Mathematics in Art and Architecture GEK1518

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Symmetry and Patterns

Introduction

In this chapter, we are interested in the underlying structure of aesthetically pleasing plane figures. By a *plane figure*, we mean any subset of the plane. To understand the underlying structure, we examine the symmetries of the figure. An *isometry* of the plane is a distance-preserving transformation of the plane. This means that for any pair of points P and Q, the distance between the images under the transformation is the same as the distance between P and Q. A *symmetry* of a figure is an isometry that maps the figure back onto itself. There are four types of isometries, namely translation, reflection, rotation and glide reflection. A short description of each isometry of the plane is given in the next section. We will classify a figure in terms of its *symmetry group*, which is the set of all symmetries of the figure. If a figure admits a symmetry, it is said to be *symmetrical*.

We will first give a brief description of the four types of isometries of the plane after which we will show how to recognise the symmetry group of a figure.

1.2 The Four Isometries of the Plane

Translation

A *translation* of a plane figure shifts the figure in a given direction. Under a translation, there are no fixed points; every point moves by exactly the same distance, *d*.

@ @ @ @

Fig 1.2.1 Horizontal translation

↑
↑
↑
↑

Fig 1.2.2 Vertical translation

Note that a translation need not take place horizontally or vertically. It can take place in any direction.

Reflection

A *reflection* is a mapping of all points of the original figure onto the other side of a "mirror line" such that the distance between the image (the figure that is mapped from the original one) and the mirror line is the same as that between the original figure and the mirror line. The mirror line is known as the *axis of reflection*.

➢

Fig 1.2.3a

The above figure is obtained by reflecting the ≺ in the vertical reflection axis:

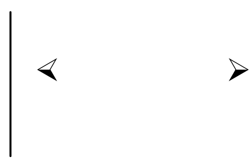


Fig 1.2.3b Reflection about a vertical axis.

Reflection can also take place in the horizontal axis as shown below:

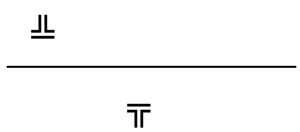


Fig 1.2.4 Reflection about a horizontal axis

In fact, the reflection axis needs not be a vertical line or a horizontal line; it can be any straight line at any angle to the horizontal.

For example,

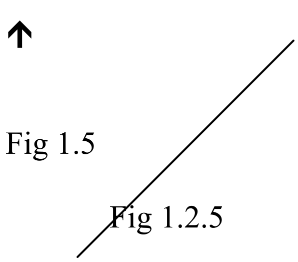


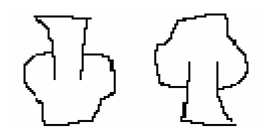
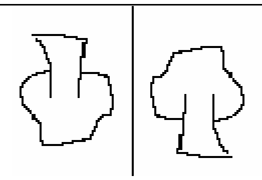
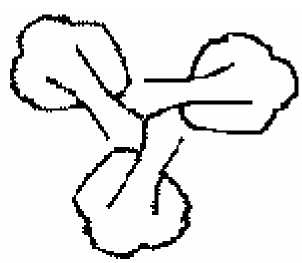
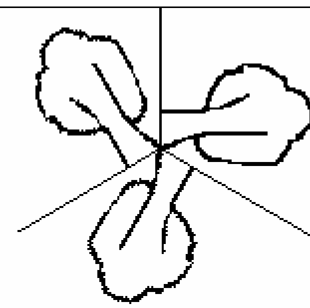
Fig 1.5

Fig 1.2.5

Rotation

We completely specify a *rotation* when we know its *centre* and *angle of rotation*. A figure with an angle of rotation θ is said to have an *order of rotation n* if $n = 360^{\circ}/\theta$ and *n* is a natural number. We define a *symmetry region* as a subset of the figure which generates the whole figure by rotations.

The table below illustrates the concepts of angle, order and centre of rotation. Please refer to the explanation beneath the table to understand what is illustrated in the table.

Order of Rotation	**Angle of Rotation**	**Figure**	**Symmetry Regions**
2	180°	 1.2.6a	 1.2.6b
3	120°	 1.2.7a	 1.2.7b

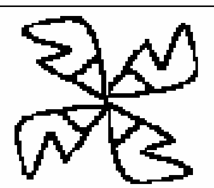
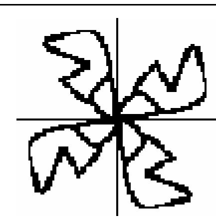
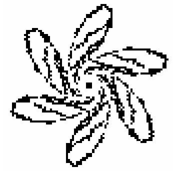
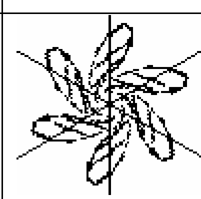
Order of Rotation	**Angle of Rotation**	**Figure**	**Symmetry Regions**
4	90°	 1.2.8a	 1.2.8b
6	60°	 1.2.9a	 1.2.9b

Table 1.2.1

For each figure in the third column, we obtain information about its angle of rotation by drawing lines as shown in the forth column. The lines show three details:

The lines divide the figure into smaller regions, such that each region is identical (except in the orientation of the figure in each region) and each region is the smallest possible such that sufficient rotations of it will generate the whole figure. The number of such regions as partitioned by the lines will give the *order of rotation* of the figure.

The angle of intersection of the lines shows the *angle of rotation*.

The point of intersection of the lines shows the *centre of rotation*.

Note that for Fig 1.2.6a, there is only one line dividing the figure. This line divides the figure into two identical regions (apart from the orientation of the figure in each region), so the order is two and the angle of rotation is 180°. Now, the question is: where is the centre of rotation since there is no intersection of lines? Take a point *p* in the original figure and locate the image of *p*. Call this image *p'*. Draw the line *pp'* and the midpoint of this line is the centre of rotation of the figure.

Glide Reflection

A *glide reflection* is a combination of two transformations: a reflection and a translation. Whether the original figure is reflected or translated first does not affect the final result; the final figure generated will be the same.

Consider the following figure, in which the original figure is ╩.

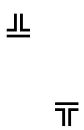
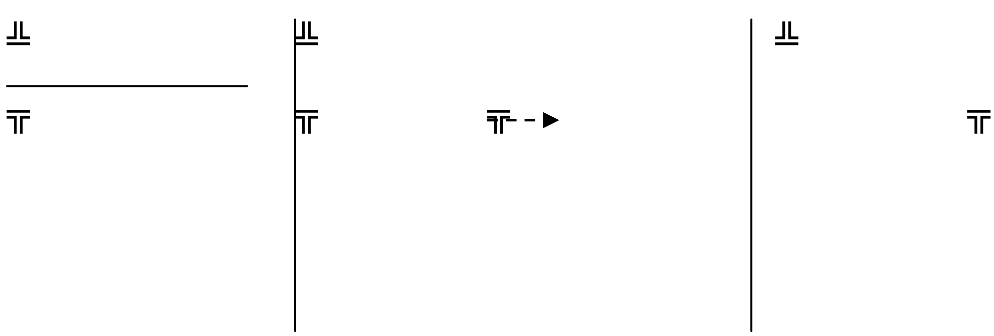


