Mathematics in Art and Architecture GEK1518K

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The Golden Ratio

The Golden Ratio

Suppose we want to divide a line AB at a point C such that AC is longer than BC and the ratio of AC to BC is equal to the ratio of AB to AC. In other words,

whole / long = long / short.

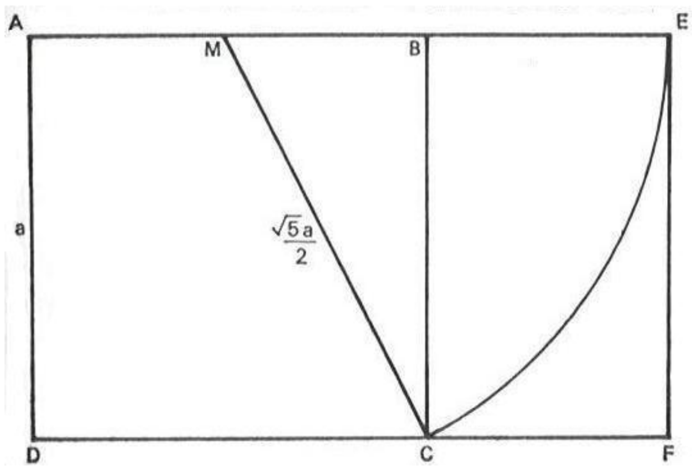
We will determine the value of this common ratio $\varphi = \frac{AC}{BC} = \frac{AB}{AC}$.

If CB = *a,* then AC = $a\,\varphi$ and AB = $a\,\varphi^2$. Hence,

$$a\,\varphi^2 = a\,\varphi + a$$
$$\varphi^2 = \varphi + 1$$

The positive solution is $\varphi = {}^{(1+\sqrt{5})}/_{2} \approx 1.618$. This value of φ is known as the *golden ratio*. It is also known as the *golden section* or the *divine ratio*.

Conversely, given a line segment AB, we may want to extend it to a line AE such that $\frac{AE}{AB} = \varphi$. We will call this a golden ration extension. To construct this extension geometrically, start with a line AB of length *a* and construct a square ABCD. Bisect AB and let the midpoint be M. With M as a center, draw an arc with radius MC to cut AB produced at E. Let us now check the ratios AE/AB and AB/BE. To do so, we need to find the lengths MC and AE.



By Pythagoras Theorem,

$$(MC)^2 = (MB)^2 + (BC)^2$$
$$= (^a/_2)^2 + a^2$$
$$= (^5/_4)\, a^2$$
$$MC = (^{\sqrt{5}}/_2)\, a$$

$$AE = AM + ME$$
$$= AM + MC$$
$$= (^1/_2)\, a + (^{\sqrt{5}}/_2)\, a$$
$$= a\,(1 + \sqrt{5})/2$$

Hence

$$\frac{AE}{AB} = \frac{a(1+\sqrt{5}/2\,)}{a} = (1 + \sqrt{5})/2,$$

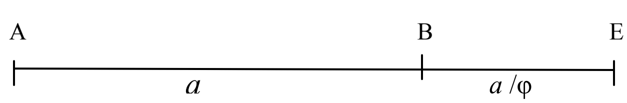
and since

$$\frac{AB}{BE} = \frac{a}{a(\sqrt{5}/2 - 1/2)} = (1 + \sqrt{5})/2,$$

we see that $\frac{AE}{AB} = \frac{AB}{BE} = \varphi$.

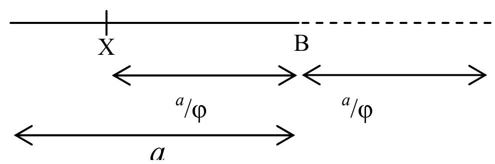
Observe that the rectangle AEFD has length $a\,\varphi$ and height a. We call a rectangle with dimensions a by $a\,\varphi$ a *golden rectangle*.

