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# THE CHEBYSHEV–LEGENDRE METHOD: IMPLEMENTING LEGENDRE METHODS ON CHEBYSHEV POINTS*

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*Dedicated to Seymour Parter on the occasion of his 65th birthday.*

**Abstract.** A new collocation method for the numerical solution of partial differential equations is presented. This method uses the Chebyshev collocation points, but, because of the way the boundary conditions are implemented, it has all the advantages of the Legendre methods. In particular $L_2$ estimates can be easily obtained for hyperbolic and parabolic problems.

**Key words.** Chebyshev–Legendre, penalty method, differentiation matrix, stability

**AMS subject classifications.** 65M12, 65M70

## 1. Introduction.

Polynomial pseudospectral (or collocation) methods have been extensively used in the numerical solutions of partial differential equations (PDEs). The underlying idea in those methods is to approximate the unknown function by an interpolation polynomial at some predescribed (collocation) points. The polynomial is then required to satisfy the PDEs at the collocation points. This procedure yields a system of ordinary differential equations (ODEs) to be solved.

Historically (see [10]), the first such points to be used were the *Chebyshev* collocation points

$$x_j = \cos\left(\frac{\pi j}{N}\right), \qquad 0 \le j \le N\,.$$

Those points were chosen because they allowed the use of fast Fourier transforms (FFTs) in the computations. It was only later (see [7]) that those points were identified with the nodes of the Gauss–Lobatto–*Chebyshev* (GLC) quadrature formula. This observation is the key in the stability analysis of the pseudospectral Chebyshev methods. The GLC quadrature formula led to the weighted $L_2$ norm

$$\int_{-1}^{1} f^2(x)\frac{dx}{\sqrt{1-x^2}}.$$

However, it has been noted in [8] that this is not a natural norm for hyperbolic equations. In fact the differential equation is not well posed in this norm. Also it complicated the stability analysis even for parabolic equations. The theory (and therefore the confidence in applying those methods) is not complete.

Once the connection between the collocation points and the Gauss–Lobatto points is established, it is natural to use the nodes of the Gauss–Lobatto–*Legendre* (GLL) quadrature formula. We refer the reader to [2] for review of those methods. Recently an $O(N \log N)$ method was proposed for the Legendre points [1]. The main problem with those points is that they are not given explicitly, and their evaluation for large $N$ is not robust due to roundoff errors.

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In this paper we present a method (and name it *the Chebyshev–Legendre* (CL) *Method*) that has the advantages of both the Chebyshev and Legendre methods. The method utilizes the Chebyshev collocation points allowing the use of fast Fourier algorithms and avoiding the roundoff error associated with computing the Legendre grid points. The boundary conditions are imposed via a new penalty technique in such a way that the method is stable in the usual $L_2$ norm (rather than the weighted $L_2$ norm). Hence the CL method enjoys the advantages of the Chebyshev method as well as those of the Legendre method.

The implementation of the boundary conditions is done by a penalty method. A penalty term is added to the PDE *at all grid points* in such a way that, as the limit of number of grid points tends to infinity, the boundary conditions are satisfied. This procedure seems to be better than the direct imposition of the boundary conditions, and in our case has the extra advantage of yielding the Legendre method at the Chebyshev points.

A similar idea had been tried by Reyna [11]. The difference between his approach and ours is in the imposition of the boundary conditions. Instead of transforming from the Chebyshev basis to the Legendre one as in [11], we impose the boundary conditions via penalty method and through that automatically switch to the Legendre basis *without using it in the differentiation procedure.*

The paper is organized as follows. In §2 we quote the essential formulas for the use of Chebyshev and Legendre methods. In §3 we present the CL method for hyperbolic equations. In §3.1 we describe the method and prove (in Theorem 3.1.1) an energy estimate to show stability. In Theorem 3.1.2 we bring another version of this energy estimate. In §3.2 we consider the relationship of the new method to the Legendre penalty (LP) method and show that the differentiation matrices of the two methods are related via a *similarity transformation.* This fact is proved in Theorem 3.2.2. In §4 we discuss and prove the stability of the CL method for the heat equation with Robin boundary conditions. Section 5 concludes the paper with some numerical experimentations with the new method.

In future work we will report on the convergence results of the new method for nonlinear hyperbolic equations.