Fig 1.2.10

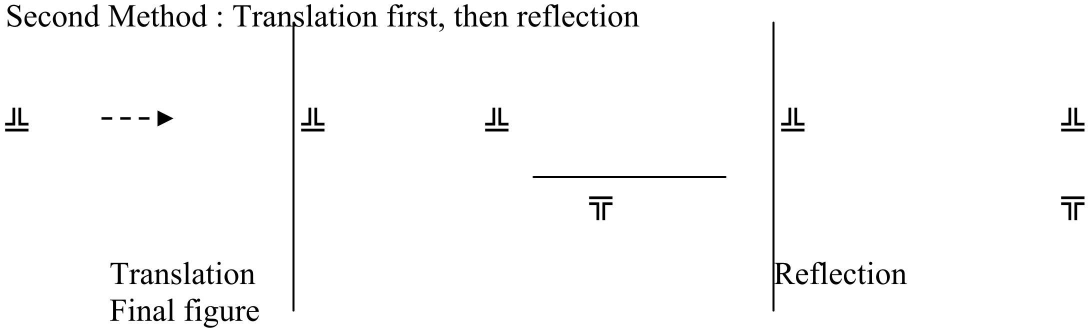
Let us try to obtain Figure 1.2.10 from ╩ using two methods, which differ only in the sequence of reflection and translation transformations.

First Method : Reflection first, then translation



Reflection Translation
Final figure

Second Method : Translation first, then reflection

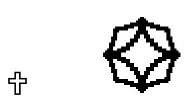


A glide reflection is *non-trivial* if its translation component and reflection component are not symmetries of the figure. We say a figure admits a glide reflection if and only if the glide reflection is non-trivial. The *glide reflection axis* is the axis of its reflection component.

We are done with the description of the four transformations.

1.3 Designs, Patterns, Motifs and Fundamental Regions

In our context, we will call figures with at least one (non-trivial[1]) symmetry *designs*. We will call designs that have a translation symmetry *patterns*. Note that patterns are unbounded figures, since with translation, the figures must extend to infinity. We will be concerned with only two types of patterns, namely *frieze patterns* which admit translation in only one direction and *wallpaper patterns* which admit translations in two or more directions. In particular, each pattern has a *basic unit* which is a smallest region of the plane such that the set of its images under translations of the pattern generates the whole pattern. Finally, we call designs that do not admit translation symmetry *finite designs*. Note that since the square of a glide reflection is a translation, therefore a finite design can only admit rotation and reflection symmetries.



a b

Fig 1.3.1 Examples of finite designs

[1] The trivial symmetry is the identity transformation, which maps the figure back to itself.

Fig 1.3.2 A frieze pattern

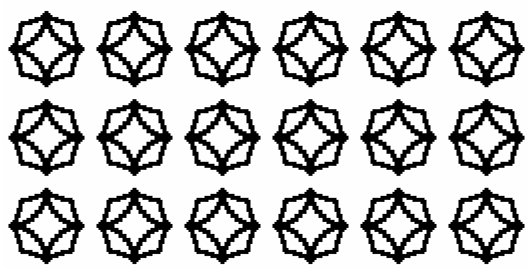


Fig 1.3.3 A wallpaper pattern

Some comments have to be made about Figures 1.3.2 and 1.3.3. Though by definition patterns extend to infinity, it is not possible in real life to have infinitely many translations of a figure. Therefore, as a general rule, we consider a figure to be a frieze pattern if it has at least the basic unit and a copy of it by translation. For a plane figure to qualify as a wallpaper pattern, it must have at least the basic unit, one copy by translation, and a copy of these two by translation in a second direction. There must be at least two rows, each one at least two units long. Hence, it can be seen easily that Figure 1.3.2 is a frieze pattern with as a basic unit and Figure 1.3.3 is a wallpaper pattern with the basic unit .

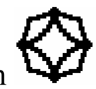
Let us now discuss the generation of a wallpaper pattern. Within each basic unit, we can find a smallest region in the basic unit whose images under the full symmetry group of the pattern cover the plane. This smallest region is known as a *fundamental region*. If the pattern consists of a figure on a plain background, we can instead focus on the part of the figure that generates the pattern. A *motif* is a subset of a fundamental region which has no symmetry but which generates the whole pattern under the symmetry group of the pattern. To understand the difference between a fundamental region and a motif, think of the fundamental region as a region, which together with the symmetry group, determines the structure of the patterns. The motif on the other hand, determines the appearance of the pattern. To make the idea clearer, let us use Figure 1.3.3 to illustrate what is a fundamental region and a motif.

Firstly, observe that the symmetry group of Figure 1.3.3 comprises vertical reflection, horizontal reflection, reflection with axis at 45^o clockwise from horizontal, reflection at 135^o clockwise from horizontal, 90^o rotation, vertical translation and horizontal translation. A fundamental region of this wallpaper pattern is a triangular region with

the motif . So the fundamental region is indeed . Note that the grey shading of the triangular fundamental region has no role in the generation of the pattern; the shading is done to allow the reader to see the shape of the fundamental region. The following sequence of steps is just one of the ways the region can generate the pattern.

Reflect  horizontally to form .

Rotate by 90^o to form  where the dot represents the center of rotation.

Reflect about the axis that is 135^o counter-clockwise from horizontal to form . So we get a unit of the pattern.
Translate the unit vertically and horizontally and we get the desired pattern.

1.4 Symmetry Groups of Finite Designs

Recall that a *finite design* is a figure that

has at least one of rotation symmetry or reflection symmetry
does not have translation symmetry.

We can categorise finite designs into two classes of symmetry groups C_n and D_n. C_n refers to cyclic groups of order n. Designs which fall into type C_n are those which have rotation of order n but no reflection symmetry. On the other hand, D_n refers to dihedral groups of order n. Designs which fall into type D_n have exactly n distinct reflection axes and rotation of order n.

A note of caution though: C_1 is the group for a finite figure which has no symmetry at all- neither rotational nor reflection. D_1 is the group for designs which have reflection symmetry but no other symmetries.

Examples of designs in cyclic groups are shown in Table 1.2.1:

Fig 1.2.6a is of C_2,
Fig 1.2.7a is of C_3,
Fig 1.2.8a is of C_4,
Fig 1.2.9a is of C_6.

We can have designs in C_{50}, C_{999}, or of even higher orders! Therefore, cyclic groups form an infinite class.

Let us now move on to dihedral groups. The following table illustrates some examples of designs in dihedral groups.

Order of Dihedral Group	Figure	Figure with Reflection Lines

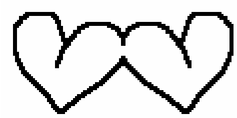
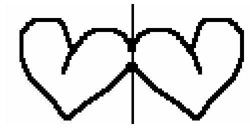
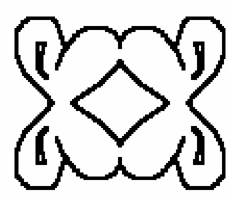
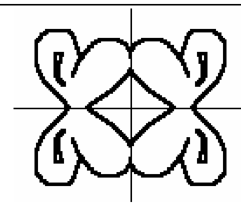
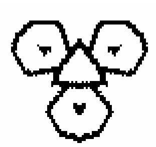
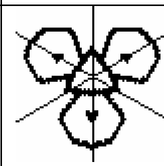
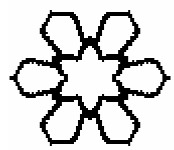
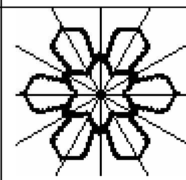
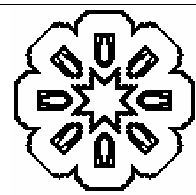
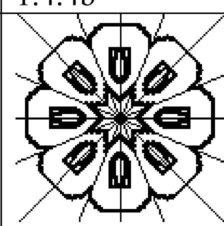
1	 1.4.1a	 1.4.1b
2	 1.4.2a	 1.4.2b
3	 1.4.3a	 1.4.3b
6	 1.4.4a	 1.4.4b
8	 1.4.5a	 1.4.5b

Table 1.4.1

For each dihedral group, we a[illegible]etries the way we do for cyclic designs. Observe that the reflection axes also partition the figure into the symmetry regions.

