Sample solutions to Tutorial 6

Q.1
(i) Multiplying both sides of the equation
\[ \frac{dy(t)}{dt} = \frac{\sin t}{y} \]
by \( y \) and integrating the obtained equation with respect to \( t \), we get
\[ \frac{1}{2} y^2(t) = \cos t + C, \]
where \( C \) is some constant to be determined.

(a) If \( y(0) = 2 \), we can find \( C = 1 \). Then the solution is
\[ y(t) = 2 |\cos \left( \frac{t}{2} \right)|. \]

(b) If \( y(0) = 1 \), we can find \( C = -1/2 \). Then the solution is
\[ y(t) = \sqrt{2 \cos t - 1}. \]

(c) If \( y(0) = 3 \), we can find \( C = 7/2 \). Then the solution is
\[ y(t) = \sqrt{2 \cos t + 7}. \]

(ii) The solution to the equation with \( y(0) = 2 \) is \( y(t) = 2 |\cos \left( \frac{t}{2} \right)| \), its plot is shown in Figure 1.

Q.2

1
Let $t$ be the time and $v = v(t)$ be the velocity of this sport car, then we have the following initial value problem

$$\frac{dv}{dt} = k(250 - v), \quad t > 0, \quad v(0) = 0,$$

where $k$ is some constant. Solving the problem, we find

$$v(t) = 250 \left(1 - e^{-kt}\right).$$

In addition, we get $k = -\frac{3600}{10} \ln(3/5)$ because $v(t = 10/3600) = 100$.

Assuming $v(t=b) = 200$ (km/h), we obtain $b = \frac{\ln(1/5)}{\ln(3/5)} \times 10/3600$ hours. Thus when $t = \frac{\ln(1/5)}{\ln(3/5)} \times 10 \approx 31.5066$ (seconds), the sports car will accelerate from rest to 200 (km/h).

Let $f(v)=k(250-v)=0$, we find the terminal velocity is 250 (km/h) from the solution $v(t) = 250 \left(1 - e^{-kt}\right)$.

Q.3
We denote $T_0$ the time at which the rocket reaches the highest altitude, $T$ the time when all the gas in the engines have been exhausted and $v = v(t)$ the velocity of the rocket. The time period when the rocket flies can be divided into two periods: $[0, T]$ and $[T, T_0]$.

For the first period, the model for the velocity of the rocket is
\[
\begin{align*}
\frac{dv}{dt} &= \frac{\alpha M}{M-\beta t} - g, \quad 0 < t \leq T, \\
v(t = 0) &= 0.
\end{align*}
\] (0.1)

For the second period, the model for the velocity of the rocket is
\[
\begin{align*}
\frac{dv}{dt} &= -g, \quad T \leq t \leq T_0, \\
v(t = T) \text{ is obtained from (0.1)}.
\end{align*}
\] (0.2)

In the above models, $M = 28000$, $\alpha = 500$, $\beta = 800$, $T = 20000/800 = 25$ and $g = 9.81$. Solving Eq.(0.1), we find
\[
v(t) = 500\ln(28000) - 500\ln(28000 - 800t) - 9.81t, \quad 0 \leq t \leq T.
\] (0.3)

Therefore, $v(t = T = 25) = 500\ln(28000) - 500\ln(8000) - 9.81 \times 25 \approx 381.1315$. After solving Eq.(0.2), we find
\[
v(t) = 500\ln(28000) - 500\ln(8000) - 9.8t, \quad T \leq t \leq T_0.
\] (0.4)

From Eq.(0.4), we obtain $T_0 = 63.8513$ because $v(t = T_0) = 0$.

Let $y = y(t)$ be the altitude of the rocket with respect to $t$ when it is flying vertically. From $\frac{dy}{dt} = v$, we can find
\[
y(T_0) - y(0) = \int_0^{T_0} v(t)dt = \int_0^T v(t)dt + \int_T^{T_0} v(t)dt, \quad (0.5)
\]
with $y(0) = 0$.

(i) Through calculation of (0.5), we get
\[
y(t = T_0) = 3.1706 + 7.4037 = 10.574(km)
\]

(ii) The acceleration of gravity is not constant in light of the computation result shown in (i). The explanation is as follows:
If we treat the Earth as a non-rotating perfect sphere, the following formula approximates the Earth’s gravity variation with altitude:

\[ g_h = 9.81 \left( \frac{1}{1 + h/r} \right)^2, \]

where \( g_h \) is the gravity measure at height above sea level, \( r = 6400\text{km} \) is the Earth’s mean radius and \( h \) is the altitude above the sea level. The formula is taken from http://en.wikipedia.org/wiki/Earth%27s_gravity.

At time \( t = T_0 \), the rocket reaches 10.574km. \( 10.574/r \approx 0.0017 \). we get \( g_h = 9.7767 \text{ m/s}^2 \) at this altitude. The \( g_h \) have the difference about 0.0333 \( \text{m/s}^2 \) from 9.81\( \text{m/s}^2 \). The difference may not be ignored.