1. Consider the Poisson equation

\[ \Delta u = f, \quad \text{in} \quad \Omega = [a, b] \times [c, d], \]

with boundary condition

\[ u = u_b(x, y), \quad \text{on} \quad \Gamma = \partial \Omega. \]

Choose \( h_x = \frac{b-a}{M} \), \( h_y = \frac{d-c}{N} \) be the mesh sizes and \((x_i, y_j)\) with \( x_i = a + i \, h_x, \ y_j = c + j \, h_y, \)
\( i = 0, 1, \ldots, M; \ j = 0, 1, \ldots, N \) be grid points.

\( a) \) Write down the second order central difference discretization for the above problem.

\( b) \) Develop a code to implement the fast Poisson solver for the above problem.

\( c) \) Apply your code to solve the following problem

\[ \Delta u = -5 \sin(2x) \sin y + 4, \quad \text{in} \quad \Omega = [0, 2\pi] \times [0, 2\pi], \]
\[ u = x^2 + y^2, \quad \text{on} \quad \partial \Omega. \]

Draw the contour plot and surface plot of your numerical solution under \( h_x = h_y = \frac{\pi}{64} \). The exact solution of the problem is

\[ u(x, y) = x^2 + y^2 + \sin(2x) \sin y. \]

Test the second order convergence rate by compute \( \|u - u_n\|_h \) for \( h = \frac{\pi}{16}, \frac{\pi}{32}, \frac{\pi}{64}, \) and \( \frac{\pi}{128} \).

\( d) \) Repeat \( a \) and \( b \) by replacing the Dirichlet boundary condition with Neumann boundary condition

\[ \frac{\partial u}{\partial n} = u_n(x, y), \quad \text{on} \quad \Gamma = \partial \Omega. \]

Apply your code to solve the following problem

\[ \Delta u = -5 \cos x \cos(2y), \quad \text{in} \quad \Omega = [0, 2\pi] \times [0, 2\pi], \]
\[ \frac{\partial u}{\partial y}(x, 0) = \frac{\partial u}{\partial y}(x, 2\pi) = x, \quad 0 \leq x \leq 2\pi, \]
\[ \frac{\partial u}{\partial x}(0, y) = \frac{\partial u}{\partial x}(2\pi, y) = y, \quad 0 \leq y \leq 2\pi. \]

Draw the contour plot and surface plot of your numerical solution under \( h_x = h_y = \frac{\pi}{64} \).

P.S. You can download fast Sine and Cosine transform from http://www.netlib.org/cgibin/search.pl via searching for sint or cost.
2. Consider the incompressible viscous flow:

\[ u_t + u u_x + v u_y + p_x = \nu \Delta u, \]
\[ v_t + u v_x + v v_y + p_y = \nu \Delta v, \]
\[ u_x + v_y = 0, \]

a). Write down the first-order (backward Euler) projection method.

b). Write down fully discretization with central difference for spatial discretization.

c). Develop a code to implement your discretization with a fast Poisson solver.

d). Use your code to simulate the time-dependent flow with \( \nu = 10^{-2} \). Display the velocity field at different times.

e). Use your code to find steady state solution for \( \nu = 1, 10^{-1}, 10^{-2}, 10^{-3} \) and \( 10^{-4} \). Draw the velocity fields of the steady state solutions.

3. Consider the incompressible viscous flow:

\[ \omega_t + u \omega_x + v \omega_y = \nu \Delta \omega, \]
\[ \Delta \psi = -\omega, \]
\[ u = \psi_y, \quad v = -\psi_x, \]

a). Develop a code to implement a numerical method for this problem.

b). Apply your code to simulate the time dependent flow for \( \nu = 10^{-2} \).

c). Apply your code to find the steady state solution for \( \nu = 10^{-1}, 10^{-2} \) and \( 10^{-3} \).