As in the case of cyclic groups, dihedral groups also form an infinite class.

1.5 Symmetry Groups of Frieze Patterns

Like finite designs, frieze patterns can also be classified according to the kinds of symmetries they admit. There are seven classes of frieze patterns. Unlike cyclic and dihedral groups which are infinite classes, there are only finite number of classes into which frieze patterns can be put into. The reader is invited to take a look at the proof in Appendix 2 of Washburn and Crowe **[1]**.

The symmetry groups of frieze patterns are named in the form of a four-symbol notation *pxyz*. Each name begins with the letter p. The following informs the reader how to derive the rest of the four-symbol notation for each symmetry group for frieze patterns.

x = { m if there is vertical reflection
{ 1 otherwise

y = { m if there is horizontal reflection
{ a if there is a glide reflection but no horizontal reflection
{ 1 otherwise

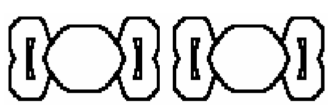
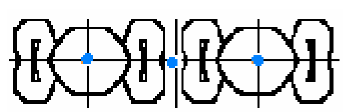
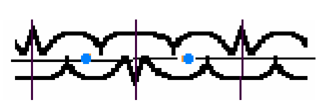
z = { 2 if there is a half turn
{ 1 otherwise

We can also use a two-symbol notation *xy* proposed by Senechal (1975):

x = { m if there is vertical reflection
{ 1 otherwise

y = { m if there is horizontal reflection
{ g if there is a glide reflection with no horizontal reflection
{ 2 if there is a half-turn with no glide nor horizontal reflection
{ 1 otherwise

The following table illustrates some examples of each of the seven symmetry groups for frieze patterns.

Symmetry Group 4-symbol/ 2-symbol	Figure	Figure with symmetry axes and centers of rotation as blue dots	Isometries present (besides translation)
pmm2 / mm	 Fig 1.5.1a	 Fig 1.5.1b	vertical and horizontal reflections; 2-fold rotation[2]
pma2 / mg	 Fig 1.5.2a	 Fig 1.5.2b	vertical reflection; horizontal glide reflection;

[2] A 2-fold rotation is a rotation of order 2. In general, a *n-fold rotation* is a rotation of order n.

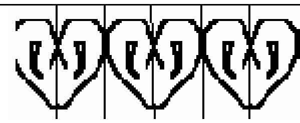
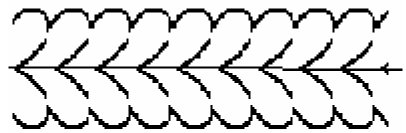
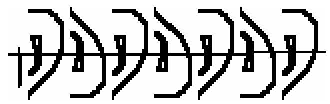
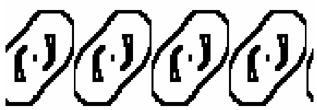
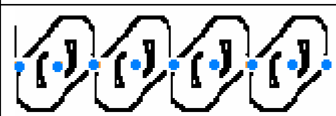
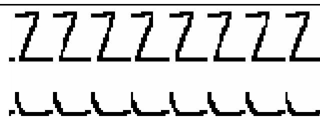
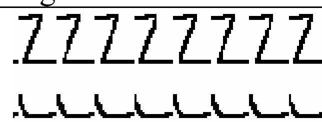
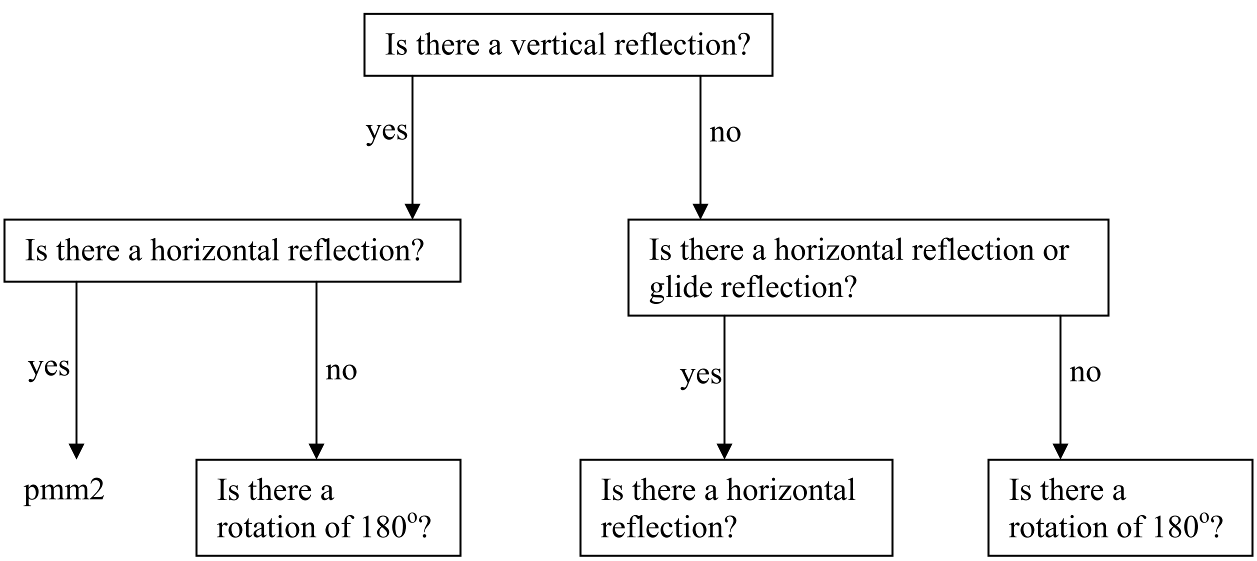
			2-fold rotation
pm11 / m1	 Fig 1.5.3a	 Fig 1.5.3b	only vertical reflection
p1m1 / 1m	 Fig 1.5.4a	 Fig 1.5.4b	only horizontal reflection
p1a1 / 1g	 Fig 1.5.5a	 Fig 1.5.5b	only horizontal glide reflection
p112 / 12	 Fig 1.5.6a	 Fig 1.5.6b	only 2-fold rotation.
p111 / 11	 Fig 1.5.7a	 Fig 1.5.7b	No other symmetry except translation.

Table 1.5.1

An Approach to Analyse the Symmetry Groups of Frieze Patterns

I would recommend the use of the following flow chart to analyse the symmetry groups of frieze patterns.