**2. Preliminaries.** This section is devoted to the definitions of the pseudospectral methods to be used later. We will discuss Chebyshev and Legendre methods, which are based on the *Chebyshev polynomials*

$$T_N(x) = \cos(N \cos^{-1} x) \tag{2.1}$$

and the *Legendre polynomials*

$$P_N(x) = \frac{1}{2^N N!} \frac{d^N}{dx^N}(x^2-1)^N, \tag{2.2}$$

respectively. Associated with these two polynomials are several Gauss-type quadrature formulas. We will consider, in this paper, the *Gauss–Lobatto*-type formulas.

We start by defining the *Chebyshev collocation points* $x_j$ by

$$x_j = \cos\left(\frac{\pi j}{N}\right), \qquad 0 \le j \le N. \tag{2.3}$$

These points are the zeros of the polynomial $(1-x^2)T_N'(x)$ and associated with it we have the following formula.

**The GLC quadrature formula.** Let $f(x)$ be a polynomial of degree $2N-1$, then

$$\sum_{j=0}^{N} f(x_j)c_j = \int_{-1}^{1} f(\xi)(1-\xi^2)^{-\frac{1}{2}} d\xi \tag{2.4}$$

where the weight $c_j$ are given by

$$c_j = \frac{\pi}{N}, \qquad 1 \le j \le N-1, \qquad c_0 = c_N = \frac{\pi}{2N}. \tag{2.5}$$

Similarly, the *Legendre collocation points* $y_j$ are defined as the roots of the polynomial $(1-x^2)P_N'(x)$. For these points we have the following formula.

**The GLL quadrature formula.** Let $f(x)$ be a polynomial of degree $2N-1$, then

$$\sum_{j=0}^{N} f(y_j)\omega_j = \int_{-1}^{1} f(\xi)d\xi \tag{2.6}$$

where the Gaus–Lobatto weights $\omega_j$ are given by

$$\omega_j = -\frac{2}{N+1}[P_N(y_j)P_{N-1}(y_j)]^{-1}, \qquad 1 \le j \le N-1, \qquad \omega_0 = \omega_N = \frac{2}{N(N+1)}. \tag{2.7}$$

Unlike the Chebyshev points that are known explicitly, there is no explicit formula for the Legendre points $y_j$; they must be computed numerically. It is interesting though that there is a simple formula, easily and robustly computed, for the values of the Legendre polynomials and their derivative at the *Chebyshev points.* In fact we have the following explicit formula for $P_N'(x_j)$ (taken from [4, p. 180]),

$$P_N'(\cos\theta) = \sum_{m=0}^{N-1} \frac{(\frac{3}{2})_m(\frac{3}{2})_{N-1-m}}{m!(N-1-m)!}\cos(N-1-2m)\theta\,. \tag{2.8}$$

Pseudospectral (or collocation) methods are based on interpolations at the points $x_j$ or $y_j$. Consider the polynomials

$$Q_L(x) = (1-x^2)P_N'(x),$$
$$Q_C(x) = (1-x^2)T_N'(x),$$

and define the Legendre–Lagrange polynomials by

$$h_j(x) = \frac{Q_L(x)}{(x-y_j)Q_L'(y_j)} \tag{2.9}$$

and the Chebyshev–Lagrange polynomials by

$$g_j(x) = \frac{Q_C(x)}{(x-x_j)Q_C'(x_j)}. \tag{2.10}$$

Then the Legendre interpolation operator $I_L$ is defined by

$$(I_L f)(x) = \sum_{j=0}^{N} f(y_j) h_j(x), \tag{2.11}$$

whereas the Chebyshev interpolation operator $I_C$ is defined by

$$(I_C f)(x) = \sum_{j=0}^{N} f(x_j) g_j(x) . \tag{2.12}$$

By definition, we have

$$(I_L f)(y_j) = f(y_j),$$
$$(I_C f)(x_j) = f(x_j).$$

From the definition of the interpolation operators $I_L$ and $I_C$ we get the *spectral differentiation matrices* $\mathcal{D}_L$ and $\mathcal{D}_C$ as follows.

The pseudospectral Legendre differentiation matrix $\mathcal{D}_L$ is defined by

$$(\mathcal{D}_L)_{j,k} = h'_k(y_j) . \tag{2.13}$$

The pseudospectral Chebyshev differentiation matrix $\mathcal{D}_C$ is defined by

$$(\mathcal{D}_C)_{j,k} = g'_k(x_j) . \tag{2.14}$$

(See [3] for explicit expressions for the matrices.)

