equator. This is called the obliquity, ε.

Figure 3 shows the earth and its orbit (its yearly course around the sun) inside a large hollow globe representing the celestial sphere. Around the sphere, at an angle to the equator, runs a belt, with 12 constellations marked on it. This belt is the **zodiac** and along its middle runs a broken line, the **ecliptic**.

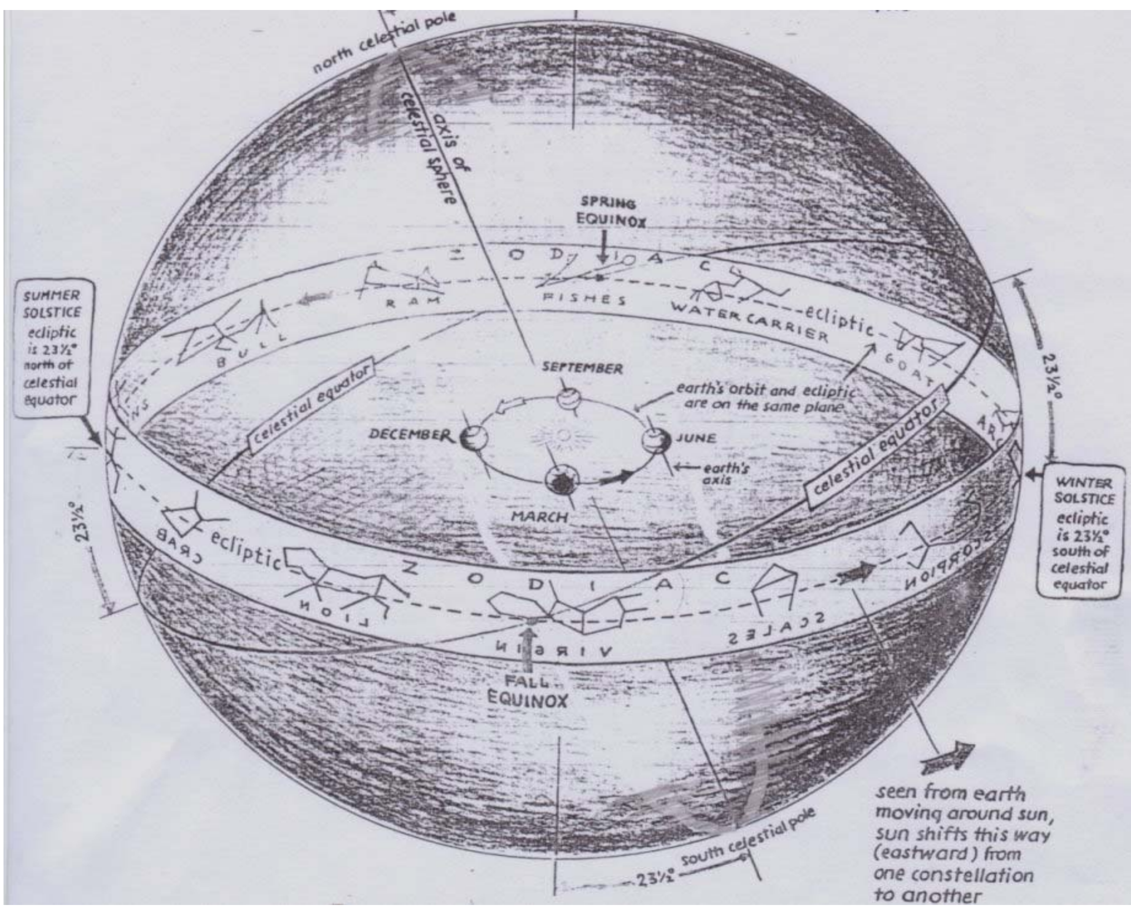
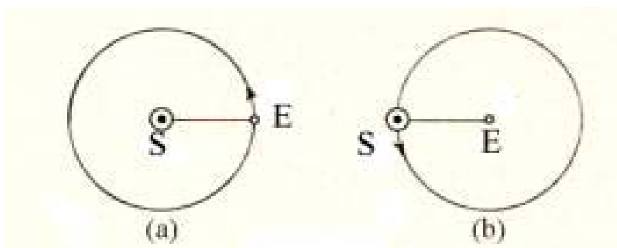


Figure 3

The earth yearly course is shown in four positions: December 21, March 21, June 21, and September 21. When the sun, on its yearly course along the ecliptic, reaches either of the two points of intersection, it is, for that moment, on the equator. At those points day and night are of equal length, and they are called the spring (vernal) equinox, when the sun is in the Fishes (Pisces) and the other, the autumnal (fall) equinox, when the sun is in the Virgin (Virgo). For the rest of the year, day and night are not of equal length. The longest day will occur during the summer solstice and the shortest day will occur on winter solstice. However, this does not correspond to the seasons in the southern hemisphere, and makes no sense in the tropics. When the northern hemisphere tilts towards the sun, it will be summer in the northern hemisphere and winter in the southern hemisphere and the converse when the northern hemisphere tilts away form the sun. Thus the earth's revolution and the tilt of the axis cause the seasons and not by what the ancient Greek astronomers perception of varying distance to the sun.

1.5 Heliocentric Model Versus Geocentric Model

In this section, we will discuss the main features of the apparent movement of the planets on a **heliocentric** (sun-in-the-center) model as well as a **geocentric** (earth-in-the-center) model. The main purpose of this section is to demonstrate that the quality of a planetary model does not depend on whether the sun or the earth is made stationary. Instead, the quality depends on the hypotheses of the model ─ the shapes and positions of the orbits, and the rule governing the variation in speed.



A system with only the Sun S and the Earth E is considered. In figure (a), the sun is made stationary with the earth revolving around it. On the other hand, in figure (b), the earth is made stationary with the sun revolving around it.

Next, we consider the motion of a superior planet, Mars, Jupiter or Saturn, which is further from the sun than the earth, on a heliocentric model.

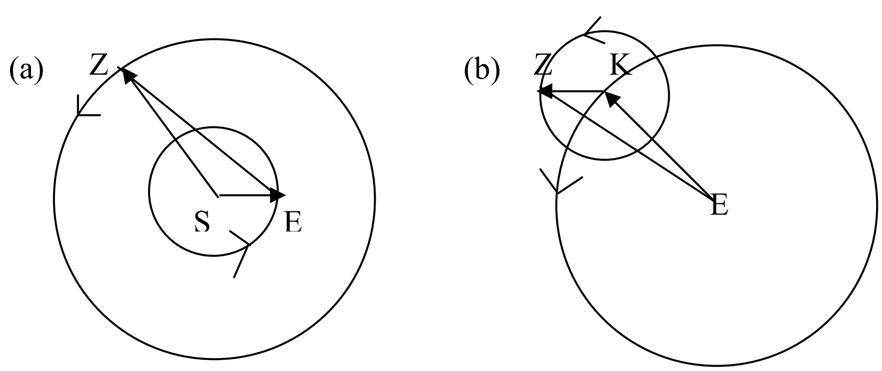


Figure 4

With reference to Figure 4(a), the Earth E executes an orbit (not necessarily circular) about the stationary Sun S. In the course of a year, the position vector ***SE*** rotates anticlockwise about S. The angular speed of ***SE*** varies slightly in the course of the year, and so does the length of the vector. Similarly, a superior planet Z revolves around the Sun in a larger orbit. Vector ***SZ*** varies at its own slightly different rate. At any instant, the line of sight from the Earth to the planet coincides with ***EZ,*** which is also equal to **–*SE+SZ***. Since these vectors may be added in either order, ***EZ*** is also equivalent to ***SZ+ (-SE).*** The new form of the addition is shown in Figure 4(b).

Here we begin at the Earth E. A vector equal to ***SZ*** is drawn with its tail at E; let the head of this vector be called K. Then ***EK*** rotates in Figure 4(b) at the same rate as ***SZ*** rotates in Figure 4(a). At K, place the tail of a second vector, equal to ***-SE*** with its head at planet Z. This series of steps brings about a transformation in a superior planet from a heliocentric model (Figure 4(a)) to a geocentric model (Figure 4(b)). Point K serves as the centre of a small circle, or an **epicycle**, upon which Z revolves while K itself moves on a large carrying circle, or **deferent**, about the Earth E. The equivalence of Figures 4(a) and 4(b) is a consequence of the commutative property of vector addition. Hence, in the case of a superior planet, the epicycle corresponds to the orbit of the Earth about the Sun, and the deferent, to the heliocentric orbit of the planet itself.

In the case of an inferior planet, one that is between the Sun and the Earth in the planetary system, the transformation is similar to that as explained for superior planets, but with adjustments made to the vectors and vector addition.