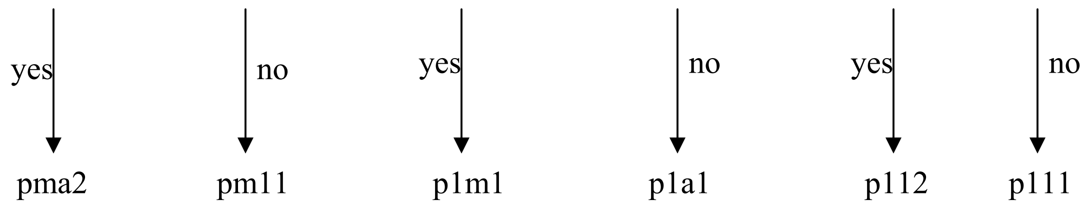


Chart 1.5.1 Flow chart for the 7 symmetry groups of frieze patterns

1.6 Symmetry Groups of Wallpaper Patterns

Before we go on to discuss the symmetry groups of wallpaper patterns, we need to learn about *lattices of points* and *primitive cells* of the patterns. We obtain a lattice of points of a wallpaper pattern by the following method:

Start by choosing a point *p*. If the pattern admits rotations, then *p* is a centre of rotation of the highest order. If the pattern does not have rotational symmetry, then *p* is any arbitrary point in the pattern.
Apply translations of the pattern on *p*. The set of all images of *p* under the translations form the lattice.

A *primitive cell* is a parallelogram such that the following hold:
Its vertices are lattice points.
It contains no other lattice points inside it or on its edge.
The vectors which form the sides of this parallelogram generate the translation group of the pattern.
There are only five types of primitive cells: parallelogram, rectangular, square, rhombic and hexagonal (the hexagonal cell is a rhombus consisting of two equilateral triangles). A descriptive proof is given by Schattschneider **[2]**.

To determine which cell the pattern actually takes, we need to obtain a lattice of points and form a parallelogram whose vertices are lattice points and whose interior and edges have no other lattice points. Sometimes, we get more than one type of parallelogram using the way just described. When such circumstances arise, we need a way to choose which parallelogram is the primitive cell. Below is a method to determine what is the primitive cell of each wallpaper pattern.

A pattern with order of rotation three or six takes a hexagonal cell.
A pattern with order of rotation four takes a square cell.
For a pattern with order of rotation one or two,
if there is neither reflection nor glide reflection, then it takes a parallelogram.
otherwise, it takes either a rectangular cell or a rhombic cell. Plot the lattice and draw a parallelogram as described initially.
It takes a rectangular cell if the four corners of the primitive cell are right angles.

Otherwise, it takes a rhombic cell.

The following table illustrates the lattice of points and the primitive cell.

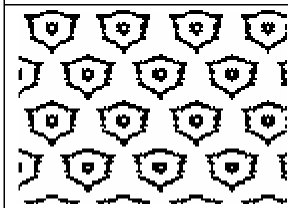
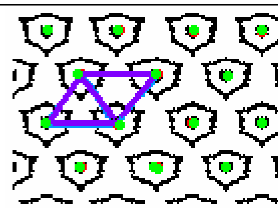
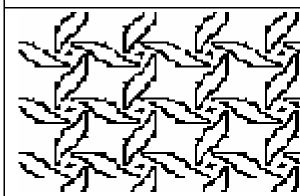
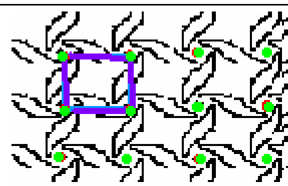
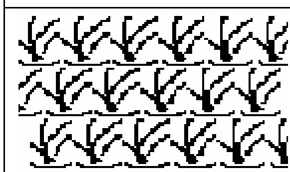
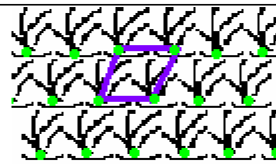
Figure	Lattices (green dots) with primitive cell (in purple)	Type of primitive cell
 1.6.1a	 1.6.1b	Hexagonal (Observe that it is made up of 2 equilateral triangles.)
 1.6.2a	 1.6.2b	Square
 1.6.3a	 1.6.3b	Parallelogram

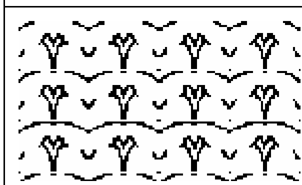
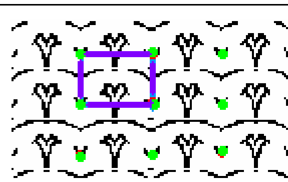
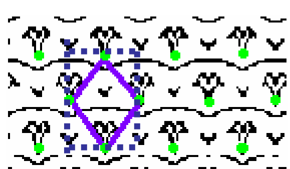
Figure	Lattices (green dots) with primitive cell (in purple)	Type of primitive cell
 1.6.4a	 1.6.4b	Rectangular
 1.6.5a	 1.6.5b	Rhombic

Table 1.6.1

Note that for a rhombic cell, we extend it to a rectangle as shown dotted in the above table. This extended cell is known as a *centred cell*. The notion of the centred cell is useful only in the notation of the symmetry groups of wallpaper patterns.

Now, we are ready to talk about the symmetry groups of wallpaper patterns. The names for the symmetry groups describing wallpaper patterns, like the frieze patterns, adopt a four-symbol notation *qrst*. The notation comes from crystallographers who use it to classify crystals. The interpretation of the crystallographic notation is as follows:

q = { p if the primitive cell is not a centred cell
{ c if the primitive cell is a centred cell

r = n, the highest order of rotation

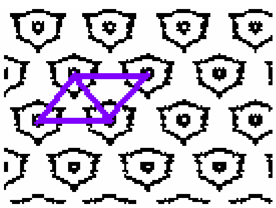
s denotes a symmetry axis normal to the left edge of the primitive or centred cell
This left edge is known as the *x-axis*.
s = { m if there is a reflection axis
{ g if there is no reflection axis, but a glide-reflection axis

{ 1 if there is no symmetry axis

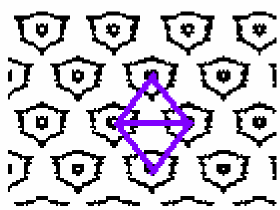
t denotes a symmetry axis at angle θ (≤ 180^o) to the x-axis.
In particular, θ = 180^o if n = 1 or 2 ; θ = 45^o if n = 4 ; θ = 60^o if n = 3 or 6.
t = { m if there is a reflection axis
{ g if there is no reflection, but a glide-reflection axis
{ 1 if there is no symmetry axis

No symbols in the third and forth positions indicate that the group contains neither reflections nor glide-reflections.

Some comments have to be made about the left edge of the cell that is known as the *x-axis*. Recall in 3-dimensional space with (x,y,z) co-ordinate system, the x-axis is the axis that points towards the reader. In other words, the x-axis is the left axis in the horizontal x-y plane. Hence, defining the left edge of cells as the x-axis makes sense to the crystallographers who deal with symmetry groups of 3-dimensional objects. However, here, we are dealing with the groups for 2-dimensional patterns. The x-axis in our wallpaper patterns is not necessarily the literal left edge of the primitive cells. To get the x-axis of the primitive cell of a wallpaper pattern requires the pattern to be aligned in a certain way or choosing the primitive cell in a certain way. For example, observe the following figures and decide what is the left edge of the primitive cell.



a



b

Fig 1.6.6 Choosing the hexagonal primitive cell in different ways for the same pattern

For Figure 1.6.6a, it is easy to see where is the left edge of the cell, but for Figure 1.6.6b, it is ambiguous where the left edge is! I suggest the following way to determine the x-axis, the third symbol and the forth symbol of the notation of the symmetry groups.