## 3. Hyperbolic equations.

**3.1. Scalar hyperbolic equation.** In this section we consider the scalar initial-boundary value hyperbolic equation

$$U_t = U_x, \quad -1 \le x \le 1, \quad t \ge 0 \tag{3.1}$$

with the initial condition

$$U(x, 0) = f(x) \tag{3.2}$$

and the boundary condition

$$U(1, t) = g(t). \tag{3.3}$$

**The Chebyshev collocation (pseudospectral) method**. This method involves seeking an $N$th degree $x$-polynomial $u_N(x,t)$ that satisfies

$$\frac{\partial u_N(x,t)}{\partial t} = \frac{\partial u_N(x,t)}{\partial x}, \quad \text{at } x = x_j, \quad 1 \le j \le N \tag{3.4}$$

with the boundary condition

$$u_N(1, t) = u_N(x_0, t) = g(t). \tag{3.5}$$

Note that the equation is satisfied at all the grid points except at the boundary point $x = 1$ where the boundary condition is satisfied.

In general, the term $\frac{\partial}{\partial x} u_N(x,t)$ is evaluated at all the grid points with the use of either FFT or matrix-vector multiplication using the matrix $\mathcal{D}_C$. Equation (3.4) is then advanced at all the grid points. The value of the solution at the boundary is then updated using (3.5).

In [5] a *penalty-type method* was introduced. In that approach we still use equation (3.4) for the inner points $x_j, 1 \leq j \leq N$; however, instead of using (3.5) for the boundary, the following equation is satisfied:

$$\frac{du_N(1,t)}{dt} = \frac{\partial u_N(x,t)}{\partial x}|_{x=1} - \tau(u_N(1,t) - g(t)) \tag{3.6}$$

where $\tau$ is determined from stability considerations. In particular it had been found that stability follows if

$$\tau \geq \frac{N^2}{2}.$$

Equations (3.4) and (3.6) can be combined into a single equation by noting that the collocation points $x_j$ defined in (2.3) are the zeros of the polynomial $(1-x^2)T'_N(x)$. Thus the penalty method [5] can be written now as

$$\frac{\partial u_N(x_j,t)}{\partial t} = \frac{\partial u_N(x,t)}{\partial x}|_{x=x_j} - \tau \frac{(1+x_j)T'_N(x_j)}{2T'_N(1)}(u_N(1,t) - g(t)) \tag{3.7}$$

for $j = 0, \ldots, N$.

The main difference between the penalty method (3.7) and the usual Chebyshev method given in (3.4) and (3.5) is that the numerical solution $u_N(x,t)$ does not satisfy the boundary condition exactly, but only in the limit as $N \longrightarrow \infty$. The boundary condition is now part of the equation.

Another penalty method based on the *Legendre points* $y_j$ is presented in [6]. Similar to (2.6) we write this method as

$$\frac{\partial u_N(y_j,t)}{\partial t} = \frac{\partial u_N(x,t)}{\partial x}|_{x=y_j} - \tau \frac{(1+y_j)P'_N(y_j)}{2P'_N(1)}(u_N(1,t) - g(t)) \tag{3.8}$$

for $j = 0, \ldots, N$.

The parameter $\tau$ is determined by the stability requirement. Thus the differential equation is satisfied at the points $y_j, j = 1, \ldots, N$. At the boundary $x_0 = 1$ one uses a combination of the boundary condition and the differential equation.

An obvious disadvantage of the method in (3.8) is that it utilizes the Legendre points. However, comparing (3.7) with (3.8) shows us how to utilize the LP method (3.8) at the Chebyshev points.

**The CL method.** Let $P_N(x)$ be the Legendre polynomial of degree $N$. In the CL method we seek a polynomial of degree $N$ in $x$ that satisfies

$$\frac{\partial u_N(x_j,t)}{\partial t} = \frac{\partial u_N(x,t)}{\partial x}|_{x=x_j} - \tau \frac{(1+x_j)P'_N(x_j)}{2P'_N(1)}(u_N(1,t) - g(t)) \tag{3.9}$$

for $j = 0, \ldots, N$.

Note that the penalty term $\frac{(1+x_j)P'_N(x_j)}{2P'_N(1)}$ is different from zero for all the Chebyshev grid points $x_j$. Note also that applying (3.9) entails the use of the differentiation matrix $\mathcal{D}_C$ at the *Chebyshev* points. In fact, given $u_N(x_j,t)$ one finds the derivative based on the Chebyshev points and then adds the penalty term with different weights at *every* grid point. The term $P'_N(x_j)$ is evaluated using the explicit formula (2.8). This is done once and for all for any grid size $N$.

The surprising fact is that the CL method, though computed at the Chebyshev points, is stable in the usual $L_2$ norm, rather than the weighted $L_2$ norm. In fact one can state the following theorem.