If the short segment has length 1, then the long segment has length φ and the whole segment has length φ^2. If the long segment has length 1, then the short segment has length 1/ φ and the whole segment has length φ. If the whole segment has length 1, then the long segment has length 1/ φ and the short segment has length $1/\varphi^2$



Extending line AB to ABE by a length of *a/*φ to get a golden ratio extension

The above construction shows how to get a golden ratio extension of AB to ABE. We now want to construct a golden ration cut, that is, start with AB and find a point C such that AB/AC=AC/CB = φ. It turns out that we have done all the hard work already. In the above golden ration extension, the length of BE is *a*/φ. But this is exactly the length of the long piece in a golden ratio cut. Using a compass centered at B with radius BE = $^a/_\varphi$, cut the line AB at X. So, BX = BE = $^a/_\varphi$. Notice that AX is the smaller part while BX is the larger part of the line segment because BX = $^a/_\varphi > \frac{1}{2}a$. Since $\frac{AB}{BE} = \varphi$ and BX = BE, $\frac{AB}{BX} = \frac{AB}{BE} = \varphi$.

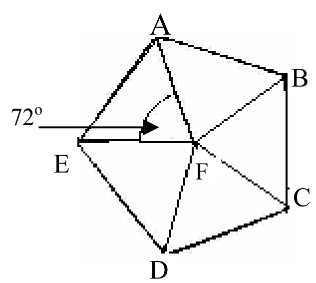


Cutting a given line AB into 2 parts to achieve the golden ratio

Now we know how to do both a golden ration extension and a golden ratio cut.

Constructing a Regular Pentagon using the Golden Ratio

We will use the golden ratio to help us construct a regular pentagon. Observe that if we divide a regular pentagon ABCDE into 5 isosceles triangles as shown in Figure 5.2.1, then, for each isosceles triangle, the interior angle at F is 72^o.



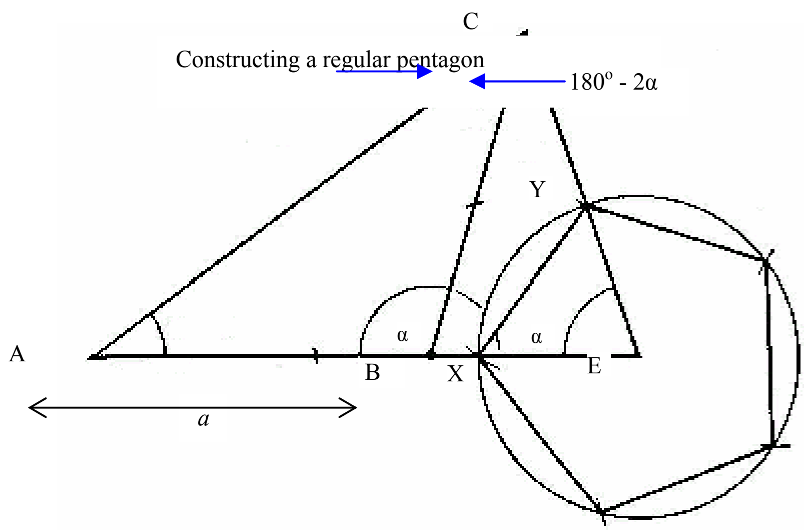
A regular pentagon

We will construct a regular pentagon within a circumscribing circle. To do that, we need a way to construct the angle of 72^o. Euclid came up with the following method to get this angle.

1) Begin with a line ABE, which is divided into the golden ratio at B, with AB longer than BE.
2) Using compasses centered at B and E and of radius AB, draw two arcs of circles to intersect at point C. So AB = BC = CE and we will prove later that AC = AE.
3) Form the triangle CEB. Since BC = CE, so $\angle CBE = \angle CEB.$ Let these two angles be α. So $\angle BCE = 180^o - 2\alpha$.
4) Consider the triangle ACE. Since AC = AE, $\angle ACE = \angle CEB = \alpha.$ So $\angle CAB = 180° - 2\alpha.$

5) Consider the triangle ACB. $\angle ACB = \angle ACE - \angle BCE = 3\alpha. - 180°$.
Since AB = BC, $\angle CAB = 3\alpha. - 180°$.
6) From (4) and (5), $\angle CAB = 3\alpha - 180° = 180° - 2\alpha$. Solving, we get $\alpha = 72°$.

