Q.1
(i) The mathematical model for the amounts \( x(t) \) and \( y(t) \) of salt in the two tanks is

\[
\frac{dx}{dt} = -\frac{x}{20},
\]
\[
\frac{dy}{dt} = \frac{x}{20} - \frac{y}{40},
\]

with the initial conditions \( x(0) = y(0) = 50 \).

(ii) The solution to the above ODE system is

\[
x(t) = 50e^{-\frac{t}{20}},
\]
\[
y(t) = 150e^{-\frac{t}{40}} - 100e^{-\frac{t}{20}}.
\]

(iii) The amount of salt in tank 2, i.e.,

\[
y(t) = 150e^{-\frac{t}{40}} - 100e^{-\frac{t}{20}},
\]
\[
= 150a - 100a^2,
\]
\[
= -100(a^2 - 1.5a + 0.75^2) + 100 \times 0.75^2,
\]

where \( a = e^{-\frac{t}{40}} \). Thus the amount of salt in the tank 2 reaches its maximum 56.25 when \( a = 0.75 \) or \( t = 11.5073 \).

Q.2
We first set \( \tau = \frac{t}{t_s} \), \( p(\tau) = \frac{p(t)}{p_s} \), \( h(\tau) = \frac{h(t)}{h_s} \) with \( t_s = \frac{1}{rC} \), \( p_s = C \) and \( h_s = \frac{c^2}{B} \). Plugging them into predator-prey model, we get

\[
\frac{dp(\tau)}{d\tau} = p \left[ (k - p) - \frac{h}{1 + p} \right],
\]
\[
\frac{dh(\tau)}{d\tau} = dh \left[ \frac{p}{1 + p} - ah \right],
\]
where \( k = \frac{K}{C} \), \( d = \frac{D}{r} \) and \( a = \frac{4C^2}{B} \).

The ODE system has two equilibrium points \((p, h) = (0, 0), (p, h) = (k, 0)\) and another equilibrium point satisfies \( k - p - \frac{p}{a(1+p)^2} = 0 \) and \( h = \frac{p}{a(1+p)} \).

To study their stability, we set

\[
\begin{align*}
f(p, h) &= p \left( k - p - \frac{h}{1+p} \right), \\
g(p, h) &= dh \left( \frac{p}{1+p} - ah \right).
\end{align*}
\]

then we get the Jacobian matrix

\[
J(p, h) := \begin{pmatrix}
\frac{\partial f}{\partial p} & \frac{\partial f}{\partial h} \\
\frac{\partial g}{\partial p} & \frac{\partial g}{\partial h}
\end{pmatrix} = \begin{pmatrix}
k - 2p + h/(1+p)^2 & -p/(1+p) \\
dh/(1+p)^2 & \frac{dp}{1+p} - 2adh
\end{pmatrix}.
\]

At the point \((p, h) = (0, 0)\), the Jacobian matrix

\[
J(p, h) := J(0, 0) = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}.
\]

It has two eigenvalues: \( k \) and 0. Both of them have positive real parts. Therefore, at this point, the system is unstable.

At the point \((p, h) = (k, 0)\), the Jacobian matrix

\[
J(p, h) := J(k, 0) = \begin{pmatrix} -k & -k/(1+k) \\ 0 & dk/(1+k) \end{pmatrix}.
\]

It has two eigenvalues: \(-k\) and \(dk/(1+k)\). One of them has positive real parts. Therefore, at this point, the system is unstable.

**Q.3.**

Suppose that \( p_n, q_n \) and \( r_n \) are the percentage of students who eat at Science Canteen, Engineering Canteen and Arts Canteen respectively and Arts Canteen at \( n \)-th period \( (n=0,1,2, \cdots) \), a discrete model can be established as follows:

\[
\begin{align*}
p_{n+1} &= 0.5p_n + 0.3q_n + 0.15r_n, \\
q_{n+1} &= 0.25p_n + 0.6q_n + 0.05r_n, \\
r_{n+1} &= 0.25p_n + 0.1q_n + 0.8r_n,
\end{align*}
\]
and \( p_n + q_n + r_n = 1 \).

Assuming that \( p_n \to p_\infty \), \( q_n \to q_\infty \) and \( r_n \to r_\infty \) when \( n \to \infty \), we then find that \( p_\infty \), \( q_\infty \) and \( r_\infty \) satisfy

\[
p_\infty = 0.5p_\infty + 0.3q_\infty + 0.15r_\infty, \]
\[
q_\infty = 0.25p_\infty + 0.6q_\infty + 0.05r_\infty, \]
\[
r_\infty = 0.25p_\infty + 0.1q_\infty + 0.8r_\infty. \]

In addition, \( p_\infty \), \( q_\infty \) and \( r_\infty \) must satisfy

\[ p_\infty + q_\infty + r_\infty = 1. \]

Solving the above equations, we can obtain

\[ p_\infty = 2/7, \quad q_\infty = 5/21, \quad r_\infty = 10/21. \]

They are the long-term percentage of students eating at Science Canteen, Engineering Canteen and Arts Canteen respectively.