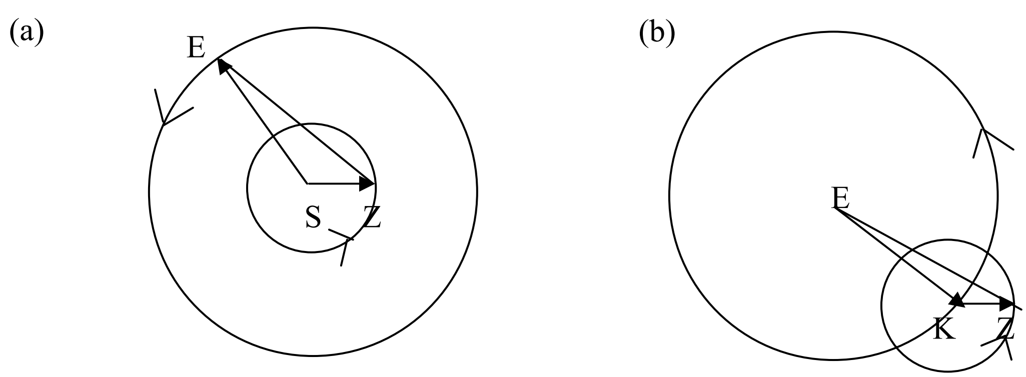


Figure 5

With reference to figures 5(a) and 5(b), S, E retains their original representations and Z now represents the inferior planet. Notice in Figure 5(a), however, that Z is on the inner orbit and vector ***EZ*** is equal to ***–SE+SZ.*** We draw a vector equivalent to ***–SE*** with its tail at E and its head at a new point called K on the deferent. At K, we add a vector equal to ***SZ*** and call it ***KZ.*** Once again; Figures 5(a) and 5(b) are equivalent to each other because of the commutative property of vector addition. Thus, for an inferior planet, the epicycle corresponds to its heliocentric orbit whilst the deferent corresponds to the orbit of the Earth about the Sun.

1.6 The zero-eccentricity deferent-and-epicycle model

The departure of a planet from uniform angular speed is known as an inequality or an anomaly. Every planet (excluding the sun) has two inequalities. The first inequality is called the zodiacal inequality because it has been observed that planets do not move with constant speed about the zodiac. They appear to move faster in some parts of the zodiac and slower in other parts.

The second inequality is a consequence of the retrograde motion of the planets. A given planet does not always retrogress at a fixed place on the ecliptic. In fact, a planet's retrogradations are connected with its position relative to the sun. For instance, the superior planets always reach the middles of their retrograde arcs when they are in conjunction with the sun. Thus this inequality in the motions of the planets that produces retrograde motion is called the inequality with respect to the sun.

Apollonios of Perge (3rd century B.C) was the first astronomer to provide an explanation for the retrograde motion of the planets using the zero-eccentricity planetary model, or the model in which the center of the deferent coincides with the earth.

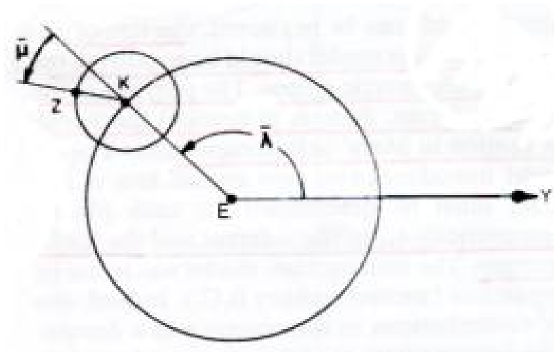


Figure 6

In Figure 6, the planet Z moves uniformly on an epicycle with centre K. K also moves uniformly around a deferent centered at Earth E. The fixed reference line EϒY points to the vernal equinoctial point. Thus angles $\bar{\lambda}$ and $\bar{\mu}$ increase uniformly with time. If $\bar{\mu}$ increases at a much higher rate than $\bar{\lambda}$, it will imply that the planet is moving at a faster rate than the earth, then the planet would appear to display retrograde motion. In other words, in the ancient planetary theory, the epicycle is the mechanism that produces retrograde motion.

However, even though the model accounts for the retrogradations qualitatively, it fails to do so quantitatively. According to the model, the retrograde loops produced would have the same size and shape, and be uniformly spaced around the ecliptic as shown in the figure below.

Yet this is overly simplified and does not occur in reality. The precise distance between one retrogradation and the next is quite variable, and thus the retrograde arcs are not equally spaced around the zodiac. There is no way for the uniformly spaced retrograde loops of the model to reproduce the unevenly spaced retrograde arcs of the planet itself.

Therefore, it was clear that the first practitioners of the deferent-and-epicycle model only considered the inequality with respect to the sun and took no account of the zodiacal inequality.

A step taken to improve this zero-eccentricity model was to displace the Earth E from the centre of the deferent C slightly.

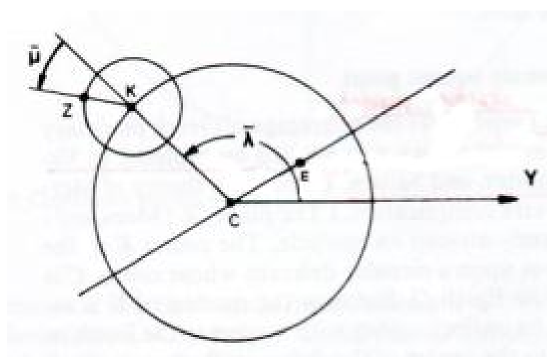


Figure 7

In Figure 7, planet Z moves uniformly on the epicycle, and the centre of the epicycle K moves around the deferent centered at earth E with a uniform speed. However, C no longer coincides with E as in the "zero-eccentricity model". This is known as the "intermediate model". However, the intermediate model provided just the kind of variable spacing between two retrogradations of the planet and was still unable to fit the width of the retrograde loops perfectly.

1.7 Ptolemy's Planetary Model

To determine planetary positions accurately and conveniently, Ptolemy introduced a model that consists of an "equant point."

Figure 8 represents the epicyclic model that Ptolemy used for any planet except Mercury. Here Z, K and E have the same meanings as in section 1.6.

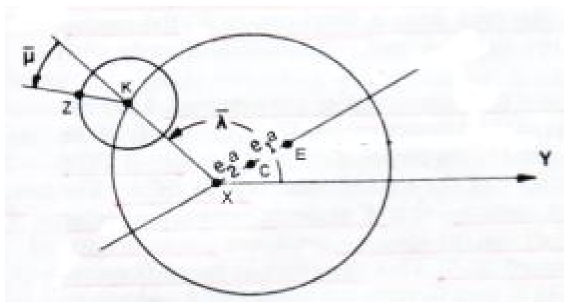


Figure 8

The motion of Z is governed in the following way: K moves uniformly counter-clockwise, not about C, but about a point X which is symmetrical to E with respect to C, and Ptolemy named the point X as the "equant point"

As a result of these two points X and E, there are now two eccentricities to be defined. Take the radius of the deferent to be *a*, the eccentricity of the Earth E with respect to centre C to be e_1 and the eccentricity of the equant point with respect to centre C to be e_2. Then,

$$e_1 = EC/a,$$

$$e_2 = XC/a.$$

The rule of equant motion – that K moves at constant angular speed as viewed from the equant – produces a physical variation in the speed of K. This variation in speed is determined by e_2 and if e_2 goes to zero, the motion of K reduces to uniform circular motion. Even if e_2 were zero, the eccentricity e_1 would cause the motion of K to appear non-uniform from the Earth E due to the same optical illusion mentioned in "Ptolemy's Solar Model". The sum $e_1 + e_2$ are called the "total eccentricity". Ptolemy always puts $e_1 = e_2$ and such a situation has come to be referred to as the "bisection of eccentricity". This notion would once again be brought up in chapter 2, section 2.4, where we make comparisons across Ptolemy's and Kepler's solar and planetary models.

1.8 Kepler's Law and Geometric Properties Of Elliptical Orbits

An ellipse is defined mathematically as the locus of all points such that the sum of the distances from two foci to any point on the ellipse is a constant (figure 9); hence

$$F'A + FA = 2a = \text{constant and}$$

$$\rho + r' = 2a = \text{constant} \qquad (1)$$

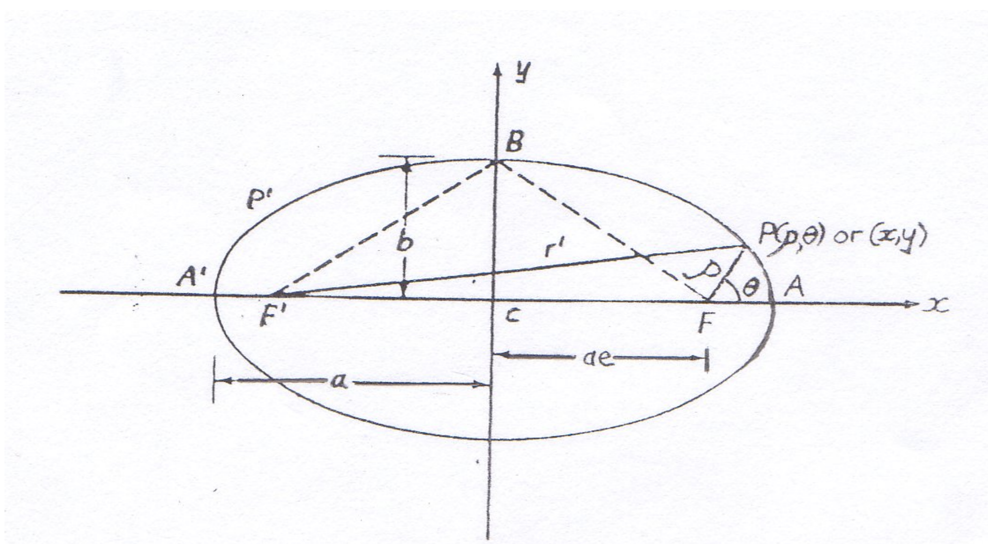


Figure 9

The line joining the two foci F and F' intersects the ellipse at the two vertices A and A'. Note that a, half the distance between the vertices A and A', denotes the semimajor axis of the ellipse. The shape of the ellipse is determined by its eccentricity e, which is defined as the ratio of the distance between the center of the ellipse and one of the foci to the semimajor axis, or

$$e = CF/a = CF'/a$$

$$CF = CF' = ae.$$

When e=0, we have a circle and for e=1, a straight line. One-half of the perpendicular bisector of the major axis is the semiminor axis b. Using, the dashed line (ρ=r'=a) in the figure 9 and the Pythagorean theorem, we find

$$b^2 = a^2 - a^2e^2 = a^2(1-e^2)$$

Kepler's First Law states that the orbit of each planet is an ellipse with the sun at one focus.