Now that we obtain the angle $\alpha = 72°$ by the above construction, construct a circle at center E, radius less than BE. The circle cuts BE and CE at X and Y respectively. X and Y form two vertices of our regular pentagon. We can get the remaining three vertices of the pentagon using compasses with radius XY and cutting the circle.

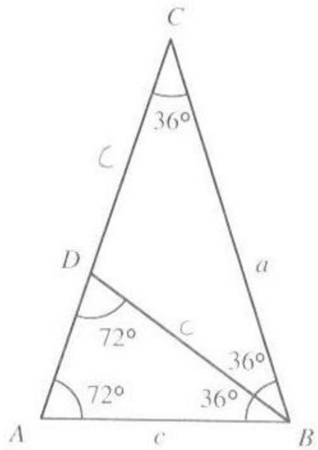


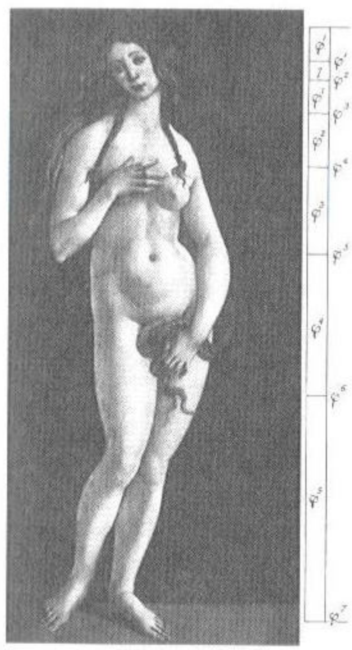
Now, we prove that AC = AE. Let the length of AB = *a* and so, BE = $^{a}/_{\varphi}$. So AE = $a\,(1 + {}^{1}/_{\varphi})$. By Equation (5.1), AE = $a\,\varphi$. So we want to prove that AC = $a\,\varphi$.