**Q.4.**

To study the stability, we set

\[
f(P, Q) = aP \left( \frac{b}{Q} - P \right), \]
\[
g(P, Q) = cQ \left( fP - Q \right). \]

We define the Jacobian matrix

\[
J(P, Q) := \begin{pmatrix}
\frac{\partial f}{\partial P} & \frac{\partial f}{\partial Q} \\
\frac{\partial g}{\partial P} & \frac{\partial g}{\partial Q}
\end{pmatrix} = \begin{pmatrix}
\frac{ab}{Q} - 2aP & -\frac{abP}{Q^2} \\
cfQ & cfP - 2cQ
\end{pmatrix}. \]

In addition we find \((P, Q) = (b/\sqrt{fb}, \sqrt{fb})\) from the equations \( f(P, Q) = 0 \) and \( g(P, Q) = 0 \).

At the point \((P, Q) = (b/\sqrt{fb}, \sqrt{fb})\), the Jacobian matrix

\[
J(b/\sqrt{fb}, \sqrt{fb}) = \begin{pmatrix}
-\frac{ab}{\sqrt{fb}} & -\frac{ab}{f\sqrt{fb}} \\
\frac{cf}{\sqrt{fb}} & -\frac{c\sqrt{fb}}{f\sqrt{fb}}
\end{pmatrix}, \]
has two eigenvalues: \( \lambda_1 = \frac{A+\sqrt{A^2-8abc}}{2} \) and \( \lambda_2 = \frac{A-\sqrt{A^2-8abc}}{2} \) where \( A = \frac{ab}{\sqrt{fb}} + c\sqrt{fb} \). Because oth of them have negative real parts when \( a, b, c \) and \( f \) are positive constants, the system at this point is stable.

(i) When \( a = 1, b = 20000, c = 1, \) and \( f = 30, \) the ODE system have one equilibrium point \( (P, Q) = (20000/\sqrt{600000}, \sqrt{600000}) \). At this point, the system is stable.

(ii) The ODE system only has one equilibrium point \( (P, Q) = (b/\sqrt{fb}, \sqrt{fb}) \). At this point, the system is stable.

(iii) When \( a = 1, b = 20, c = 1, \) and \( f = 30, \) the phase trajectory drawn by Matlab is shown in Figure 1. In this case there is an equilibrium point at \( (P, Q) = (20/\sqrt{600}, \sqrt{600} = (0.8165, 24.4949) \).

Q.5.

Let \( v = v(t) \) be the velocity of the projectile launched from the surface of the earth, then \( v = \frac{dr}{dt} \) and

\[
\frac{dv}{dr} = \frac{dv}{dr} \frac{dr}{dt} = \frac{dv}{dt} = -\frac{GM_c}{r^2} + \frac{GM_m}{(S-r)^2}.
\]

Integrating the above equation, we obtain

\[
\frac{1}{2}v^2 = \frac{GM_c}{r} - \frac{GM_m}{r-S} + C,
\]

with \( C \) being some constant.

\( r(0) = R \) and \( v(0) = v'(0) = v_0, \) we get

\[
\frac{1}{2}v_0^2 = \frac{GM_c}{R} - \frac{GM_m}{R-S} + C.
\]

Therefore

\[
C = \frac{1}{2}v_0^2 - \frac{GM_c}{R} + \frac{GM_m}{R-S}.
\]

Since \( M_c = 5.975 \times 10^{24} kg, M_m = 7.35 \times 10^{22} kg, R = 6.378 \times 10^6 m, S = 3.84 \times 10^8 m, \) and \( G = 6.67428 \times 10^{-11} m^3 kg^{-1}s^{-2}, \) we get

\[
C = \frac{1}{2}v_0^2 - 6.25126 \times 10^7.
\]
Therefore we have
\[
\frac{1}{2} v^2 = \frac{GM_c}{r} + \frac{GM_m}{S - r} + \frac{1}{2} v_0^2 - 6.25273 \times 10^8.
\]

To make the projectile reaches the moon, we must have \( r \geq S \).
In addition, we can obtain
\[
-\frac{GM_c}{r^2} + \frac{GM_m}{(S - r)^2} = 0,
\]
according to the assumption “to reach the moon, the projectile \( \cdots \) its net acceleration vanishes”. Thus \( r_1 := \frac{\sqrt{M_c}}{\sqrt{M_c - \sqrt{M_m}}} S = 3.9796 \times 10^8 \) and \( r_2 := \frac{\sqrt{M_m}}{\sqrt{M_c + \sqrt{M_m}}} S \). We note that \( r_2 < S \). We choose \( r = r_2 \approx 3.456622 \times 10^8 m \).

Based on the above analysis, at the moment that the projectile reaches the moon, \( v \geq 0 \) and \( r = r_2 \). Therefore
\[
\frac{1}{2} v^2 = \frac{GM_c}{r_2} + \frac{GM_m}{S - r_2} + \frac{1}{2} v_0^2 - 6.25273 \times 10^8 \geq 0.
\]

From the above equation, we find
\[
\frac{1}{2} v_0^2 \geq -\frac{GM_c}{r_2} - \frac{GM_m}{S - r_2} + 6.25273 \times 10^8 \approx 6.23991 \times 10^8.
\]

The minimum velocity \( v_0 \approx 1.106625 \times 10^4 m/s \).