With reference to figure 9, we place the sun at one focus, F. Then vertex A is termed the perihelion of the orbit (point nearest the sun), and vertex A' is called the aphelion (point farthest from the sun). The perihelion distance AF is a(1-e), and the aphelion distance A'F is a(1+e). The mean (average) distance from the sun to a planet in elliptical orbit is just the semimajor axis a. We prove this fact by noting that for each point P on the ellipse at a distance ρ from focus F, there is a symmetrical point P' at a distance r' from F; the average of these distances is (ρ+r')/2=a. This result holds for any arbitrary but symmetrical pair of points.

Now, let us consider the distance from one focus to a point on the ellipse (such as the sun-planet or planet-satellite distance) as a function of the position of that point. Center a polar coordinate system (ρ,θ) at F and let the line FA correspond to θ = 0. Now ρ measures the distance FP, and then θ measures the true anomaly (the counterclockwise angle AFP). Using

$$\cos(\pi-\theta) = -\cos\theta$$

and the law of cosines, we have

$$(r')^2 = \rho^2 + (2ae)^2 + 2\rho\,(2ae)\cos\theta$$

From Equation 1, however, r'= 2a - ρ, and so

$$\rho = a(1-e^2)/(1+e \cos \theta)$$

The time required for the planet to describe its orbit is called the **period**, denoted by T. In time T, the radius vector FP sweeps out an angle of 2π and thus, the mean angular velocity of the planet, ω, is $2\pi/ T$.

Kepler's Second Law states that the radius vector SZ (in Figure 10) sweeps out equal areas in equal times.

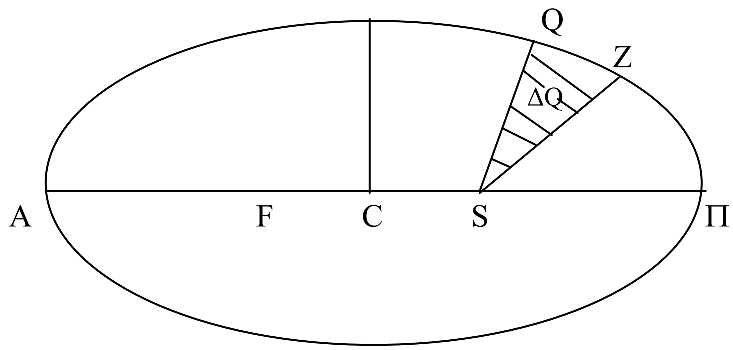


Figure 10

As a remark, note that figure 10 is reflection of figure 9, so positive angles have become negative.

Let Z corresponds to the planet's position at time t and Q its position at time $t+\Delta t$. Let $\rho+\Delta\rho$ denote the radius vector SQ and $\theta+\Delta\theta$ be $Q\hat{S}\Pi$. Hence, $Q\hat{S}Z= \Delta\theta$. If $\Delta\theta$ is sufficiently small, the arc ZQ may be regarded as a straight line and the area swept out in the infinitesimal interval Δt is simply the area of triangle QSZ which is equal to $1/2\rho(\rho+\Delta\rho)\sin\Delta\theta$ or with sufficient accuracy, $1/2\rho^2\Delta\theta$. The area velocity or the rate of description of area is the previous expression for area divided by Δt. As this rate is constant according to Kepler's second law, we can write,

$$h= 1/2\rho^2\, d\theta/dt \qquad (1)$$

where h is a constant.

Now, the whole area of the ellipse is πab and this is described in the period *T*. Hence,

$$h = \pi ab/T$$

or, $$h = \pi a^2(1-e^2)^{1/2} / T \qquad (2)$$

because $b^2 = a^2(1-e^2)$ which is equivalent to $b=a(1-e^2)^{1/2}$.

By (1) and (2), we have, $1/2\rho^2 \, d\theta/dt = [\, \pi a^2(1-e^2)^{1/2}] / T$.

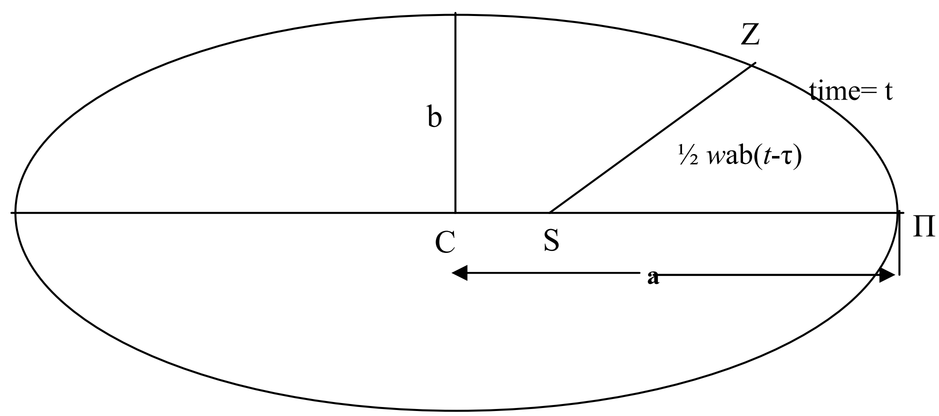


Figure11

Theoretically, if the values of the semimajor axis *a*, the eccentricity *e*, the time at which the planet passed through perigee τ and the orbital period *T* are known, Kepler's second law would enable us to determine the position of the planet in its orbit at any instant.

Referring to Figure 11, Z is the position of the planet at time t. In the interval $(t - \tau)$ the radius vector moving from SΠ to SZ sweeps out the area SZΠ. By Kepler's Second Law,

$$\text{Area SZ}\Pi : \text{Area of ellipse} = t - \tau : T.$$

Hence, $$\text{Area SZ}\Pi = [\pi ab(t - \tau)]/ T,$$

Or, $$\text{Area SZ}\Pi = 1/2\ wab(t - \tau)$$

where $w = 2\pi/T$ and $b^2 = a^2(1-e^2)$.

CHAPTER TWO

2.1 The Analemma in the Construction of San Petronio.

The true sun is the actual sun that we will see. Its path is on the ecliptic, and it takes one year to travel along the celestial sphere. The mean sun is the sun we would see if the earth was not tipped 23.5 degree on its axis. It takes one year to travel once along the celestial equator. The modern analemma is the figure-eight diagram that shows how much the apparent sun is ahead or behind the mean sun.

In the past, an analemma referred to any tool used for astronomical calculations. The analemma of San Petronio is a tool used to determine where along the meridian line the noon ray will cross at a given date of each month throughout the year. In other words, the analemma is a graphical tool for computing declination as a function of celestial longitude. Figure 12 shows a group of people in the process of laying the meridiana at San Petronio. The circle imposed on the diagram is the analemma.

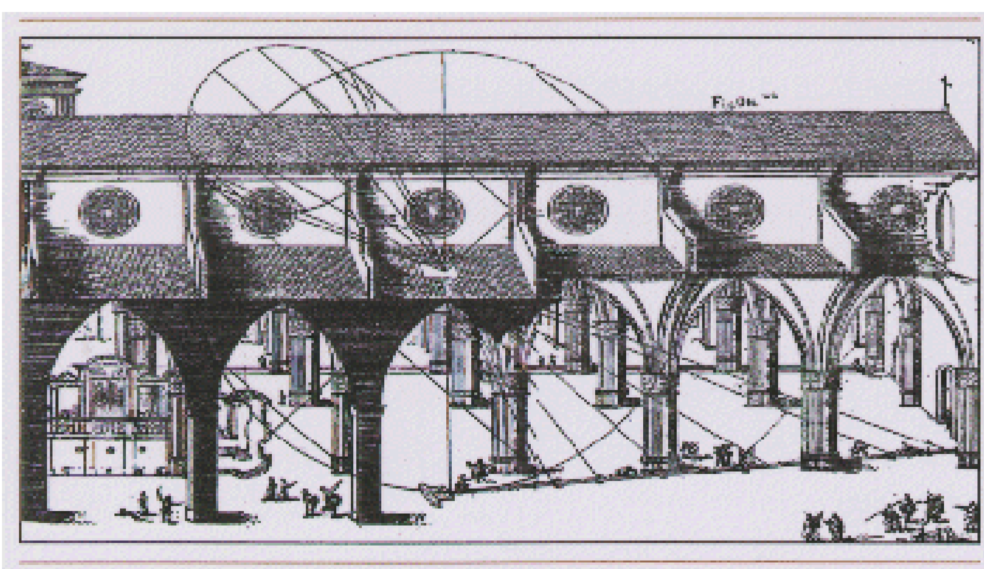


Figure 12

The usual way adopted by the ancient astronomers to determine the direction of the sun's ray at certain times throughout the year, was to observe the motion of the sun throughout the year and then mark the respective points on the meridiana. However, Gian Domenico Cassini, the builder of the meridian line in San Petronio, calculated the images in advance. The calculation involved is briefly stated in Appendix B of Heilbron's book and this section aims to elaborate.

The vertical circles and lines in the figure 12 have been redrawn in figure 13, which is shown below.

