1. Consider the iteration: \( x_{k+1} = x_k + \alpha_k d_k \) for solving the linear system \( Ax = b \), where \( d_k \) is a vector called the direction of search, and \( \alpha_k \) is a scalar. It is assumed throughout that \( d_k \) is a nonzero vector. Consider a method which determine \( x_{k+1} \) so that the residual \( \| r_{k+1} \|_2 \) with \( r_{k+1} = b - Ax_{k+1} \) is the smallest possible.

(a) Determine \( \alpha_k \) so that \( \| r_{k+1} \|_2 \) is minimal.

(b) Show that the residual vector \( r_{k+1} \) obtained in this manner is orthogonal to \( A d_k \).

(c) Show that the residual vectors satisfy the relation

\[
\| r_{k+1} \|_2 = \| r_k \|_2 \sin(\theta_k),
\]

where the angle \( \theta_k \in [0, \pi] \) is defined by

\[
\cos(\theta_k) = \frac{(r_k, A d_k)}{\| r_k \|_2 \| A d_k \|_2}.
\]

(d) Assume that at each step \( k \), we have \( (r_k, A d_k) \neq 0 \). Will the method always converge?

(e) Now assume that \( A \) is positive definite and select at each step \( d_k \equiv r_k \). Prove that the method will converge for any initial guess \( x_0 \).

2. Show how the GMRES method converge by going through (by hand) a few number of iterations of this method for the linear system \( Ax = b \) when

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}, \quad b = \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix};
\]

and \( x_0 = 0 \).

3. Given the linear system

\[
\begin{align*}
4x_1 - x_2 + x_3 &= 8, \\
2x_1 + 5x_2 + 2x_3 &= 3, \\
x_1 + 2x_2 + 4x_3 &= 11, \\
4x_1 + x_2 + 2x_3 &= 9, \\
2x_1 + 4x_2 - x_3 &= -5, \\
x_1 + x_2 - 3x_3 &= -9.
\end{align*}
\]

(a) Find the least square solution of the above problem by the normal equation (NE) method.

(b) Find the least square solution of the above problem by the singular-value-decomposition (SVD) method.
4. Consider different numerical methods for computing $\sqrt{a}$ with $a > 0$.

(a) Construct the Newton’s method based on the nonlinear equation $f(x) = x^2 - a = 0$. Prove that, for $k = 1, 2, \ldots, x_k \geq \sqrt{a}$ and the sequence $x_1, x_2, \ldots$ is decreasing. Find

$$
\lim_{k \to \infty} \frac{|\sqrt{a} - x_{k+1}|}{|\sqrt{a} - x_k|^2}, \quad \lim_{k \to \infty} \frac{|x_{k+1} - x_k|}{|x_k - x_{k-1}|^2},
$$

What is the order of convergence of this method?

(b) Construct the Newton’s method based on the nonlinear equation $f(x) = 1 - \frac{a}{x^2} = 0$. Find

$$
\lim_{k \to \infty} \frac{|\sqrt{a} - x_{k+1}|}{|\sqrt{a} - x_k|^2}, \quad \lim_{k \to \infty} \frac{|x_{k+1} - x_k|}{|x_k - x_{k-1}|^2},
$$

What is the order of convergence of this method?

(c) Show that the iterative method

$$
x_{k+1} = \frac{x_k(x_k^2 + 3a)}{3x_k^2 + a}, \quad k = 0, 1, 2, \ldots; \quad x_0 > 0,
$$

is a third-order method for computing $\sqrt{a}$ by finding

$$
\lim_{k \to \infty} \frac{|\sqrt{a} - x_{k+1}|}{|\sqrt{a} - x_k|^3}.
$$

(d) Write codes to implement the above three methods. Apply them to compute numerically $\sqrt{115}$ and $\sqrt{0.111}$ with different initial data $x_0 > 0$. Compare the convergence rates and the choices of different initial data for different numerical methods. What conclusion can you obtain?