1) Draw the primitive cell. If it is a rhombic cell, replace it by a centred cell.

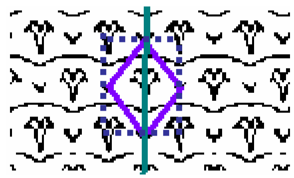
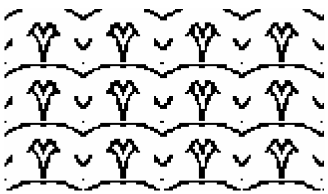
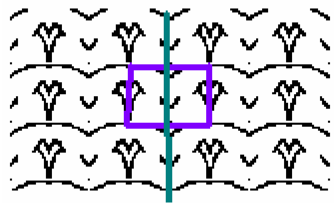
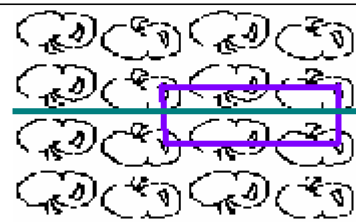
2) i) If there is a reflection that is perpendicular to an edge of the cell, that edge is the x-axis and the third symbol is *m*.

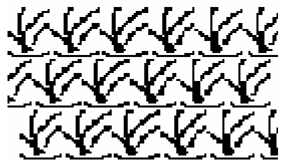
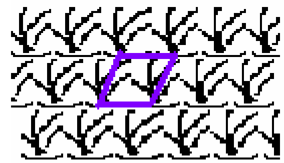
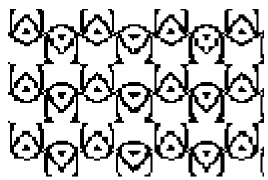
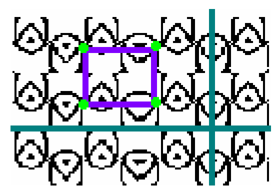
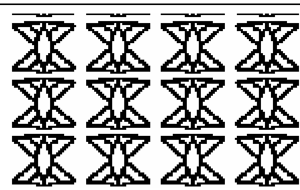
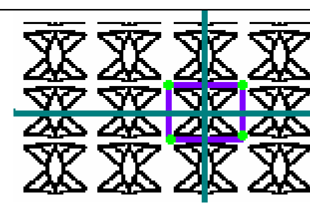
   ii) Otherwise, if there is a glide reflection axis that is perpendicular to an edge of the cell, that edge is the x-axis and the third symbol is *g*.

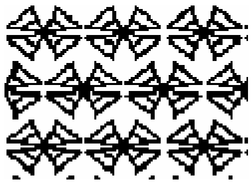
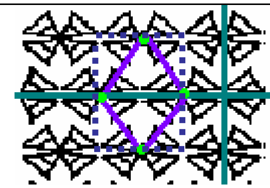
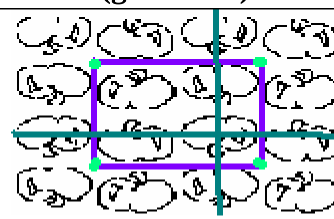
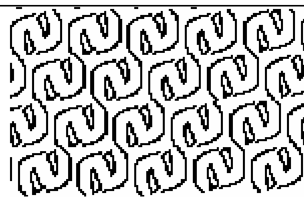
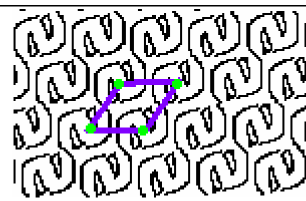
   iii) Otherwise, any edge is the x-axis and the third symbol is *1*.

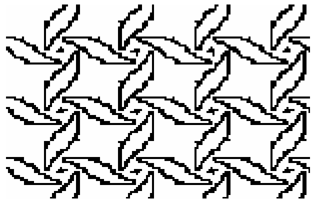
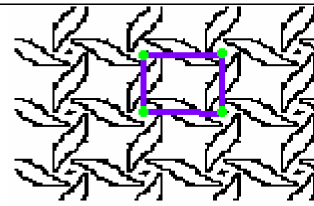
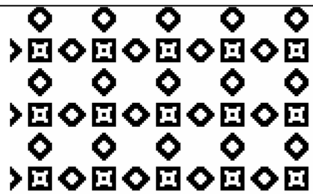
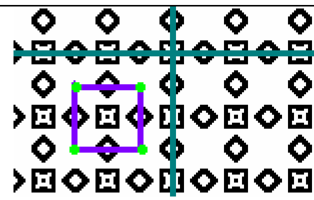
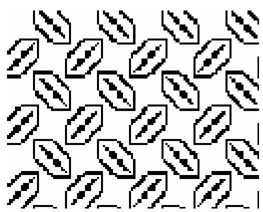
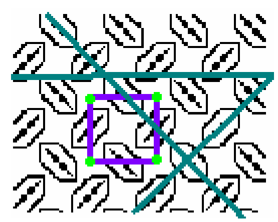
3) If there is a reflection or glide reflection axis that is not normal to the x-axis, then the forth symbol is m or g respectively. Otherwise, the forth symbol is 1.

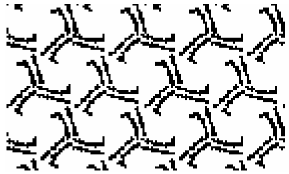
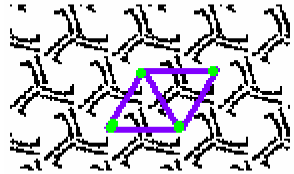
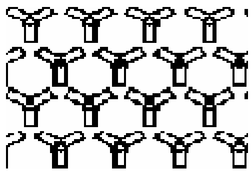
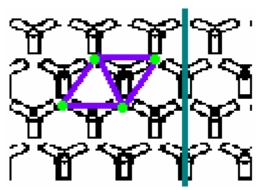
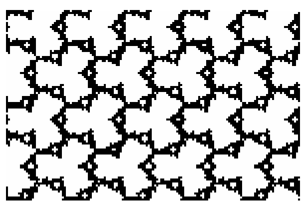
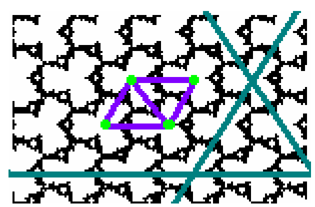
The table on the following pages illustrates examples of wallpaper patterns for each symmetry group. In all, there are seventeen wallpaper symmetry groups.

Notation For Symmetry Group (4-symbol / short form)	**Figure**	**Figure with primitive cell (in purple) symmetry axis (in dark green) centre of rotation (green dot)**	**Remarks on deriving the correct symmetry group (Note: let a glide reflection be denoted as a glide for convenience)**
c1m1/cm	 1.6.6a	 1.6.6b	Centred cell. There is a reflection axis normal to the horizontal edges of the centred cell. Pick one of these 2 edges as the x-axis. So the 3rd symbol is m. No other symmetry.
p1m1/pm	 1.6.7a	 1.6.7b	Rectangular cell. There is a reflection axis normal to the horizontal edges of the cell. Pick one of these 2 edges as the x-axis. So the 3rd symbol is m. No other symmetry.
p1g1/pg	 1.6.8a	 1.6.8b	Rectangular cell. No reflection axes. There is a glide axis normal to the vertical edges of the cell. Pick one of these 2 edges as the x-axis. So the 3rd symbol is g. No other symmetry.