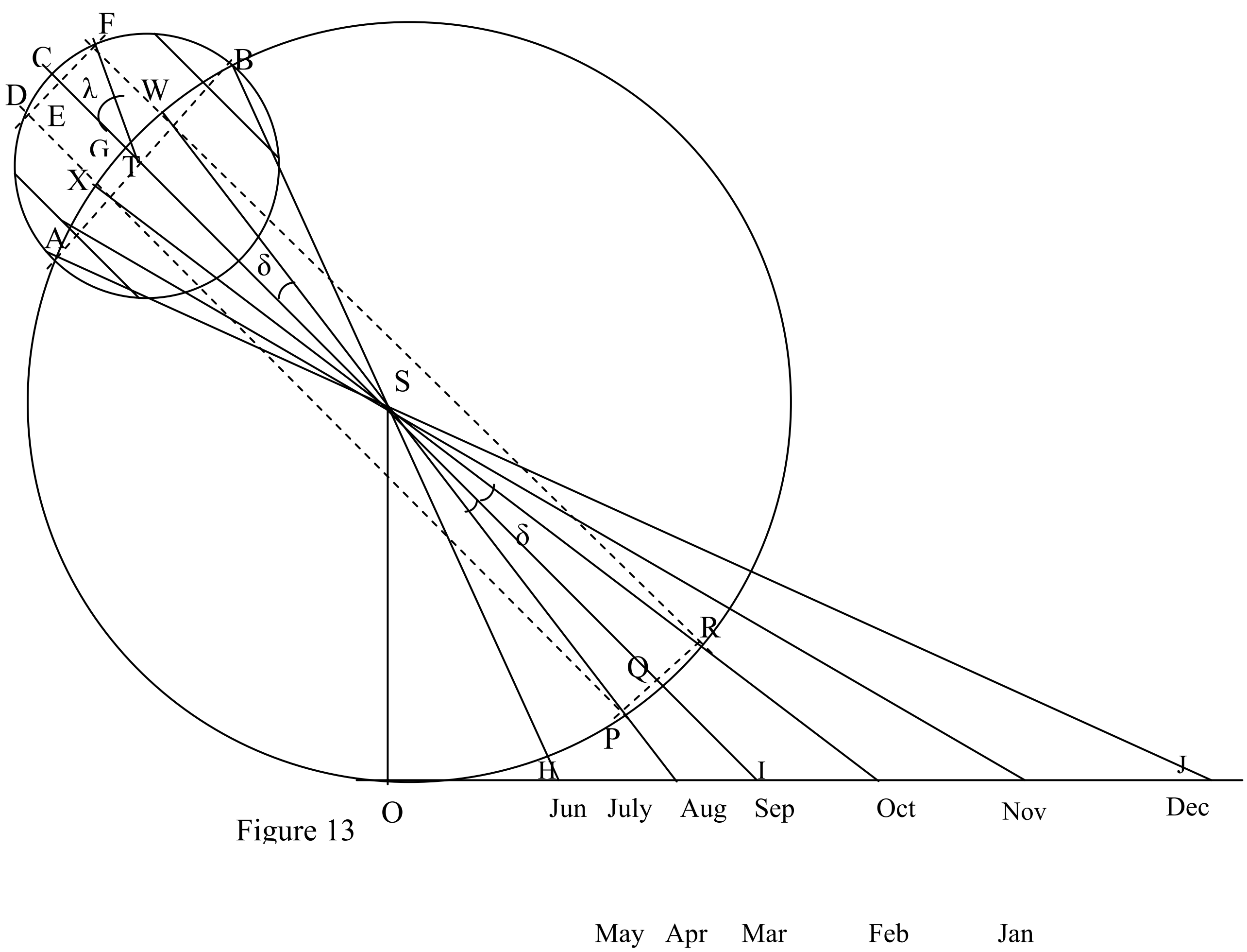


Figure 13

To construct an analemma, one has to first determine the length of the shadow cast by the gnomon at the equinox of the particular place. To do so, one needs to determine the latitude of the place. For instance, if the latitude of a place is equal to $\tan^{-1}(5/7)$, then a gnomon of whatever size that is divided into 7 parts will cast a shadow of 5 units at the time of the equinoxes. Similarly for San Petronio, Cassini first determined the latitude of Bologna, the location of San Petronio.

With reference to figure 13, OS indicates the gnomon, an instrument that serves to indicate the time of the day by casting its shadow upon a marked surface. In the case of San Petronio, the marked surface would refer to the meridiana, which is the tangent line to the bigger circle. The bigger circle is drawn such that its center is at S and its radius is OS. Having determined the latitude of Bologna, the shadow of the gnomon at the equinox is known. With reference to figure 13, the length of this shadow is OI. Then from the point I, a line is drawn through the center S until it touches the circumference of the big circle at point G. This line ISG is then the ray of the sun at the equinox.

Taking the obliquity to be 24 degrees and using the equinoctial ray ISG as a base and the point S, we can measure 24 degrees and construct the lines HSB and JSA. Accordingly, HSB and JSA are the noon ray of the sun at summer solstice and winter solstice respectively. Next, a dotted line is drawn through A and B. Let the mid-point of the dotted line AB be T, the smaller circle with center T and radius TA is then drawn. As a remark, the smaller circle should be 1/15 that of the bigger circle. Nevertheless, to give a clearer diagram, figure 13 is not drawn to scale. This smaller circle acts as an auxiliary circle that helps to determine the position of the sun at different times of the year. For instance, points A, G, B on the smaller circle will represents the position of the sun in the sky during winter solstice, equinox and summer solstice respectively. Accordingly, the first, middle and last points on the meridian line correspond to the noon ray of the sun at summer solstice, equinox and winter solstice. While the rest of the points on the horizontal line correspond to where the noon ray crosses the meridiana for the remaining months. That is to say when the sun is at summer solstice (point B), it will have a noon ray of BSH. So all the slanted lines starting from the circumference of the bigger circle, passing through S, represent the noon ray of the sun at different times of the years.

Up to this point, we have managed to determine the noon ray of the sun at the equinox, summer solstice and winter solstice. The problem now is how can we calculate the images of the sun's shadow for the rest of the months. The calculation involved will be discussed in the paragraphs that follow.

We noticed that the larger circle centered on S cuts the smaller circle centered on T at points A and B such that the chord AB of the bigger circle is identical to the diameter of the smaller circle. Let F be any arbitrary point on the small circle; then point W is constructed by dropping a perpendicular from point F onto the circumference of the bigger circle. Points P, Q, R are just extensions of the noon ray of the sun at positions W, G and X respectively until they touch the circumference of the bigger circle. Let angle CTF be λ, angle CSW = angle PSQ = angle RSQ = δ and r, R be the radii of the smaller and bigger circle respectively.

Since DF is parallel to AB, in triangle EFT, $EF/r = \sin \lambda$ and this implies $EF = r \sin \lambda$.

Next, in triangle RSQ, $QR/R = \sin \delta$ which implies $QR = R \sin \delta$.

Furthermore, since EF = QR, we will have $r \sin \lambda = R \sin \delta$ and this implies
$\sin \delta = r/R \sin \lambda$.

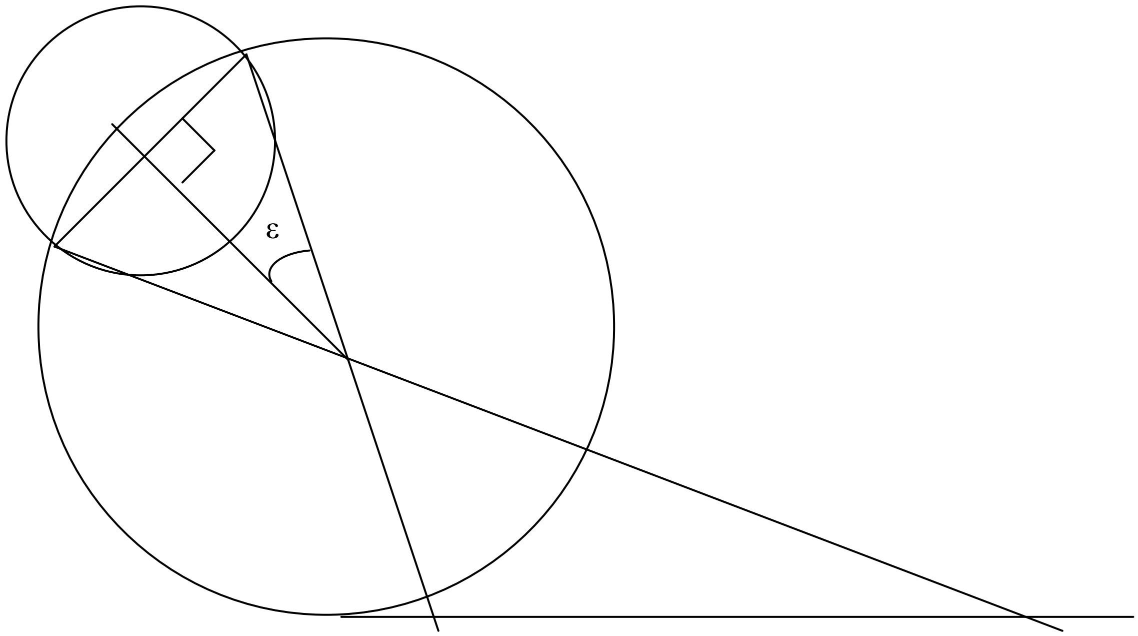


Figure 14

With reference to figure 14, taking $\sin \varepsilon = r/R$, this is so because the smaller circle is 1/15 of the bigger circle and ε is just the obliquity. As a result we have $\sin \delta = \sin \varepsilon \sin\lambda$, therefore, giving us, $\sin \lambda = \sin \delta / \sin \varepsilon$. (1)

