Q.1
Because the manager of the reserve is considering allowing controlled hunting of $E$ birds annually, the population of the bird, i.e., $N(t)$ will satisfy

$$\frac{dN}{dt} = \frac{(1 - 0.01N)N}{1 + 0.1N} - E.$$ 

Suppose that the limit of $N(t)$ exists, i.e., $\lim_{t \to +\infty} N(t) = \beta$. Then from

$$\frac{d\beta}{dt} = \frac{(1 - 0.01\beta)\beta}{1 + 0.1\beta} - E = 0,$$

we find

$$\frac{-0.01\beta^2 + (1 - 0.01E)\beta - E}{1 + 0.1\beta} = 0.$$  

(1)

Let we define $\Delta = (1 - 0.1E)^2 - 4 \times 0.01E = 0.01E^2 - 0.24E + 1$. We find that: (1) when $\Delta \geq 0$, $\beta$ exists; (2) when $\Delta < 0$, $\beta$ does not exist and

$$\frac{-0.01N^2 + (1 - 0.01E)N - E}{1 + 0.1N} = \frac{-0.01(N-50+5E)^2+25\Delta}{1+0.1N} < 0$$

for all $N$.

Thus, the critical value should be attained at the turning point: $\Delta = 0$, i.e. $0.01E^2 - 0.24E + 1 = 0$ which have two roots: $E_1 = 12 + \sqrt{44}, E_2 = 12 - \sqrt{44}$. From (1), we see that when $E = E_1$, $\beta = \frac{1 - 0.1E}{0.02} > 0$, while when $E = E_2$, $\beta = \frac{1 - 0.1E}{0.02} > 0$. So, the critical value shud be $F = E_2 = 12 - \sqrt{44}$.

Q.2
(i) Let $\tau = \frac{P}{A} t$ and $u(\tau) = \frac{N(t)}{A}$. Plugging them into the equation

$$\frac{dN(t)}{dt} = RN \left(1 - \frac{N}{K}\right) - P \left[1 - \exp \left(-\frac{N^2}{\varepsilon A^2}\right)\right],$$
we get the dimensionless formulation
\[
\frac{du(\tau)}{d\tau} = ru \left( 1 - \frac{u}{q} \right) - \left[ 1 - \exp \left(-\frac{u^2}{\varepsilon} \right) \right],
\]
where the positive constants \( r := \frac{RA}{P} \) and \( q := \frac{K}{A} \).

(ii) If we define \( f(u) := ru \left( 1 - \frac{u}{q} \right) - \left[ 1 - \exp \left(-\frac{u^2}{\varepsilon} \right) \right] \), then \( f(0) = 0 \). Therefore \( u = 0 \) is equilibrium; in addition, we know \( f'(0) = r > 0 \) because \( f'(u) = r - \frac{2ru}{q} - \frac{2u}{\varepsilon} \exp^{-\frac{u^2}{\varepsilon}} \). Therefore at this point the equilibrium is unstable.

(iii) Let \( r \) and \( q \) lie in a domain in \( r, q \) space given approximately by \( rq > 4 \). In addition we assume that \( 0 < \varepsilon << 1 \). Our purpose is to find four points \( u_0 < u_1 < u_2 < u_3 \) such that \( f(u) \) has different sign at any adjacent points.

(1) For \( u_0 \). Since \( f'(0) > 0 \) and \( f(u) \) is smooth, by the continuity of \( f'(u) \), we see that \( \exists \eta_0 \), s.t. \( f'(u) > 0 \) for \( u \in [0, \eta_0] \). Hence, \( f(u) \) monotone increasing in \([0, \eta_0]\), which indicate \( f(\eta_0) > f(0) = 0 \). We can simply let \( u_0 = \eta_0 \).

(2) Let \( u_1 = \sqrt{\varepsilon} \), we find that:
\[
f(\sqrt{\varepsilon}) = r\sqrt{\varepsilon} - \frac{r}{q}\varepsilon - 1 + e^{-1}
\]
\[= -\frac{r}{q}\left[\sqrt{\varepsilon} - \left(\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{q}{r}(1 - e^{-1})}\right)\right] - \left(\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{q}{r}(1 - e^{-1})}\right)\]
Since \( rq > 4 \), we have \( \frac{q^2}{4} - \frac{q}{r}(1 - e^{-1}) > 0 \) which imply that \( \sqrt{\frac{q^2}{4} - \frac{q}{r}(1 - e^{-1})} \) is a real number, now together with \( \frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{q}{r}(1 - e^{-1})} > 0 \) and \( \sqrt{\varepsilon} > \eta_0 \), say sufficiently small, we get \( f(\sqrt{\varepsilon}) < 0 \). Moreover, \( \sqrt{\varepsilon} > \eta_0 \) since when \( u \leq \eta_0 \), \( f'(u) > 0 \).

(3)Let \( u_2 = \frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{q}{r}} \); (which is real by the condition \( rq > 4 \)).We see:
\[
f(u_2) = -\frac{ru_2^2}{q} + ru_2 - 1 + e^{-\frac{u_2^2}{\varepsilon}}
\]
\[= -\frac{r}{q}\left[u_2 - \left(\frac{q}{2} - \sqrt{\frac{q^2}{4} - \frac{q}{r}}\right)\right] + \left(\frac{q}{2} + \sqrt{\frac{q^2}{4} - \frac{q}{r}}\right) + e^{-\frac{u_2^2}{\varepsilon}}
\]
Moreover, $u_2 > u_1$ since $0 < \varepsilon << 1$.

(4) Setting $u_3 = \frac{q}{2} + \sqrt{q^2/4 - (1 - e^{-1})q} > u_2$, we find that

$$f(u_3) = \frac{r}{4} q - \frac{r}{q} \varepsilon - 1 + e^{u_3^2/\varepsilon} = -e^{-1} + e^{-u_3^2/\varepsilon}.$$

Now that $0 < \varepsilon << 1$, we have $f(u_3) < 0$.

In conclusion, we find $0 < u_0 < u_1 < u_2 < u_3$ s.t. $f(u_0) > 0$, $f(u_1) < 0$, $f(u_2) > 0$, $f(u_3) < 0$ and $f(u_4) < 0$, by the continuity of function $f(x)$ we can prove that there are at least three nonzero solution for $f(u) = 0$. 