Notation For Symmetry Group (4-symbol / short form)	**Figure**	**Figure with primitive cell (in purple) symmetry axis (in dark green) centre of rotation (green dot)**	**Remarks on deriving the correct symmetry group (Note: let a glide reflection be denoted as a glide for convenience)**
p1/p1	 1.6.9a	 1.6.9b	Parallelogram cell. No other symmetry besides translations.
p2mg/pmg	 1.6.10a	 1.6.10b	Rectangular cell. 2-fold rotation. There is a reflection axis normal to the horizontal edges of the cell. Pick one of these 2 edges as the x-axis. So the 3rd symbol is m. There is a glide axis parallel to the x-axis. So the 4th symbol is g.
p2mm/pmm	 1.6.11a	 1.6.11b	Rectangular cell. 2-fold rotation. Both reflection axes are normal to some edges of the cell. So the x-axis is any of the edges and both the 3rd and 4th symbols are m.

c2mm/cmm	 1.6.12a	 1.6.12b	Centred cell. 2-fold rotation. Both reflection axes are normal to some edges of the centred cell. So the x-axis is any of the edges and both the 3rd and 4th symbols are m.
Notation For Symmetry Group (4-symbol / short form)	**Figure**	**Figure with primitive cell (in purple) symmetry axis (in dark green) centre of rotation (green dot)**	**Remarks on deriving the correct symmetry group (Note: let a glide reflection be denoted as a glide for convenience)**
p2gg/pgg	 1.6.13a	 1.6.13b	Rectangular cell. 2-fold rotation. No reflection. Both glide axes are normal to some edges of the cell. So the x-axis is any of the edges and both the 3rd and 4th symbols are g.
p211/p2	 1.6.14a	 1.6.14b	Parallelogram cell. 2-fold rotation. No other symmetry.

p4/p4	 1.6.15a	 1.6.15b	Square cell. 4-fold rotation. No other symmetry.
Notation For Symmetry Group (4-symbol / short form)	**Figure**	**Figure with primitive cell (in purple) symmetry axis (in dark green) centre of rotation (green dot)**	**Remarks on deriving the correct symmetry group (Note: let a glide reflection be denoted as a glide for convenience)**
p4mm/p4m	 1.6.16a	 1.6.16b	Square cell. 4-fold rotation. Both reflection axes are normal to some edges of the cell. So the x-axis is any of the edges and both the 3rd and 4th symbols are m.
p4gm/p4g	 1.6.17a	 1.6.17b	Square cell. 4-fold rotation. No reflection axis that is normal to any edges of the cell. There is a glide axis normal to the vertical edges of the cell. Pick one of these 2 edges as the x-axis. So the 3rd symbol is g. There is a reflection axis that is not normal to the x-axis. So the 4th symbol is m.

p3/p3	 1.6.18a	 1.6.18b	Hexagonal cell. 3-fold rotation. No other symmetry.
Notation For Symmetry Group (4-symbol / short form)	**Figure**	**Figure with primitive cell (in purple) symmetry axis (in dark green) centre of rotation (green dot)**	**Remarks on deriving the correct symmetry group (Note: let a glide reflection be denoted as a glide for convenience)**
p3m1/ p3m1	 1.6.19a	 1.6.19b	Hexagonal cell. 3-fold rotation. There is a reflection axis normal to the horizontal edges of the cell. Pick one of these 2 edges as the x-axis. So the 3rd symbol is m. No other symmetry.
p31m/ p31m	 1.6.20a	 1.6.20b	Hexagonal cell. 3-fold rotation. No reflection or glide axis that is normal to any edges of the cell. So the 3rd symbol is 1. There is a reflection axis not normal to all edges. So the 4th symbol is m.

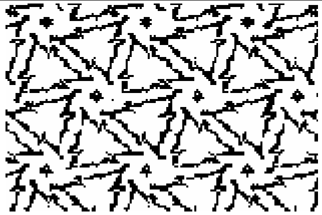
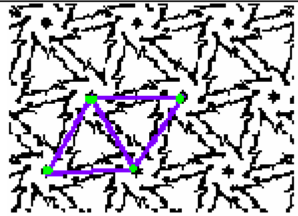
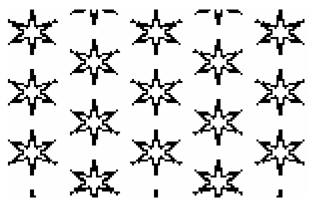
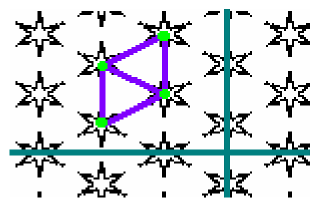
p6/p6	 1.6.21a	 1.6.21b	Hexagonal cell. 6-fold rotation. No other symmetry.
Notation For Symmetry Group (4-symbol / short form)	**Figure**	**Figure with primitive cell (in purple) symmetry axis (in dark green) centre of rotation (green dot)**	**Remarks on deriving the correct symmetry group (Note: let a glide reflection be denoted as a glide for convenience)**
p6mm/p6m	 1.6.22a	 1.6.22b	Hexagonal cell. There is a reflection axis normal to the vertical edges of the cell. Pick one of these 2 edges as the x-axis. So the 3rd symbol is m. There is a reflection axis not normal to the x-axis. So the 4th symbol is m.

Table 1.6.2

An Approach to Analyse the Symmetry Groups of Wallpaper Patterns

Like frieze patterns, we can use flow charts to determine the symmetry groups of the wallpaper patterns. Here, we will give two flow charts. The first chart is from Washburn and Crowe **[1]** and the second chart is a modification of the first. The differences in the charts are highlighted in yellow in the second chart. Explanatory notes are given after the second chart.