THEOREM 3.1.1 (The $L_2$ stability of the CL method). *Let $u_N(x,t)$ be the solution of* (3.9). *Let $\omega_j$ be the weights of the* GLL *quadrature formula and $y_j$ the nodes of the same quadrature formula. Let $g(t) = 0$ in* (3.3) *and* (3.9), *then for*

$$\tau \geq \frac{1}{2\omega_0} = N(N+1)$$

*the* CL *method is stable in the $L_2$ norm. More specifically,*

$$\sum_{j=0}^{N} u_N^2(y_j,t)\omega_j = \sum_{j=0}^{N} u_N^2(y_j,0)\omega_j - \int_0^t [u_N^2(1,t)(2\omega_0\tau - 1) + u_N^2(-1,t)]\, dt. \tag{3.10}$$

*Proof.* It follows from (3.9) that

$$\frac{\partial u_N(x,t)}{\partial t} = \frac{\partial u_N(x,t)}{\partial x} - \tau\frac{(1+x)P'_N(x)}{2P'_N(1)}(u_N(1,t) - g(t)). \tag{3.11}$$

This is because both sides of (3.9) are polynomials of degree $N$ that agree at $N+1$ points, namely, at the Chebyshev collocation points $x_j, 0 \leq j \leq N$.

We now read (3.11) at the *Legendre* points $y_j$ to get

$$\frac{d}{dt}\sum_{j=0}^{N} u_N^2(y_j,t)\omega_j = 2\sum_{j=0}^{N} u_N(y_j,t)\frac{\partial u_N(y_j,t)}{\partial x}\omega_j - 2\tau\omega_0 u_N^2(1,t).$$

Since the GLL quadrature formula is exact for polynomials of degree $2N-1$ it follows that

$$2\sum_{j=0}^{N} u_N(y_j,t)\frac{\partial u_N(y_j,t)}{\partial x}\omega_j = \int_{-1}^{1} (u_N^2(\xi,t))_\xi d\xi = u_N^2(1,t) - u_N^2(-1,t)$$

and thus the stability estimate (3.10) follows. □

Note that unlike the LP method (3.8), in which one needs to use the Legendre points $y_j$ in the computations, these points do not appear in the computations in the CL method. They are just introduced for the sake of the proof. The actual computations are done using the Chebyshev grid points $x_j$.

An energy estimate based on the Chebyshev points $x_j$ can be derived by using (2.11) and (2.12) as follows in Theorem 3.1.2.

THEOREM 3.1.2. *Let*

$$||u_N(\cdot,t)||^2 = \sum_{j=0}^{N}\sum_{l=0}^{N} H_{j,l}\, u_N(x_l,t)u_N(x_j,t) \tag{3.12}$$

*with*

$$H_{j,l} = \sum_{k=0}^{N} g_j(y_k) g_l(y_k) \omega_k$$

*where the* CL *polynomials* $g_j(x)$ *are defined in* (2.10).

*Then* $u_N(\cdot, t)$ *satisfies the energy estimate*

$$||u_N(\cdot,t)||^2 = ||u_N(\cdot,0)||^2 - \int_0^t \{u_N^2(1,t)(2\omega_0\tau - 1) + u_N^2(-1,t)\}dt. \tag{3.13}$$

*Proof.* Equation (3.13) is really a restatement of (3.10). Since $u_N(x,t)$ is a polynomial of degree $N$ in $x$, it can be represented exactly by

$$u_N(x,t) = I_C u_N = \sum_{l=0}^{N} u_N(x_l, t) g_l(x) .$$

Thus

$$u_N(y_j, t) = \sum_{l=0}^{N} u_N(x_l, t) g_l(y_j) . \tag{3.14}$$

The estimate (3.13) follows from (3.10) upon substituting (3.14) for the values of $u_N(y_j, t)$. □

The CL method can be viewed from many different points of view. Theorem 3.1.1 shows that this method, in the constant coefficient case, is equivalent to the LP method introduced in [5]. *Thus the* CL *method is the realization of the Legendre method at the Chebyshev points.*

We close this section by pointing out that the estimate (3.13) enables one to pass to systems easily. Actually it had been done in [5]. One must follow the same steps. It shows that the CL method is stable for constant coefficients systems of hyperbolic equations.

**3.2. The differentiation matrices.** Perhaps more insight can be gained if one compares the differentiation matrix induced by the CL method $\mathcal{D}_{CL}$ with the differentiation matrices induced by the Chebyshev penalty (CP) method (3.7) $\mathcal{D}_C$ and the LP method (3.8) $\mathcal{D}_L$.

