Question 1. Denote \( x_n \) \((n = 0, 1, 2, \ldots)\) the balance at the \( n \)th month (after withdrawing \$1,000), then
\[
x_{n+1} = x_n + 0.01x_n - 1,000 = 1.01x_n - 1,000, \quad x(0) = 50,000. \tag{1}
\]
Let \( u_n = x_n - a \), such that \( u_{n+1} = 1.01u_n \), i.e.,
\[
x_{n+1} - a = 1.01(x_n - a) = 1.01x_n - 1.01a
\]
\[\implies x_{n+1} = 1.01x_n - 1.01a + a = 1.01x_n - 0.01a. \tag{2}\]

From (1) and (2), we get \( a = 100,000 \). So,
\[
u_n = 1.01u_{n-1} = (1.01)^2u_{n-2} = \cdots = (1.01)^nu_0 = (1.01)^n(x_0 - a)
\]
\[= (1.01)^n(-50,000) \implies x_n = 100,000 - (1.01)^n50,000.
\]

Whether the annuity would run out of money means whether there exists a \( n \), such that \( x_n \leq 0 \), and this \( n \) (if exists) indicates 'when' it happens. Since
\[
100,000 \leq 50,000(1.01)^n \iff 2 \leq (1.01)^n,
\]
and
\[
1 = \log_2 2 \leq \log_2(1.01)^n = n\log_2 1.01 \iff n \geq \frac{1}{\log_2 1.01} \approx 69.6607,
\]
the annuity will run out of money after 70 months. The below table shows the balance information

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\cdots</th>
<th>68</th>
<th>69</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_n )</td>
<td>50,000</td>
<td>49,500</td>
<td>48,485</td>
<td>\cdots</td>
<td>1,638.9</td>
<td>655.3</td>
</tr>
</tbody>
</table>

Question 2. Denote \( C_n \) the concentration at \( n \)th hour, then
\[
C_{n+1} = C_n - 0.2C_n = 0.8C_n, \quad C_0 = 640[mg/L].
\]
So,
\[
C_n = 0.8C_{n-1} = (0.8)^2C_{n-2} = \cdots = (0.8)^nC_0 = 640 \cdot (0.8)^n.
\]

When the drug is not effective, then
\[
C_n \leq 100 \implies n \geq \frac{\log_{0.8} \frac{100}{640}}{\log_{0.8}} \approx 8.3189.
\]

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>\cdots</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C_n )</td>
<td>640</td>
<td>512</td>
<td>409.6</td>
<td>\cdots</td>
<td>134.22</td>
<td>107.37</td>
<td>85.90</td>
</tr>
</tbody>
</table>
Question 3. Denote $G_n$, $D_n$ the GNP and national debt at the end of the $n$th year from now, and let the national debt is increasing at a rate $r$ proportional to the GNP, then,

$$G_{n+1} = G_n + 0.03G_n = 1.03G_n, \quad D_{n+1} = D_n + rG_{n+1}.$$ 

Note that here we assume the debt increase proportionally to GNP of the current year. But the assumption that dependence is on the previous year is also fine, since $G_{n+1} = 1.03G_n$.

Since

$$G_n = 1.03G_{n-1} = (1.03)^2G_{n-2} = \cdots = (1.03)^nG_0,$$

when

$$(1.03)^n = 2 \implies n = \frac{1}{\log_{1.03} 2} \approx 23.45,$$

the GNP will double after 24 years.

Question 4. The modeling task requires some basic knowledge in classical mechanics and Newton’s laws of motion. To solve the problem, necessary prepare of calculus and ordinary differential equation is also needed.

The following assumptions are put:

- Only the motion in the direction perpendicular to the ground is considered, i.e., one-dimensional problem;
- direction of air resistance is parallel to the motion direction, opposite to the body’s gravity;
- opening parachute is completed instantly.

Denote $x(t)$ the height she has fallen after bailing out, i.e., $x(0) = 0$ and eventually $x = 10,000$ [ft], $v(x)$ the velocity, then

$$v(t) = \frac{d}{dt} x(t), \quad x(t_2) - x(t_1) = \int_{t_1}^{t_2} v(t) dt. \quad (3)$$

When she falls (both freely and with parachute open), by Newton’s laws of motion,

$$m \frac{dv}{dt} = mF = mg - \rho v(t), \implies \frac{dv}{v - \frac{mg}{\rho}} = \frac{g}{m} dt.$$

Recall how to solve the above ODE:

$$\frac{dv}{v - \frac{mg}{\rho}} = \frac{g}{m} dt \implies \ln \left( \frac{mg}{\rho} - v \right) = -\frac{\rho t}{m} + C_1 \implies v(t) = \frac{mg}{\rho} + Ce^{-\frac{m}{\rho} t},$$

where $C$ is determined by initial data.
• When falling freely, from $v(0) = 0$,

$$v(t) = \frac{mg}{\rho} \left( 1 - e^{-\frac{\rho}{m} t} \right),$$

plugging $\frac{\rho}{m} = 0.15$ into (5) to get $v(20)$, and integrating (5) from 0 to $t = 20$,

$$v(20) \approx 203.6832 \, [ft/s], \quad x(20) = \int_0^{20} v(t)dt \approx 2,929.2186 \, [ft].$$

• After opening the parachute $\frac{\rho}{m} = 1.5$, with $v(20)$ as initial data, similar as (5),

$$v(t) = 21.4355 + 182.2477 e^{-1.5(t-20)}.$$  

Let $T$ be the time when she reaches the ground, from (3) and (7),

$$7,070.7814 = x(T) - x(20) = \int_{20}^{T} 21.4355 + 182.2477 e^{-1.5(t-20)}dt$$

$$\Rightarrow 121.4985 e^{-1.5(T-20)} - 21.4355(T-20) + 6,949.2829 = 0$$

$$\Rightarrow T \approx 344.1950.$$ 

Therefore, the woman will reach the ground in about 344 [s] after bailing out.

**REMARK:** sometimes finding a root of $f(x) = 0$ is required. In most cases it is difficult to do by hands (like in this problem), then you may use any numerical technique to solve it, for example, Newton method, or just use the built-in function 'fzero' in Matlab. For example, in this problem using 'fzero' in Matlab:

```matlab
>> f=@(t)121.4985*exp(-1.5*(t-20))-21.4355*(t-20)+6949.2829
f =
 @(t)121.4985*exp(-1.5*(t-20))-21.4355*(t-20)+6949.2829

>> T=fzero(f,300)
T =
344.1950
```

To check the solution,

```matlab
feval(f,T)
an =
-9.0949e-013
```