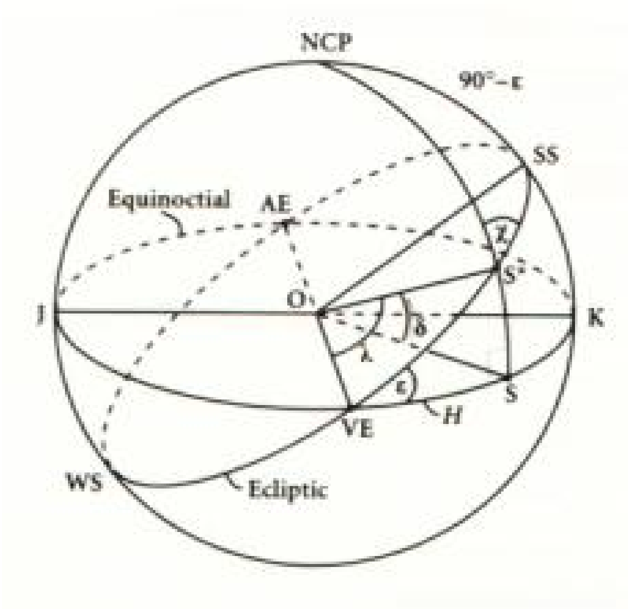


Figure 15

The above figure gives the earlier-mentioned angles λ and δ on the celestial sphere. S* marks the true Sun while S the projection of the true Sun on the equinoctial. δ is the Sun's declination and ε the obliquity of the ecliptic. Looking at the spherical triangle S*SVE, we have sin ε /sin SS* = sin 90^0/sin S*VE and this in turn implies that sin λ / sin 90^0 = sin δ / sin ε and this is equivalent to (1).

Thus, the calculation is complete. Since λ is the ecliptic longitude, it is sufficient to mark the point where the noon ray falls on the meridian line at an equinox, and then by increasing λ in steps of 30 degrees, the rest of the points whereby the sun's image crosses the meridiana could be positioned accordingly.

The idea behind this is that since the sun moves approximately about 1 degree each day, its position will differ by approximately 30 degrees each month, thus by marking the point where the noon ray falls on the meridian line at an equinox whereby the longitude is 0 degree, and by increasingly λ in steps of 30 degrees, one can approximately estimate the position of the sun and thus calculate the position of the sun in advance.

2.2 Ptolemy’s Solar Eccentricity (Notes to “The Sun in the Church”, page 105)

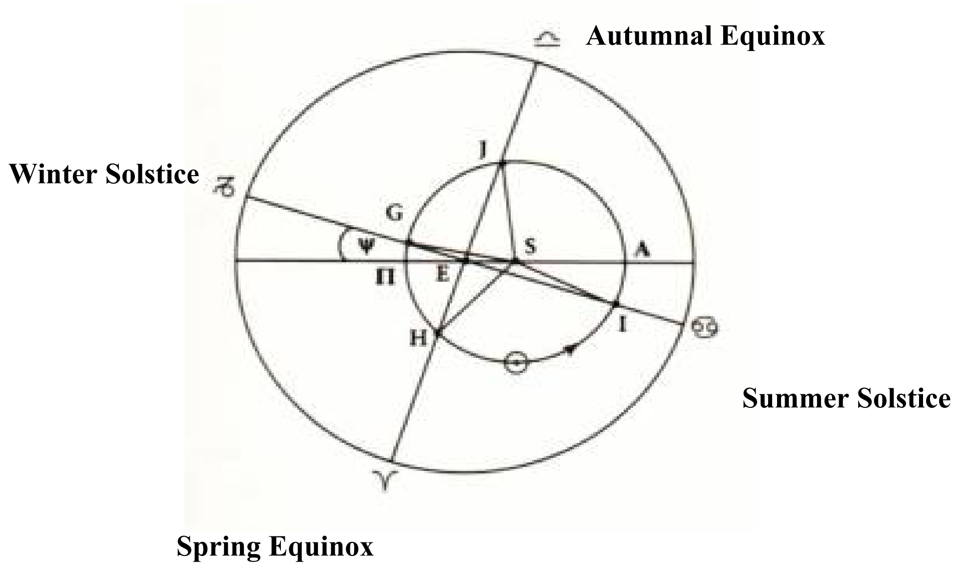


Figure 16

As highlighted in "Ptolemy's Solar Models", section 1.3, Ptolemy’s method for obtaining the solar eccentricity, and the angle between the line of apsides and the line joining the solstices ψ will be illustrated and discussed in this section. Figure 16 shows the model that Ptolemy used to work out the aforesaid values. This section aims to provide a detailed set of working to obtain the result quoted at the end of the page: e = 0.0334, $\psi = 12^\circ\ 58'$, by applying modern seasonal lengths.

The modern values for the lengths of the seasons are as follows: Spring – 92 days, 18 hours, 20 minutes or 92.764 days; Summer – 93 days, 15 hours, 31 minutes or 93.647 days; Autumn – 89 days, 20 hours, 4 minutes or 89.836 days; and Winter – 88 days, 23 hours, 56 minutes or 88.997 days.

Using modern seasonal lengths, since summer is the longest season of the year, EA must point somewhere between the summer solstice and autumnal equinox so that, when viewed from the earth, E, the sun’s orbit, S, between points I and J will be longer than 1/4 arc of the entire path. Similarly, since winter is the shortest season of the year, EΠ must point between the winter solstice and the spring equinox so that, when viewed from the earth, E, the sun's orbit, S, between points G and H will be shorter than 1/4 arc of the entire path.

The Julian calendar introduced a solar year of 365 days and the year was divided into 12 months. April, June, September and November had 30 days, February 28 days and all other months, 31 days. In every 4th year, an extra day was added into the calendar, making the year 366 days long. Thus, the average length of a Julian solar year became 365.25 days. Taking the Julian value for the number of days in a year (denoted by y), the mean angular motion of the sun, w, is equal to $360^\circ/y$ per day, or

$$w = 360^\circ / 365.25 \approx 0.9856^\circ \text{ per day}$$

Using the modern values for the lengths of seasons and the mean angular motion of the sun, w, we can work out the following angles:

Angle GSH = length of winter × w = 88.997(360/365.25) ≈ 87.7178^0

Angle HSI = length of spring × w = 92.764(360/365.25) ≈ 91.4306^0

Angle JSI = length of summer × w = 93.647(360/365.25) ≈ 92.3009^0

Angle GSJ = length of autumn × w = 89.836(360/365.25) ≈ 88.5447^0

However, in Heilbron's book, the order of listing for the value of the modern length of seasons is incorrect. Had readers overlooked this and proceeded with the calculations for e and ψ, errors would have been unavoidable.

Now, we continue with the calculation to obtain the values for *e* and *ψ*.For easy reference, parts of figure 16 has been enlarged and redrawn in figure 17.

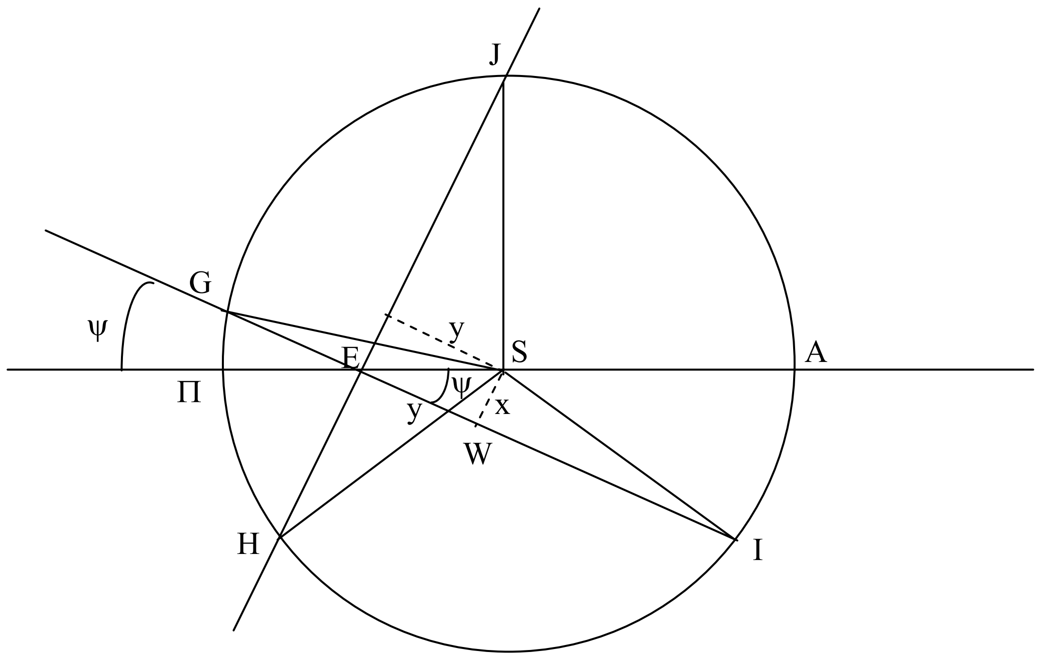


Figure 17

With reference to figure 17, consider triangle JSH, with angle JSH=angle GSJ+ angle GSH Since it is an isosceles triangle, the perpendicular, of length *y*, when dropped from point S to the base JH, would bisect the angle JSH. In addition, take $JS = HS = a$ where a is the radius of circle centered at S. Then,

$$\cos(\angle JSH/2) = y/a$$

$$\Rightarrow \cos(\angle GSJ + \angle GSH)/2 = y/a$$

Similarly, in isosceles triangle GSI with angle GSI = angle GSH + angle HSI, taking x as the length of the perpendicular dropped from point S to the base GI, we have,

$$\cos(\angle GSI / 2) = x/a$$

$$\Rightarrow \cos(\angle GSH + \angle HSI / 2) = x/a$$

Looking at the right-angled triangle, WES, we have

$$\tan\psi = x/y = (x/a)/(y/a)$$

that is,

$$\tan\psi = [\cos\{(G\hat{S}H + H\hat{S}I)/2\}] \div [\cos\{(G\hat{S}J + G\hat{S}H)/2\}]$$

Hence,

$$\psi = \tan^{-1}([\cos\{(G\hat{S}H + H\hat{S}I)/2\}] \div [\cos\{(G\hat{S}J + G\hat{S}H)/2\}])$$

By substituting the values of the angles we obtained previously into the above equation, we obtain the value of ψ to be approximately 12˚58', equivalent to the value quoted in Heilbron's book.