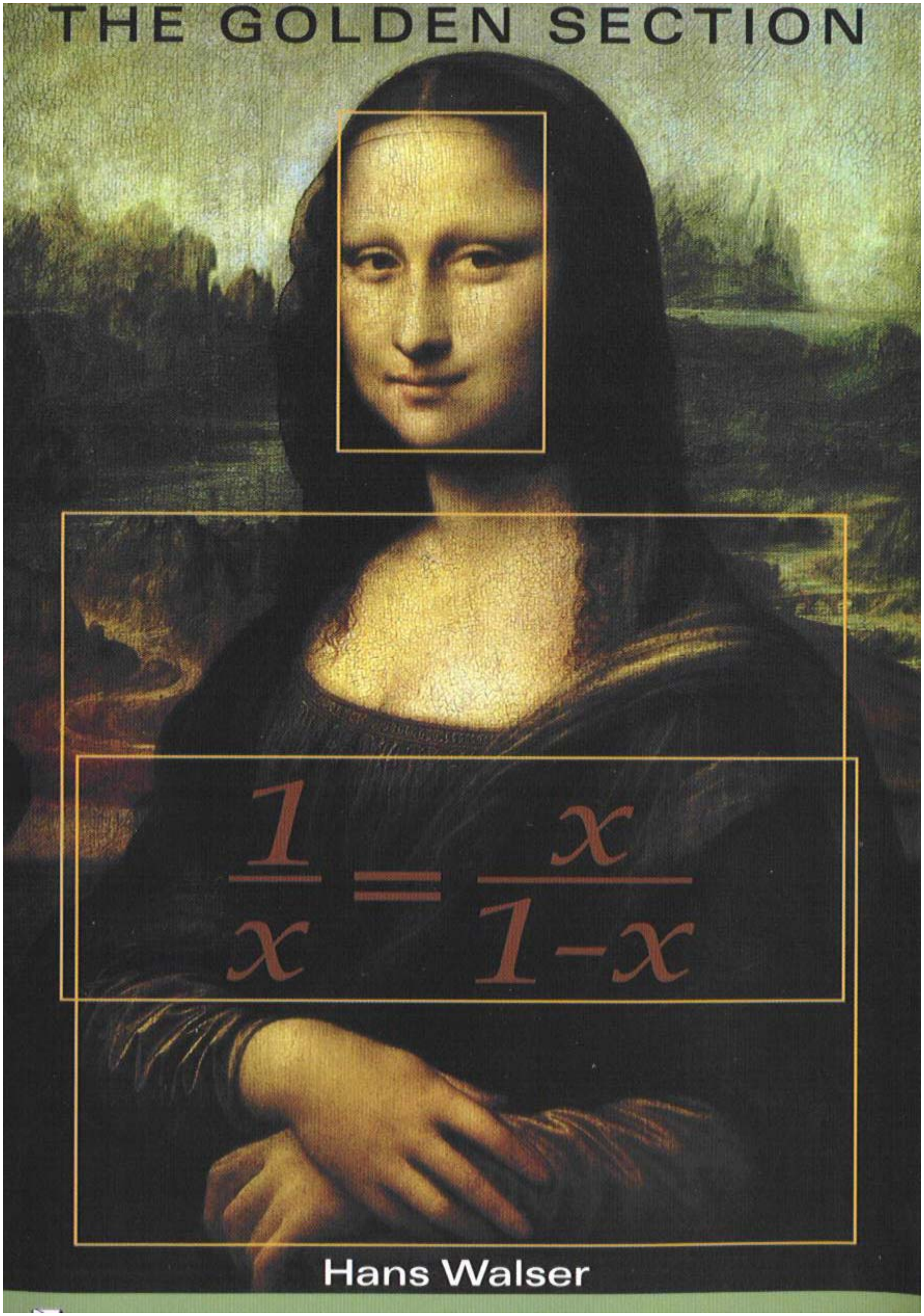
Find the mid-point *N* of BE and the line CN is perpendicular to BE. We apply Pythagoras Theorem to get AC.

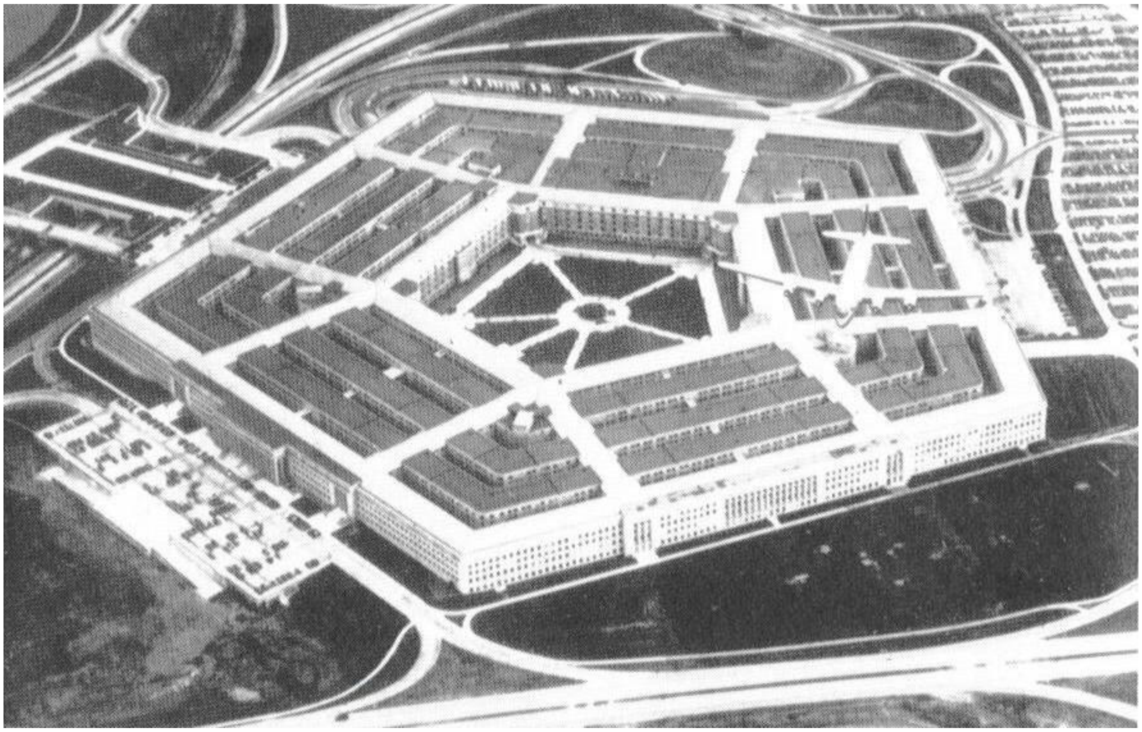
$$\begin{aligned}(AC)^2 &= CN^2 + AN^2 \\ &= (CE^2 - NE^2) + (AB + BN)^2 \\ &= (a^2 - (^{a}/_{2\varphi})^2) + (a + (^{a}/_{2\varphi}))^2 \\ &= a^2(2 + {}^{1}/_{\varphi}) \\ &= a^2\,(1 + \varphi) \quad \text{by Equation (5.1)} \\ &= a^2\,\varphi^2 \qquad\qquad \text{by Equation (5.2)}\end{aligned}$$