Chart 1.6.1 First Flow Chart for the Symmetry Groups of Wallpaper Patterns

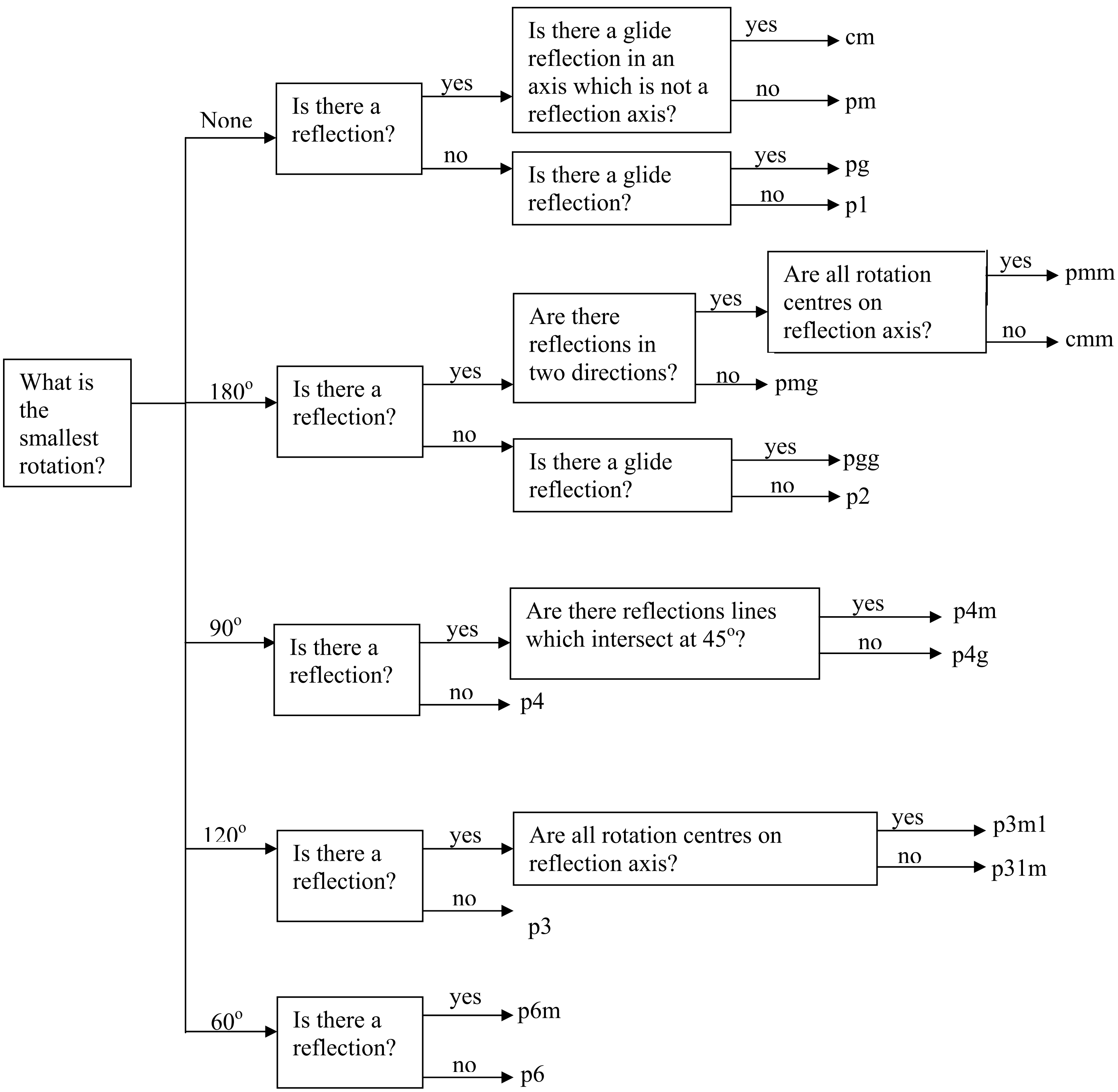
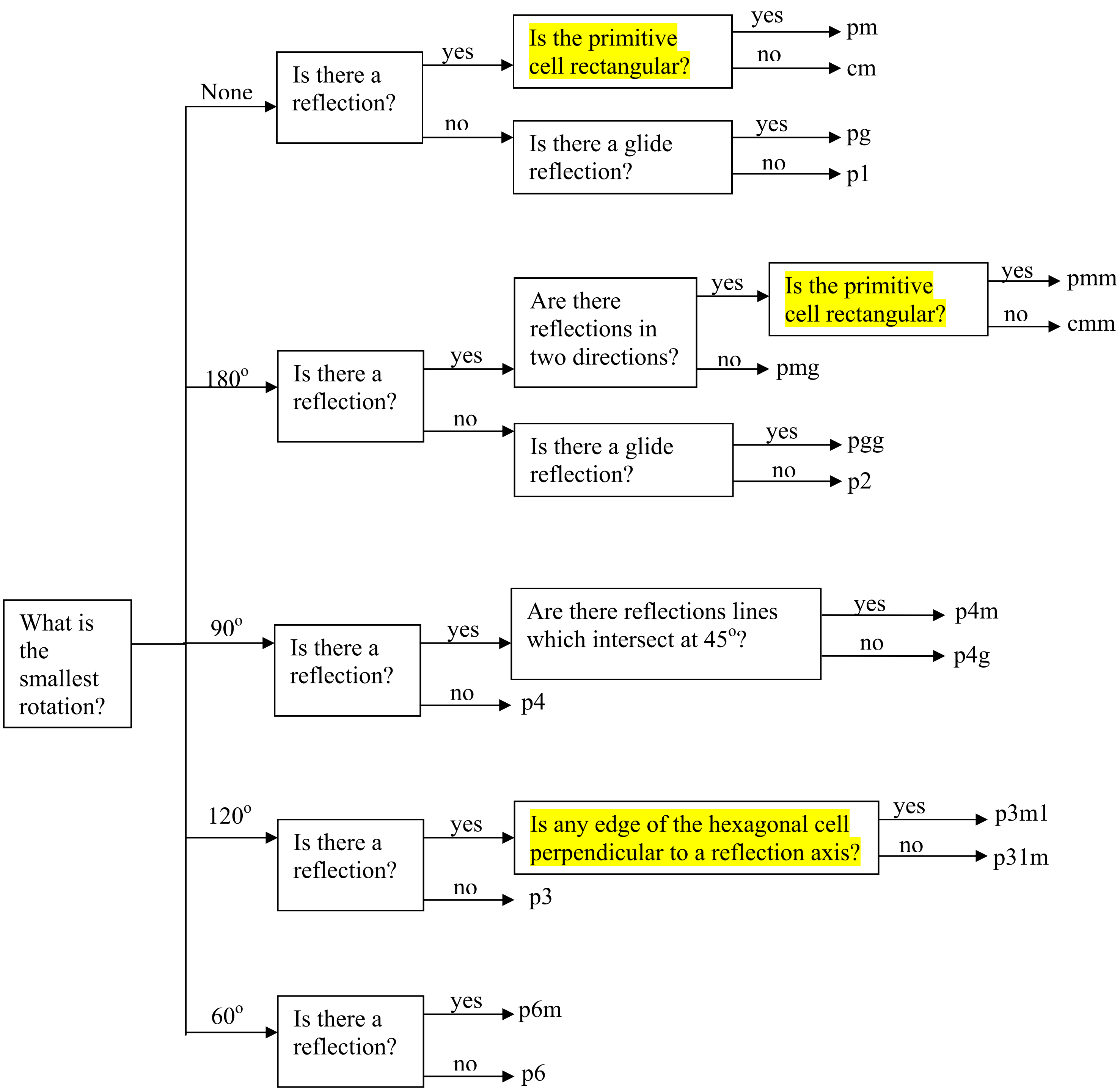


Chart 1.6.2 Second Flow Chart for the Symmetry Groups of Wallpaper Patterns



Examples to Illustrate the Difference in Approach in Each Flow Chart

(I) *cm* patterns versus *pm* patterns


Fig 1.6.23


Fig 1.6.24

Steps to determine the correct symmetry group of Figures 1.6.23 and 1.6.24 using the charts:

1) Determine the smallest rotation. Both have no rotations.
2) Both have a vertical reflection.
3) a)Using Chart 1.6.1, to check for glide reflections and reflections:

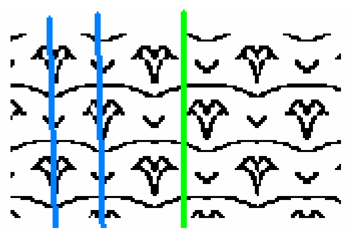


Fig 1.6.23a

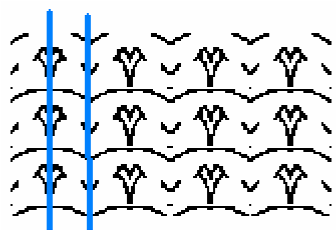


Fig 1.6.24a

The blue axes denote reflection axes and the green axis denotes a glide axis. Hence, Figure 1.6.23 is a *cm* pattern while Figure 1.6.24 is a *pm* pattern.