To work out the value of eccentricity, e, we note that e = ES/a. But

$$\sin\psi = x/ES, \text{ this implies that } ES = x/\sin\psi.$$

Therefore,

$$e = [x/\sin\psi]/a = [x/a]/\sin\psi$$

Using the result from above, we have,

$$e = [\cos\{(G\hat{S}H + H\hat{S}I)/2\}]/\sin\psi$$

After substituting the relevant values into the above relation, we obtain the value of *e* to be around 0.0334, which is the same value as that quoted in Heilbron's book.

However, Ptolemy's value of solar eccentricity of 0.0334 exceeded that found by Kepler which is 0.0167. The details that account for this factor-of-two difference will be explained in the next section (section 2.3).

2.3 Bisection Of Eccentricity

The notion of "bisection of eccentricity" has been briefly mentioned in earlier sections. Here, we provide a more detailed discussion of this concept and aim towards clarifying the double meaning it holds. Most of the content presented in this section is not new but having the information on bisection of eccentricity grouped in one place should boost our understanding of it significantly.

The meaning of "bisection of eccentricity" arises when we turn towards comparing Kepler's eccentricity value with Ptolemy's value. The former is half the value of the latter because of the difference in the way Kepler and Ptolemy had measured the separation between Sun and Earth on their respective solar models

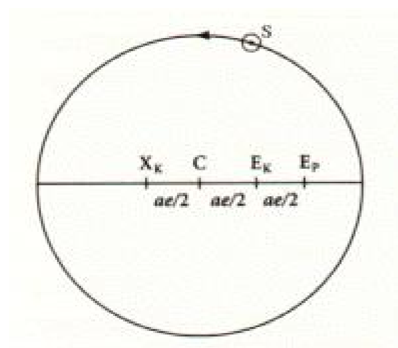


Figure 18

With reference to figure 18, C represents the centre of the sun's orbit, E_P denotes where Ptolemy positions the earth, E_K denotes where Kepler positions the earth and X_K marks Kepler's equant point. Consider radius of the sun's orbit to be *a*, then the intervals between X_K, C, E_K and E_P will each be ae/2.

Both Ptolemy and Kepler had calculated the solar eccentricity as the ratio of the separation between Earth and centre of Sun's orbit to the orbital radius of the Sun. However, with reference to Figure 18, we see that Kepler would have measured that particular separation as $CE_k = ae/2$ whereas Ptolemy would have measured it as $CE_p = (ae/2)+(ae/2) = ae$. The resulting value of eccentricity found by Kepler is thus equivalent to the bisected value of Ptolemy's eccentricity.

We can also recall that on Ptolemy's planetary model, he had added a third device called the "equant point" to the intermediate deferent-epicycle model such that the width of the retrograde loops could fit exactly onto it. The angular velocity of a body on orbit, when observed from the equant point, is constant. Figure 19 is repeated below for easy reference.

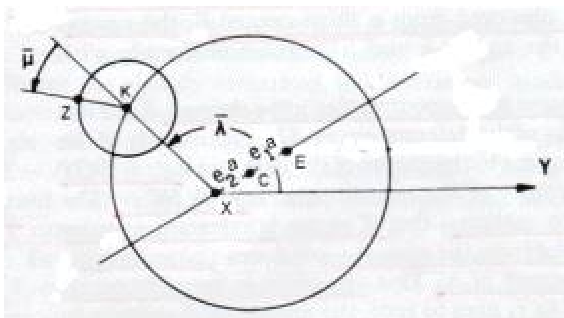


Figure 19

As a result of the addition, two eccentricities were defined. Taking the radius of the deferent to be *a*, e_1 represented the eccentricity of the Earth E with respect to centre C and e_2 represented the eccentricity of the equant point X with respect to C. That is,

$$e_1 = EC/a,$$

$$e_2 = XC/a.$$

The total eccentricity refers to the sum of e_1 and e_2. In this case, "bisection of eccentricity" refers to the situation where Ptolemy places the equant point on the mirror image of E along the line of apsides such that $e_1 = e_2$ is obtained.

In summary, we see that "bisection of eccentricity" either refers to Ptolemy splitting the total eccentricity exactly into two on his planetary theory, or refers to Kepler dividing the solar eccentricity as defined in Ptolemy's solar theory.

2.4 Comparison of Models

In this paper, Ptolemy's planetary model, Ptolemy's solar model and Kepler motion (law) have been bought into discussion. We observed that Ptolemy's equant theory for the planets is a close approximation to Kepler's solar theory. In this section, we aim to find out mathematically how this has been possible. Prior to that, we will derive the equations defining the position of a body in its orbit for each of the models. The defining parameters of the body's position are the true anomaly and the radius vector. In finding out the equations of these parameters, we will also make use of the eccentricity.

In Heilbron's book, the value of eccentricity in Ptolemy's planetary theory and solar theory is e whereas that in Kepler's is e/2. However, in most of the astronomical textbooks, the value of eccentricity in Ptolemy's planetary theory and solar theory is quoted as 2e while that in Kepler's solar theory is e. Since this paper is written as a mathematical supplement to Heilbron's book, we will follow the eccentricity values used by Heilbron.

As described in section 1.9, the equation for the elliptical orbit is

$$\rho = a(1-e^2)/\,(1+ e \cos \theta) \qquad (1)$$

where ρ is the planet's distance from the Sun, θ is the true anomaly and e is the eccentricity.

By means of the binomial theorem, the above equation may be expanded as the following:

$$\rho(\theta) = a\,[1- e^2][1 + e \cos\theta]^{-1}$$
$$= a[1-e^2][1-e \cos\theta + e^2 \cos^2\theta - \ldots\ldots\ldots]$$
$$= a[1-e^2]\,[1-e \cos\theta + e^2 (1- \sin^2\theta)-\ldots\ldots]$$
$$= a[1-e^2][1-e \cos\theta + e^2 - e^2\sin^2\theta-\ldots\ldots.]$$
$$= a[1-e \cos\theta + e^2- e^2 \sin^2\theta-\ldots..-e^2 + e^3 \cos\theta-\ldots]$$

Hence, $\rho(\theta) = a[\ 1- e \cos\theta - e^2 \sin^2\theta + \ldots\ldots\ldots\ldots]$ (2)

Furthermore, taking the semi-major axis, a, of the ellipse to be unity, we have

$$\rho(\theta) = [\ 1- e \cos\theta - e^2 \sin^2\theta + \ldots\ldots\ldots\ldots]$$

Also derived in section 1.9 from Kepler's 2nd law, we have

$$(1/2\rho^2)d\theta /dt = (\ \pi a^2[1- e^2\ ]^{1/2}) / T$$

When (2) is substituted into the above, and the resulting differential equation for $\theta(t)$ is expanded and integrated through order e^2, we obtain

$$\theta(t) = wt + 2e \sin wt + 5/4\, e^2 \sin 2wt, \qquad (3)$$

where $w= 2\pi/ T$ is the angular velocity.

Equations (2) and (3) are the defining equations for a body on the Keplerian model. We next derive similar equations for a body in Ptolemy's equant theory. Figure 20 is the reference diagram for the derivation steps that follow. In figure 20, the radius of the eccentric circle has been set equal to unity. Here, e_1 and e_2 denote the eccentricity of the Earth, O, and the equant X.

Error!