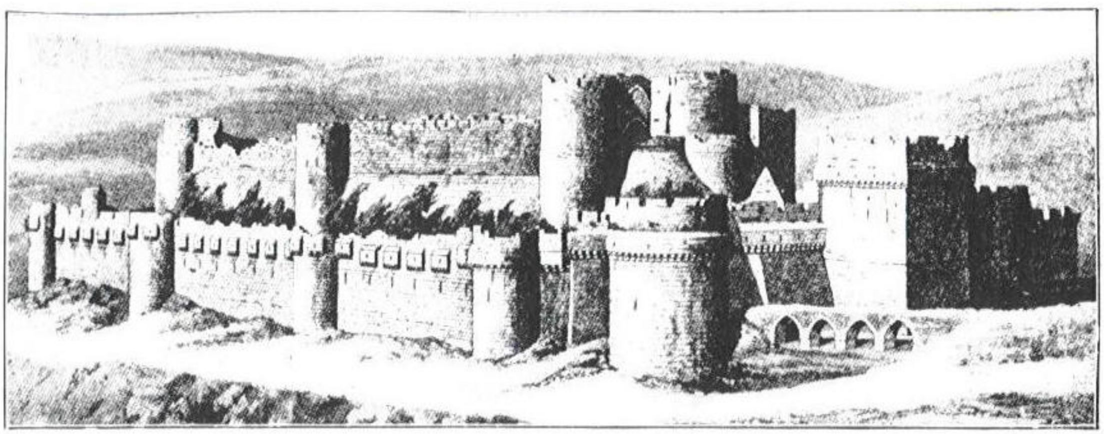
Therefore, AC = $a\,\varphi$ = AE.