We start by noting that the differentiation matrices $\mathcal{D}_L$ and $\mathcal{D}_C$ defined in (2.13) and (2.14) do not take into account any boundary conditions. The differentiation matrices induced by (3.7)–(3.9) are variations of the basic matrices $\mathcal{D}_L$ and $\mathcal{D}_C$, differing only in the method of imposing boundary condition.

Not surprisingly, $\mathcal{D}_L$ and $\mathcal{D}_C$ are similar because both differentiate *exactly* polynomials of degree $N$. Since for these polynomials the operators $I_L$ and $I_C$ are the same, the matrices $\mathcal{D}_L$ and $\mathcal{D}_C$ represent the same operation in a different basis. This implies a similarity relationship. More specifically, this relationship can be written explicitly in Theorem 3.2.1.

THEOREM 3.2.1. *Let* $S$ *be the matrix whose elements* $S_{j,k}$ *are given by*

$$S_{j,k} = h_j(x_k) \tag{3.15}$$

*where* $h_j(x)$ *are the Legendre–Lagrange polynomials defined in* (2.9). *Again* $x_j$ *are the Chebyshev points.*

*Let $\mathcal{T}$ be the matrix whose elements $T_{j,k}$ are given by*

$$T_{j,k} = g_j(y_k) \tag{3.16}$$

*where $g_j(x)$ the* CL *polynomials are defined in* (2.10). *$y_k$ are the Legendre collocation points.*

*Then*

$$\mathcal{S} = \mathcal{T}^{-1} \tag{3.17}$$

*and*

$$\mathcal{D}_C = \mathcal{S}\mathcal{D}_L\mathcal{T}. \tag{3.18}$$

*Proof.* Since $g_j(x)$ is a polynomial of degree $N$ it is given by

$$g_j(x) = \sum_{l=0}^{N} g_j(y_l)h_l(x). \tag{3.19}$$

Substituting $x_k$ and making use of the fact that $g_j(x_k) = \delta_{j,k}$, we get

$$\delta_{j,k} = \sum_{l=0}^{N} g_j(y_l)h_l(x_k),$$

proving that

$$\mathcal{I} = \mathcal{S}\mathcal{T}.$$

Differentiating (3.19) we get

$$g_j'(x) = \sum_{l=0}^{N} g_j(y_l)h_l'(x).$$

However $h_l'(x)$ is itself a polynomial of degree $N$ and therefore it can be expressed as

$$h_l'(x) = \sum_{m=0}^{N} h_l'(y_m)h_m(x),$$

which leads to

$$g_j'(x_k) = \sum_{l=0}^{N}\sum_{m=0}^{N} g_j(y_l)h_l'(y_m)h_m(x_k). \qquad \square$$

We will show now that the differentiation matrix induced by the CL method $\mathcal{D}_{CL}$ is similar to the differentiation matrix $\mathcal{D}_{CP}$ induced by the LP method (3.8). This will demonstrate the fact that the CL method is the realization of the Legendre method on the Chebyshev grid.

THEOREM 3.2.2. *Let $\mathcal{D}_{CL}$ be the differentiation matrix induced by the* CL *method* (3.9) *and $\mathcal{D}_{LP}$ the differentiation method induced by the* LP *method* (3.8) *then*

$$\mathcal{D}_{CL} = \mathcal{S}\mathcal{D}_{LP}\mathcal{T} \tag{3.20}$$

*where the matrices* $\mathcal{S}, \mathcal{T}$ *defined in* (3.15), (3.16) *are the transformation matrices between the Chebyshev points and the Legendre points.*

*Proof.* Note that the differentiation matrix $\mathcal{D}_{LP}$ is essentially the matrix $\mathcal{D}_L$ introduced in (2.13), modified to take into account the boundary conditions, imposed via penalty in (3.8). Thus

$$(\mathcal{D}_{LP})_{j,k} = (\mathcal{D}_L)_{j,k} - \tau\delta_{0,k}\delta_{j,0}\,. \tag{3.21}$$

In the same manner we can write explicitly

$$(\mathcal{D}_{CL})_{j,k} = (\mathcal{D}_C)_{j,k} - \tau\frac{(1+x_k)P_N'(x_k)}{2P_N'(1)}\delta_{j,0}\,. \tag{3.22}$$

Equation (3.22) is a direct consequence of (3.9). Note that in (3.21) only the element $(0,0)$ of the matrix $\mathcal{D}_L$ is modified whereas in (3.22) the 0th column of the matrix $\mathcal{D}_C$ is modified.