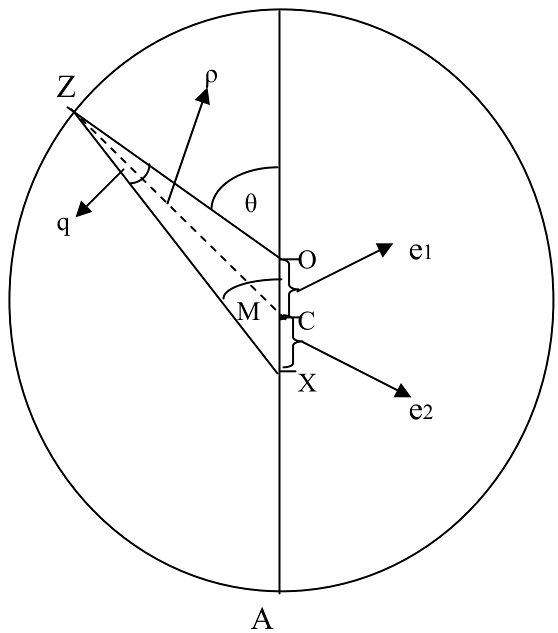


Figure 20

The calculation of the equation of the orbit in polar coordinates for a body Z is as follows:

Using the Law of Cosine for triangles, we have,

$$1^2 = e_1^2 + \rho^2 - 2e_1 \rho \cos(\pi-\theta)$$

$$\Longrightarrow 1^2 = e_1^2 + \rho^2 + 2e_1\rho\cos\theta \qquad [\text{since } \cos(\pi-\theta) = -\cos\theta]$$

$$\Longrightarrow \rho^2 + 2e_1\rho\cos\theta + e_1^2 - 1 = 0$$

Using formula for quadratic equations, we have,

$$\rho(\theta) = \{ -2e_1 \cos\theta + [\, 4e_1^2 \cos^2\theta - 4(e_1^2 - 1)]^{1/2} \}/2$$

$$= - e_1 \cos\theta + [\, e_1^2 \cos^2\theta - e_1^2 + 1]^{1/2}$$

$$= - e_1 \cos\theta + [\, e_1^2(1 - \sin^2\theta) - e_1^2 + 1]^{1/2}$$

$= -e_1 \cos\theta + [1 - e_1^2 \sin^2\theta]^{1/2}$

Thus, the equation of the orbit in polar coordinates for a body Z is that of a circle eccentric to C, which is,

$$\rho(\theta) = -e_1 \cos\theta + [1 - e_1^2 \sin^2\theta]^{1/2}$$

Hence, by binomial expansion,

$$\rho(\theta) \approx 1 - e_1 \cos\theta - 1/2\, e_1^2 \sin^2\theta \qquad (4)$$

Since mean anomaly M is equivalent to wt and $q = \theta - M \Rightarrow \theta = M + q$ therefore, $\theta = wt + q$.

Applying Sine Rule to triangle OZX, we have

$$OX/\sin q = OZ/\sin M$$

$$\Longrightarrow [e_1 + e_2]/\sin q = \rho/\sin wt$$

$$\Longrightarrow \sin q = ([e_1 + e_2] \sin wt)/\rho$$

This gives

$$q = \sin^{-1}\{([e_1 + e_2] \sin wt)/\rho\}.$$

.Hence,

$$\theta(t) = M + q$$

$$\theta(t) = wt + \sin^{-1}\{([e_1 + e_2] \sin wt)/\rho\}.$$

By substituting the expression of ρ obtained in (4) into this equation, and expanding to second order in e, we get

$$\theta(t) = wt + [e_1 + e_2] \sin wt + 1/2 e_1(e_1 + e_2) \sin 2wt \qquad (5)$$

With reference to Heilbron's book, the value of eccentricity in Ptolemy's planetary and solar model is e while that in Kepler's is e/2. However, on Ptolemy's planetary model, he added a 3rd device called the equant point, as a result, two eccentricities were defined and we have $e_1+e_2 = e$ and this implies that $e_1=e_2=e/2$. As a remark, for Ptolemy's solar theory, there is no equant point and this will implies that the value of eccentricity in this model will still be e. Therefore, by substituting the three sets of eccentricity values into equations (5) and (3) respectively, we will get the corresponding equations for true anomaly, θ(t), as shown in the table below. Similarly, by substituting Ptolemy's eccentricity value into equation (4) and Kepler's eccentricity value into equation (2), we will obtain the values for the radius vector, ρ(θ), as shown in the table below.

Model	True anomaly, θ(t)	Radius Vector, ρ(θ)
Ptolemy's Planetary Model **Eccentricity values:** $e_1=e_2=e/2$	$wt + e \sin wt + (e/2)^2 \sin 2wt$	$1- e/2 \cos \theta - 1/2\, (e/2)^2 \sin^2\theta$
Ptolemy's **Solar Model** **Eccentricity values:** $e_1=e,\ e_2=0$	$wt + e \sin wt + 1/2\, e^2 \sin 2wt$	$1- e \cos \theta - 1/2\, e^2 \sin^2\theta$
Kepler's Solar Model **Eccentricity values:** $e=e/2$	$wt + e \sin wt + 5/4 (e/2)^2 \sin 2wt$	$1- e/2 \cos \theta - (e/2)^2 \sin^2\theta$

With reference to the above data, we notice the first discrepancy between the radius vector of Ptolemy's solar model and Kepler's solar model is displayed in order e. This implies that Ptolemy's solar model was not accurate. On the other hand, we can see that the difference between the true anomalies of Ptolemy's planetary model and Kepler's solar model is only a mere $(-1/4)\ (e/2)^{2} \sin 2wt$. This implies that Ptolemy's and Kepler's models have approximated each other so well that up to first order terms in *e*, the empty focus of the Keplerian ellipse is indistinguishable from the equant point in Ptolemy's model. The working steps that follow provide the mathematics behind such an approximation.

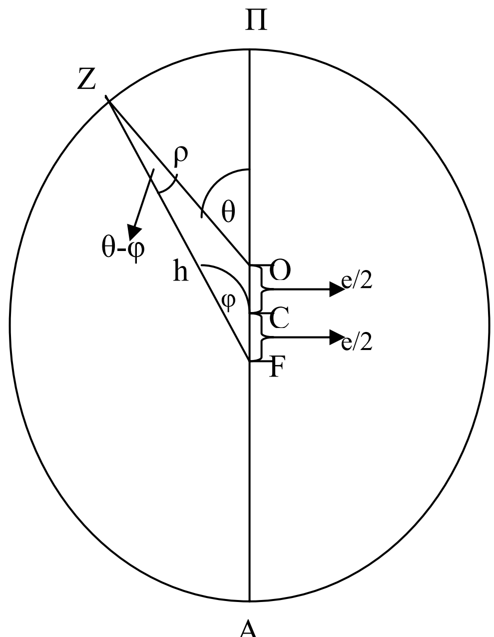


Figure 21

In the Keplerian ellipse of figure 21, FZ makes an angle ϕ with the line of apsides AΠ; θ and ρ have their usual meanings. Take FZ = *h.* By properties of ellipses, FZ = AΠ - OZ. That is, $h=2a-\rho$ or if the semi-major axis has been set to unity, $h=2-\rho$. In addition, angle OZF= $\theta-\varphi$ We aim to find an expression for $\varphi(t)$ to represent the motion of the planet as observed from the empty focus. Applying Sine Rule to triangle OZF, we have,

$$e / \sin(\theta-\varphi) = h / \sin(\pi-\theta) = h / \sin\theta$$

$$\Longrightarrow \quad \sin(\theta-\varphi) = (e \sin\theta) / h$$

$$\Longrightarrow \quad \sin(\theta-\varphi) = (e \sin\theta) / 2- \rho$$

Upon substituting the Keplerian expressions for $\theta(t)$ and $\rho(\theta)$,that were calculated earlier in this section, and expanding, we yield the following result:

$$\varphi(t) = wt + 0 + 1/4\ (e/2)^2 \sin 2wt$$

Therefore, if we only consider terms in *e* up to first order, $\varphi(t)$ is equivalent to *wt* which in turn gives the constant motion of the planet Z when observed at the empty focus F. This is why Kepler's empty focus is said to behave as Ptolemy's equant point, to order *e*.

Thus in conclusion, considering the low precision in instruments used to make measurement of celestial bodies in the ancient days; it would have been hard to recognize the difference between Kepler's solar model and Ptolemy's planetary model. As a result, mathematicians and astronomers had to struggle with deciding which of the models was the correct one. This motivated Cassini to find out which theory was the accurate one by using the meridiana of San Petronio; and he succeeded in concluding that Kepler's solar model was the correct model. The basis of and working steps to his conclusion can be found in Appendix C of Heilbron's book.

APPENDIX

APPENDIX 1

ELLIPSES

Many mathematical textbooks contain comprehensive accounts of what an ellipse is and what its properties are. A recommended reference text is "Basic Calculus: from Archimedes to Newton to its Role in Science" written by Alexander J Hahn, and in particular, pages 54 – 55 and 90 – 94.

Here, some of the properties have been selected and stated below for easy reference.