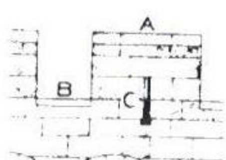

C
36°
c
D
a
72°
c
36°
72°
36°
A
c
B



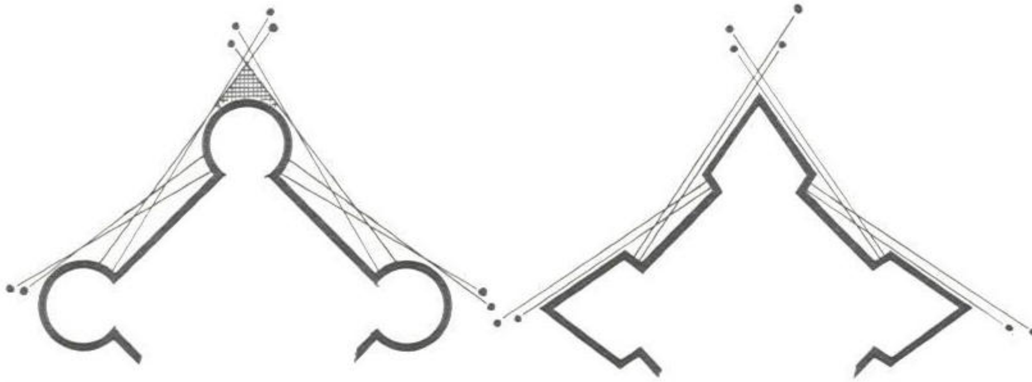

THE GOLDEN SECTION
$\frac{1}{x} = \frac{x}{1-x}$
Hans Walser







Battlement
A, Merlon; B, Crenel,
C, Loophole; D, Ma-
chicolation



Comparison of (left) *ground plan of a medieval castle with circular towers, and* (right) *a bastioned plan, showing how the new design eliminated the dead ground* (shaded) *at the foot of the tower.*

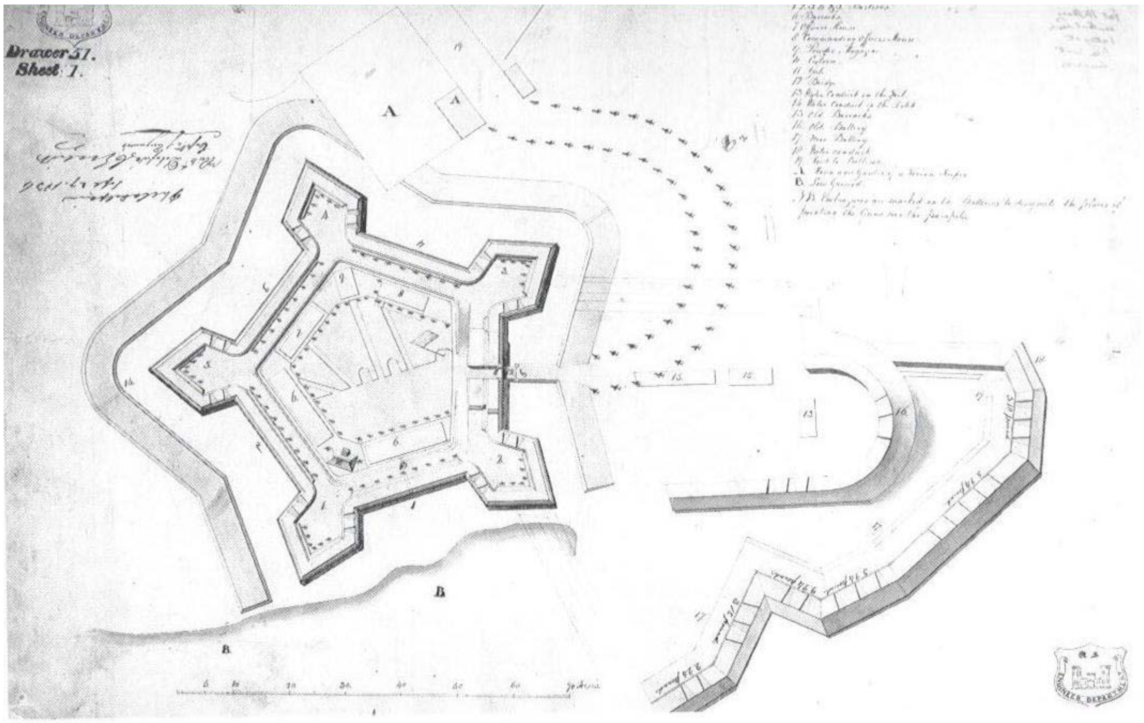


Figure 4.3 Fort McHenry, Maryland (1798). Jean Foncin, engineer. Plan (1803). *National Archives, Washington, D.C.*