Probability paradoxes

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The Car and Goat Game

You're on a game show and you're faced with three doors. Behind one of them is a car. You're asked to pick a door, and you choose door 1. Before the host opens it, she first opens door 2, and you see that there's nothing behind it. She then asks you if you are sure about door 1, or if you want to switch your guess.

Suppose you pick door 1 and that the host opens door 2. The probability that the car is behind 1 is 1/3, so you're probably wrong, but after the host opened door 2, you know that if the car isn't behind 1 (and it probably isn't), then it must be behind 3, so you should switch!

Simpson's Paradox in University admission

UC Berkeley admitted 44% of males and 35% of females who applied in 1973. Data from the six largest departments.

Department	Male acceptance rate	Female acceptance rate
A	62%	82%
B	63%	68%
C	37%	34%
D	33%	35%
E	28%	24%
F	6%	7%

	Male		Female	
	Applicants	%	Applicants	%
A	825	62%	108	82%
B	560	63%	25	68%
C	325	37%	593	34%
D	417	33%	375	35%
E	191	28%	393	24%
F	373	6%	341	7%

Boys and girls

Suppose all families continue having babies until they get a boy, at which time they stop. Suppose also that you're not allowed to have more than 4 children.

<table>
<tr><th>children</th><th>boys</th><th>girls</th><th>probability</th></tr>
<tr><td>1</td><td>1</td><td>0</td><td>1/2</td></tr>
<tr><td>2</td><td>1</td><td>1</td><td>1/4</td></tr>
<tr><td>3</td><td>1</td><td>2</td><td>1/8</td></tr>
<tr><td rowspan="2">4</td><td>1</td><td>3</td><td>1/16</td></tr>
<tr><td>0</td><td>4</td><td>1/16</td></tr>
</table>

Boys: $1 \cdot 1/2 + 1 \cdot 1/4 + 1 \cdot 1/8 + 1 \cdot 1/16 = 15/16$,
Girls: $1 \cdot 1/4 + 2 \cdot 1/8 + 3 \cdot 1/16 + 4 \cdot 1/16 = 15/16$.

There is nothing special about 4, the conclusion will be the same if we cap at another number, of if we don't cap at all.

Birthdays

Suppose you have n people. How likely is it that two people have the same birthday?

The probability of n different birthdays is:

$$\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)$$

When $n = 23$, this dips below $1/2$.

Bus Frequency

Both 97 and 197 have 9 minutes frequency.
23 times with 97 (85.2%) and 4 times with 197.
197 comes 2 min after 97!

AIDS Testing

Suppose a test is such that if you are HIV infected, the test will be positive 99.9% of the time, and if you are not infected, then you will test negative 99.5% of the time. Suppose you test positive, how likely is it that you are infected?

We define the conditional probability by

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

We will use Bayes's formula

$$\begin{aligned} P(E) &= P(E \cap F) + P(E \cap F^c) = \\ & P(E|F)P(F) + P(E|F^C)P(F^C) \end{aligned}$$

Suppose the rate of infection is r, I denotes that you are infected, and T denotes that you tested positive.

We are trying to find $P(I|T)$.

$$P(I|T) = \frac{P(I \cap T)}{P(T)} = \frac{P(T|I)P(I)}{P(T|I)P(I) + P(T|I^c)P(I^c)}$$

This equals

$$\frac{0.999r}{0.999r + 0.005(1-r)} = \frac{999r}{5 + 994r}$$

If $r = 0.001$, then $P(I|T) = 0.167$.

Benford's Law

Benford's Law states that in "naturally occuring" sets of numbers, the probability that the first digit is d is $\log_{10}(1 + 1/d)$.

digit	1	2	3	4	5	6	7	8	9
%	30.1	17.6	12.5	9.7	7.9	6.7	5.8	5.1	4.6

Imagine a stock index that starts at 100. To get to a first digit of 2, the index must increase to 200, a 100% increase. Let's say that the index goes up at a rate of about 10% a year. That means that it would take seven years to go from 1 to 2 as a first digit. But suppose we start at 900. It then takes only a little more than a year to reach 1,000 and a first digit a 1.