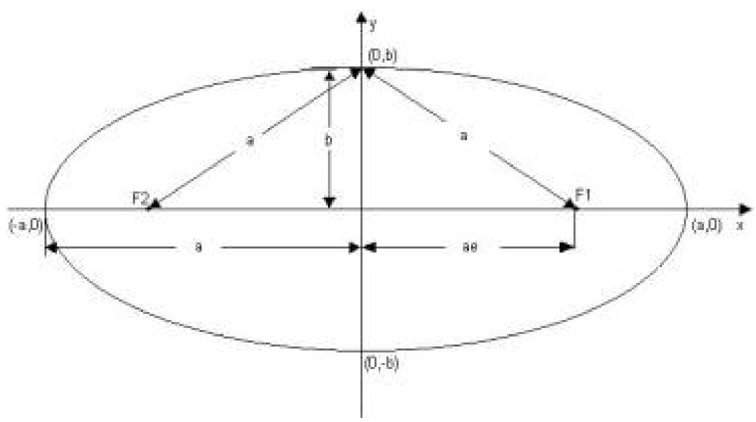


Figure 1

With reference to Figure 1, the **standard equation of the ellipse** is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Eccentricity, *e*, is defined as the ratio of the distance between the centre of the ellipse and one of the foci to the semimajor axis, or $e = OF_1/a = OF_2/a$

Then,

$$OF_1 = OF_2 = ae.$$

Since the point $B = (0,b)$ is on the ellipse, $2BF_1 = BF_1 + BF_2 = 2a$ and hence $BF_1 = a$. Similarly, $BF_2 = a$. By Pythagoras' Theorem,

$$\begin{aligned} a^2 &= b^2 + (ae)^2 \\ a^2 &= b^2 + a^2e^2 \\ b^2 &= a^2 - a^2e^2 \\ b^2 &= a^2(1-e^2). \end{aligned}$$

APPENDIX 2

Cavalieri's Principle

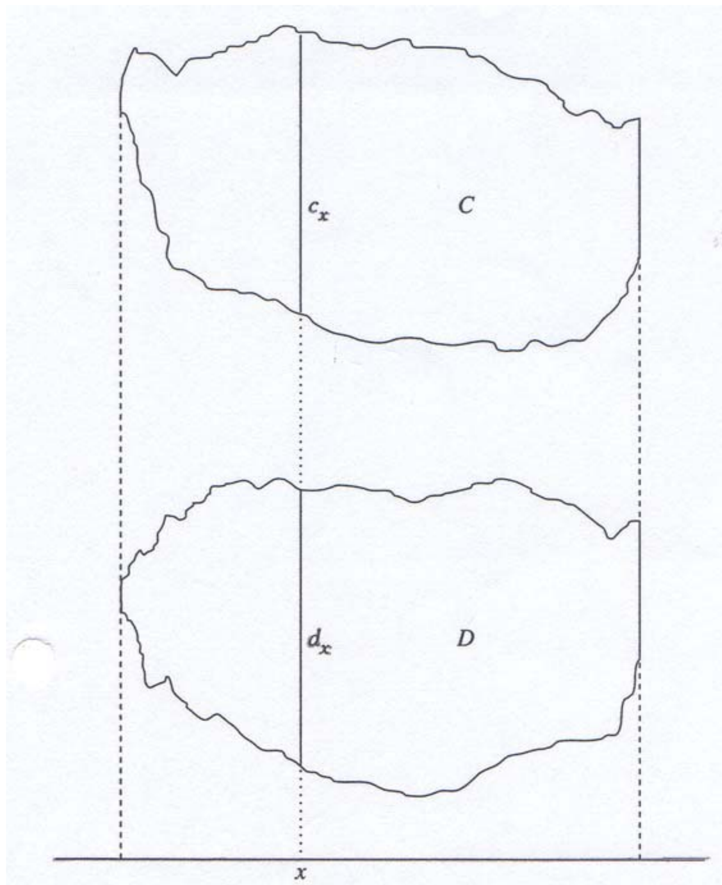


Figure 2

Referring to Figure 2, **Cavalieri's Principle** states that if $d_x = kc_x$ for all x and for a fixed positive number k, then D = kC.

Now, consider simultaneously the graph of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and that of the circle $x^2 + y^2 = a^2$,as shown in figure 3.

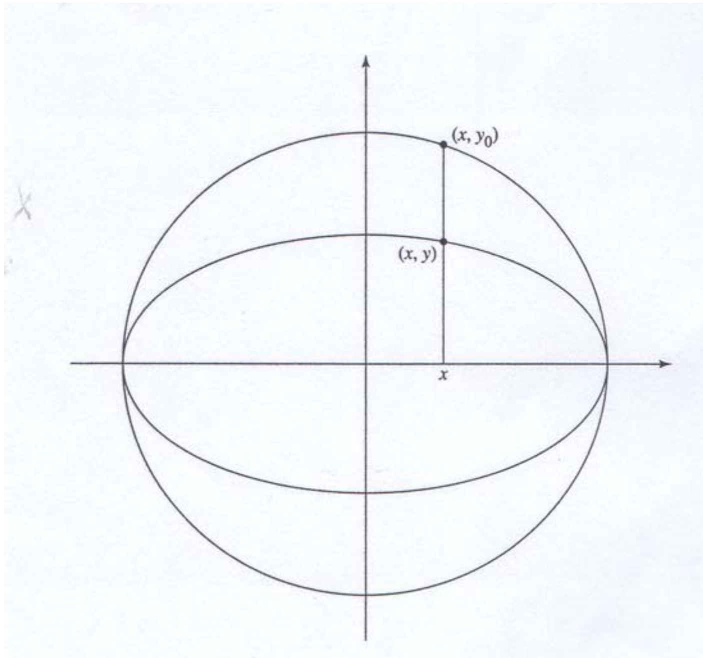


Figure 3

Let x satisfy $-a \le x \le a$ and, let (x, y) and (x, y_0) be the indicated points on the ellipse and circle, respectively. Since (x, y_0) satisfies $x^2 + y^2 = a^2$ and $y_0 \ge 0$ it follows that $y_0 = \sqrt{a^2 - x^2}$.

Since (x, y) is on the ellipse,

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2}$$
$$y^2 = \frac{b^2 a^2 - b^2 x^2}{a^2} = \frac{b^2}{a^2}\left(a^2 - x^2\right)$$
$$y = \frac{b}{a}\sqrt{a^2 - x^2} = \frac{b}{a} y_0.$$

The above relation is frequently used in later calculations. In addition, if we suppose that the upper semicircles and the upper part of the ellipse are separated as shown in Fig. 4(a), we would then have demonstrated that $d_x = kc_x$ for all x.

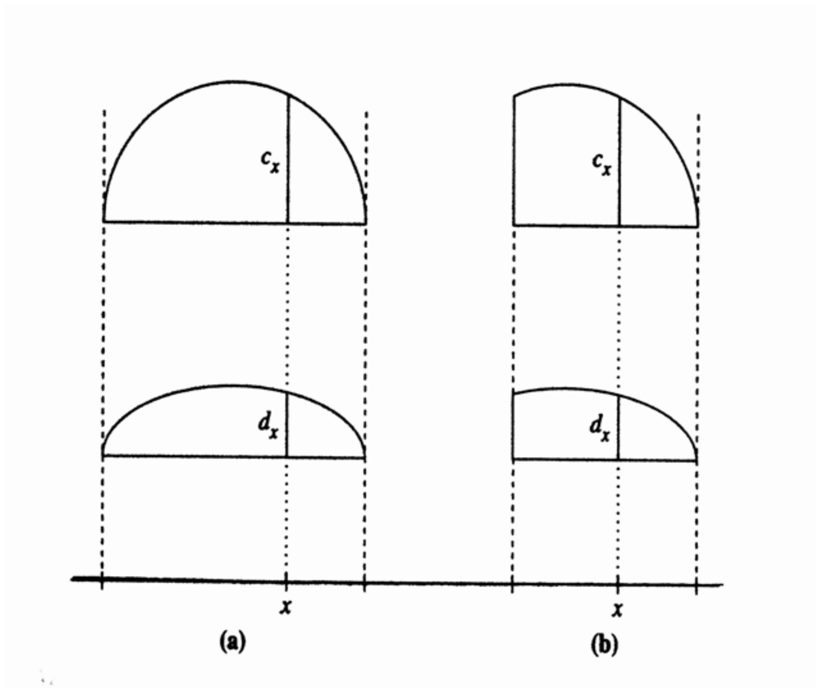


Figure 4

Since the area of a semicircle of radius *a* is 1/2ла^2, it follows by Cavelieri's principle that the area of the upper half of the ellipse is equal to b/a(1/2ла^2)= 1/2лab.

Therefore, the full ellipse with semimajor axis *a* and semiminor axis *b* has area πab. Note that Cavalieri's principle also applies to Figure 4(b). In particular, the area of the elliptical section has area $\frac{b}{a}$ times that of the semicircular section.

APPENDIX 3

SPHERICAL TRIGNOMETRY

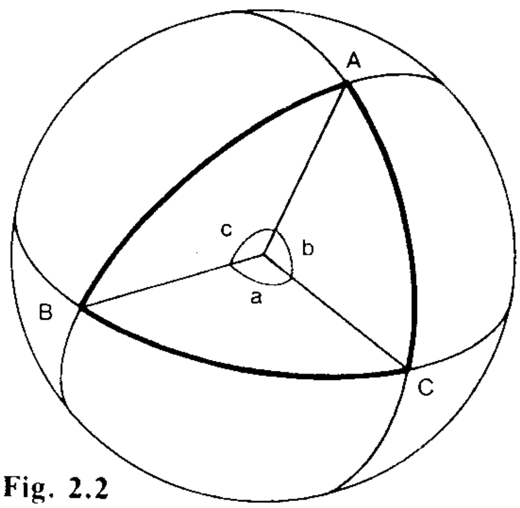


Fig. 2.2

With reference to figure 2.2, we have

sin BC/ sin a = sin AC/ sin b = sin AB/ sin c.

APPENDIX 4

SMALL ANGLE APPROXIMATIONS

Some angles dealt with in astronomy are small and may be approximated with little error caused. Consider θ being a small angle. The Maclaurin series or Taylor series about the origin for sin θ, cos θ and tan θ up to terms in θ^2 are recorded below.

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan\theta \approx \theta.$$

APPENDIX 5

BINOMIAL THEOREM

The Binomial Theorem states that for real numbers *x* and *n*,

$(1+x)^n = 1 + nx + \{[n(n-1)]/2!\}x^2 + \ldots\ldots + \{ [n(n-1)\ldots.(n-k+1)]/k!\}x^k + \ldots\ldots\ldots.$

valid for $|x| < 1$

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