We proceed by writing explicitly the elements of the matrix $\mathcal{S}\mathcal{D}_{LP}\mathcal{T}$. In fact

$$(\mathcal{S}\mathcal{D}_{LP}\mathcal{T})_{j,k} = (\mathcal{S}\mathcal{D}_L\mathcal{T})_{j,k} - \tau\sum_{l=0}^{N}\sum_{m=0}^{N} g_j(y_l)\delta_{0,l}\delta_{0,m}h_m(x_k).$$

Thus using (3.18) we get

$$(\mathcal{S}\mathcal{D}_{LP}\mathcal{T})_{j,k} = (\mathcal{D}_C)_{j,k} - \tau g_j(y_0)h_0(x_k). \tag{3.23}$$

From (2.9) we get

$$h_0(x_k) = \frac{(1+x_k)P_N'(x_k)}{2P_N'(1)}$$

and since $y_0 = x_0 = 1$ $g_j(y_0) = \delta_{j,0}$ so the right-hand side of (3.23) is exactly the same as that of (3.22).

Thus (3.20) is established. □

**4. Parabolic equations.** In this section we present the CL method for the parabolic equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad -1 \le x \le 1, \quad t > 0 \tag{4.1}$$

with Robin boundary condition

$$\begin{aligned} \alpha u(1,t) + \beta u_x(1,t) &= g^+(t), \\ \gamma u(-1,t) + \delta u_x(-1,t) &= g^-(t). \end{aligned} \tag{4.2}$$

We will assume that $\alpha, \beta, \gamma$ are nonnegative and $\delta$ is nonpositive. This assures the time decay (or nongrowth) of $u(x,t)$.

We note that by now there is a very limited stability theory for the Chebyshev method. In fact stability had been proved first for the Dirichlet case $\beta = 0$, $\delta = 0$ (see [7]) and then for Neumann case $\alpha = \gamma = 0$. (See [9].) Here we present the CL method and prove stability for the approximation to (4.1) and (4.2) for the general Robin case.

Denote by $\mathcal{P}_N$ the finite-dimensional space of polynomial of degree at most $N$. We define the operator $\mathcal{A}$ to be

$$\mathcal{A} : \mathcal{P}_N \to \mathcal{P}_N$$

by

$$\mathcal{A}v(x,t) = -\frac{\partial^2 v(x,t)}{\partial x^2} + R(x,t) \tag{4.3}$$

where

$$R(x,t) = \tau_0 Q^+(x)[B^+(t) - g^+(t)] + \tau_N Q^-(x)[B^-(t) - g^-(t)] \tag{4.4}$$

with

$$Q^+(x) = \frac{(1+x)P_N'(x)}{2P_N'(1)}, \qquad B^+(t) = \alpha v(1,t) + \beta v_x(1,t),$$
$$Q^-(x) = \frac{(1-x)P_N'(x)}{2P_N'(1)}, \qquad B^-(t) = \gamma v(-1,t) + \delta v_x(-1,t).$$

The numbers $\tau_0, \tau_N$ will be determined later to assure stability. We define also the following scalar product

$$(v,w)_N = \sum_{j=0}^{N} v(y_j)w(y_j)\omega_j \tag{4.5}$$

where $y_j, \omega_j$ are the Legendre points and weights, respectively.

**The CL method for parabolic equations.** We seek the polynomial of degree $N$ in $x$, $v(x,t)$ that satisfies

$$\frac{\partial v(x_j,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2}|_{x=x_j} - R(x_j,t), \qquad 0 \le j \le N \tag{4.6}$$

where $x_j$ are Chebyshev collocation points.

Note that, again, the work is done on the Chebyshev points $x_j$; the penalty values $Q^+(x_j), Q^-(x_j)$ are computed by (2.8) and are nonzero for *any* $x_j$.

To prove the stability of (4.6), we set $g^+(t) = g^-(t) = 0$. In the following lemma, we will find conditions on $\tau_0$ and $\tau_N$ such that the operator $\mathcal{A}$ is semibounded.

LEMMA 4.1. *Let* $v \in \mathcal{P}_N$ *and*

$$\tau_{a,b}^+ = \frac{1}{\omega_0 b}[(1+2\kappa) + 2\sqrt{\kappa+\kappa^2}],$$
$$\tau_{a,b}^- = \frac{1}{\omega_0 b}[(1+2\kappa) - 2\sqrt{\kappa+\kappa^2}]$$

*with* $\kappa = \omega_0 a/b$.