b) Using Chart 1.6.2, to check the primitive cells:

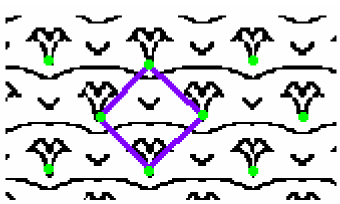


Fig 1.6.23b

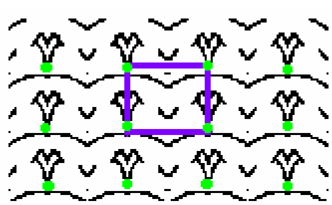


Fig 1.6.24b

The green dots form the lattice points of the figures. The blue parallelograms form the primitive cells. Hence, Figure 1.6.23 is a *cm* pattern while Figure 1.6.24 is a *pm* pattern.

(II) *cmm* patterns versus *pmm* patterns

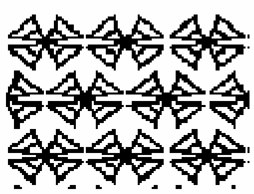


Fig 1.6.25

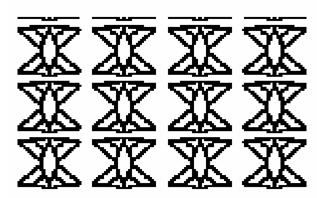


Fig 1.6.26

Steps to determine the correct symmetry group of Figures 1.6.25 and 1.6.26 using the charts:

1) Determine the smallest rotation. Both have rotations of 180^o.
2) Both have vertical and horizontal reflections.
3) a) Using Chart 1.6.1, to check for all rotation centres and reflection axes:

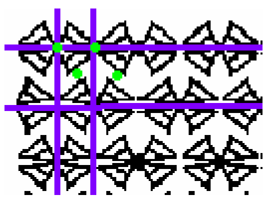


Fig 1.6.25a

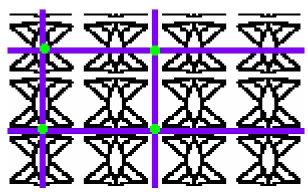


Fig 1.6.26a

The blue axes denote reflection axes and the green dots denote rotation centres. In Figure 1.6.25, some rotation centres are not on reflection axes but in Figure 1.6.26, all rotation centres are on reflection axes. So, Figure 1.6.25 is of class *cmm* while Figure 1.6.26 is of class *pmm*.

b) Using Chart 1.6.2, to check the primitve cells:

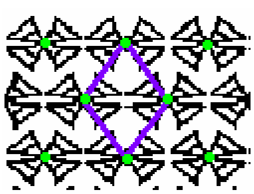


Fig 1.6.25b

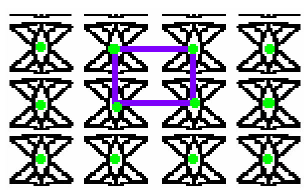


Fig 1.6.26b

The green dots in the above figures denote lattice points and the blue parallelograms denote primitive cells. Hence Figure 1.6.25 is a *cmm* pattern while Figure 1.6.26 is a *pmm* pattern.

(III) *p31m* patterns versus *p3m1* patterns

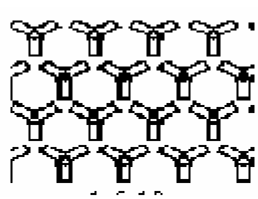


Fig 1.6.27


Fig 1.6.28

Steps to determine the correct symmetry group of Figures 1.6.25 and 1.6.26 using the charts:

1) Determine the smallest rotation. Both have rotations of 120°.
2) Both have reflection. Figure 1.6.27 has a vertical reflection while Figure 1.6.28 has a horizontal reflection.
3) a) Using Chart 1.6.1, to check for rotation centres and reflection axes:

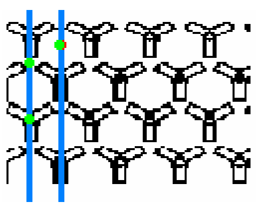


Fig 1.6.27a

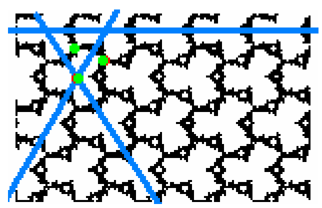


Fig 1.6.28a

The blue axes denote reflection axes and the green dots denote rotation centres. So Figure 1.6.27 is a *p3m1* pattern while Figure 1.6.28 is a *p31m* pattern.

b) Using Chart 1.6.2, to check if any edge of the hexagonal cell is perpendicular to any reflection axes:

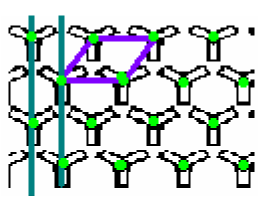


Fig 1.6.27b

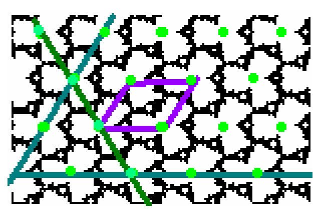


Fig 1.6.28b

The blue parallelograms are the primitive cells while the dark green lines denote reflection axes. The light green dots denote lattice points. In Figure 1.6.27, the horizontal edge is perpendicular to a reflection axis while in Figure 1.6.28, none of the edges of the primitive cell is perpendicular to any reflection axes. Hence Figure 1.6.27 is a *p3m1* pattern while Figure 1.6.28 is a *p31m* pattern.

Alternative Approach to Determine p31m Patterns and p3m1 Patterns

The two charts shown previously recommend two methods to determine *p31m* patterns and *p3m1* patterns. Here, we offer another method to tell the difference between the two patterns.

Select a type of rotation centre with a three-legged object.
Observe if the legs of this object point towards the centres of the nearest three-legged objects.
If yes, then it is *p31m*. Otherwise, it is *p3m1*.

Let us use the methodology on Figures 1.6.27 and 1.6.28.

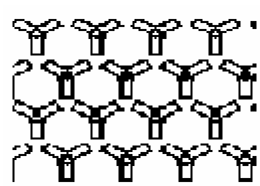


Fig 1.6.27

Fig 1.6.28

So for each figure, we select the rotation centre, shown as green dots and colour the "legs" of the three-legged object blue, as shown below.

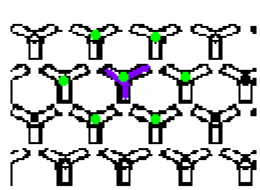


Fig 1.6.27c

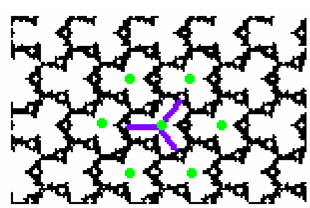


Fig 1.6.28c

For Figure 1.6.27c, the blue legs do not point towards the green dots, which represent the rotation centres of the blue-legged objects. For Figure 1.6.28c, the blue legs point towards the green dots. Hence, Figure 1.6.27 is of *p3m1* while Figure 1.6.28 is of *p31m*.

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