*Let the operator* $\mathcal{A}$ *be defined in* (4.3). *Then*

$$(\mathcal{A}v, v)_N \ge \sum_{j=1}^{N-1} v_x^2(y_j)\omega_j \tag{4.7}$$

*provided that*

$$\tau^-_{\alpha,\beta} \le \tau_0 \le \tau^+_{\alpha,\beta}, \tag{4.8}$$

$$\tau^-_{\gamma,|\delta|} \le \tau_N \le \tau^+_{\gamma,|\delta|}. \tag{4.9}$$

*Proof.* Since the Gauss–Lobatto quadrature formula (2.6) is exact for polynomials of degree $2N-1$ and since $v(x,t)$ is a polynomial of degree $N$, we have

$$\sum_{j=0}^{N} v(y_j)v_{xx}(y_j)\omega_j = \int_{-1}^{1} v(x)v_{xx}(x)dx$$
$$= v(1)v_x(1) - v(-1)v_x(-1) - \int_{-1}^{1} v_x(x)v_x(x)dx$$

using the standard integration-by-parts technique.

Again using the Gauss–Lobatto formula, one would get

$$\begin{aligned} -\sum_{j=0}^{N} v(y_j)v_{xx}(y_j)\omega_j &= \sum_{j=0}^{N} v_x^2(y_j)\omega_j - v(1)v_x(1) + v(-1)v_x(-1) \\ &= \sum_{j=1}^{N-1} v_x^2(y_j)\omega_j \\ &\quad + v_x^2(1)\omega_0 + v_x^2(-1)\omega_N - v(1)v_x(1) + v(-1)v_x(-1). \end{aligned} \tag{4.10}$$

Thus making use of (4.10) we can write

$$(\mathcal{A}v, v)_N = F(1,\alpha,\beta,0) + F(-1,\gamma,|\delta|,N) + \sum_{j=1}^{N-1} v_x^2(y_j)\omega_j \tag{4.11}$$

where

$$F(x,a,b,k) = x(\tau_k b\,\omega_k - 1)v(x)v_x(x) + \tau_k a\,\omega_k v^2(x) + \omega_k v_x^2(x)\,. \tag{4.12}$$

For $\mathcal{A}$ to be positive we need to choose $\tau_0$ and $\tau_N$ such that $F(1,\alpha,\beta,0)$ and $F(-1,\gamma,|\delta|,N)$ are nonnegative. For $F(1,\alpha,\beta,0)$ to be positive, we need

$$(\tau_0\beta\omega_0 - 1)^2 \le 4\alpha\tau_0\omega_0^2$$

or

$$\tau_0^2\beta^2\omega_0^2 - 2\tau_0\omega_0(\beta + 2\alpha\omega_0) + 1 \le 0.$$

Thus $\tau_0$ must lie between the roots of the parabola described in the left-hand side, namely, $\tau^-_{\alpha,\beta}$ and $\tau^+_{\alpha,\beta}$.

The same kind of consideration holds for $\tau_N$. Thus $F(1,\alpha,\beta,0)$ and $F(-1,\gamma,|\delta|,N)$ are nonnegative for the range of $\tau_0$ and $\tau_N$ given in (4.8) and (4.9), respectively. (4.7) follows from (4.11).

*Remark* 1. The Dirichlet boundary condition for $x = 1$ is obtained from (4.2) by setting $\alpha = 1, \beta = 0$. In this case

$$\tau^+_{1,0} = \infty,$$
$$\tau^-_{1,0} = \frac{1}{4\omega_0^2},$$

which yields the condition for the penalty amplitude

$$\tau_0 \geq \frac{1}{16} N^4 (N+1)^4.$$

*Remark* 2. The Neumann boundary condition for $x = 1$ corresponds to the case $\alpha = 0, \beta = 1$. In this case $\tau_{0,1}^{+} = \tau_{0,1}^{-}$, yielding the condition

$$\tau_0 = \frac{1}{\omega_0} = \frac{N(N+1)}{2}.$$

We are now ready to state the stability theorem for the CL method when applied to parabolic equations with Robin boundary conditions.

THEOREM 4.1. Let $\tau_0$ and $\tau_N$ satisfy (4.8) and (4.9), respectively. Let $v(x,t) \in \mathcal{P}_N$ be the CL approximation to $u(x,t)$, obtained by (4.6). Assuming that $g^{+}(t) = g^{-}(t) = 0$, $v(x,t)$ satisfies the energy estimate

$$(v(x,t), v(x,t))_N \leq (v(x,0), v(x,0))_N - 2 \int_0^T \sum_{j=1}^{N-1} v_x^2(y_j, t) dt \tag{4.13}$$

where the scalar product $(f, g)_N$ is defined in (3.5).

*Proof.* Since (4.6) holds for $j = 0, \ldots, N$ and since $v, v_{xx}$, and $R$ are polynomials of degree at most $N$, we conclude that both sides of (4.6) are equal not only at the grid points but also for every $x$.

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} - R(x,t), \qquad -1 \leq x \leq 1$$

where $R(x,t)$ is defined in (4.4).

Noting the definition of $\mathcal{A}$ in (4.3), we get

$$\frac{\partial v}{\partial t} = \mathcal{A} v\,.$$

Thus

$$\left(v, \frac{\partial v}{\partial t}\right)_N = (v, \mathcal{A} v)_N.$$

Using Lemma 4.1 yields

$$\frac{1}{2} \frac{d}{dt} (v, v)_N \leq - \sum_{j=1}^{N-1} v_x^2 (y_j, t) \omega_j$$

and integration yields the stability result (4.13).

We stress again that the Legendre collocation points $y_j$ are "ghost points," which are never used in the computations but only in the proof of the stability. Actually we could restate the proof in terms of the Chebyshev collocation points $x_j$ as in Theorem 3.1.2.

**5. Numerical results.**

*Case* 1 (Linear scalar PDE). In this section, we will consider some numerical examples that verify our claims stated in previous sections. Consider the scalar linear initial-boundary value hyperbolic PDE

$$U_t = U_x, \quad -1 \leq x \leq 1, \quad t > 0 \tag{5.1}$$

with initial condition

$$U(x,0) = \sin(2\pi k x)$$

and boundary condition at $x = 1$,

$$U(1,t) = g(t) = \sin(2\pi k(1+t)).$$

We seek an $N$ degree $x$-polynomial $v(x,t)$ that satisfies

$$\frac{d}{dt} v(x_j,t) = Dv(x_j,t) - \tau Q(x_j)(v(1,t) - g(t)) \tag{5.2}$$

at Chebyshev collocation point $x_j = \cos(\pi j/N)$, $j = 0, \ldots, N$, and $D$ is the differentiation operator (matrix).

For different construction of the $N$th degree polynomial $Q(x)$, one could have different types of boundary treatments. For example,

1. if $Q(x) = \frac{(1+x)P_N'(x)}{2P_N'(1)}$, $D = D_C$, and $x = x_j$ are the GLC points, then we have the CL method;
2. if $Q(x) = \frac{(1+x)T_N'(x)}{2T_N'(1)}$, $D = D_C$, and $x = x_j$ are the GLC points, then we have the CP method;
3. if $Q(x) = \frac{(1+x)P_N'(x)}{2P_N'(1)}$, $D = D_L$, and $x = y_j$ are the GLL points, then we have the LP method.

Let us denote $v_j^{(n)} = v(x_j, t_n)$ and let $\Delta t$ be the timestep increment, then for $j = 0, \ldots, N$, we would advance the system of ODEs (5.2) in time by the third order Heun–Runge–Kutta scheme that has the following form.

For $j = 0, 1, \ldots, N$, and $v_j^{(0)} = g(0)$,

$$\begin{aligned}
v_j^{(1)} &= v_j^{(n)} + \frac{\Delta t}{3}(Dv_j^{(n)} - \tau Q(x_j)(v_j^{(n)} - g(t_n))),\\
v_j^{(2)} &= v_j^{(n)} + \frac{2\Delta t}{3}(Dv_j^{(1)} - \tau Q(x_j)(v_j^{(1)} - g(t_n) - \frac{\Delta t}{3} g'(t_n))),\\
v_j^{(n+1)} &= \frac{1}{4} v_j^{(n)} + \frac{3}{4} v_j^{(1)}\\
&\quad + \frac{3\Delta t}{4}(Dv_j^{(2)} - \tau Q(x_j)(v_j^{(2)} - g(t_n) - \frac{\Delta t}{3} g'(t_n) - \frac{2\Delta t^2}{9} g''(t_n))),
\end{aligned} \tag{5.3}$$

where $g'(t_n)$ and $g''(t_n)$ are the derivatives of the time-dependent boundary conditions in time at $t = t_n$.

It has been observed before that if one imposes boundary condition at each intermediate stages of the Runge–Kutta scheme, a larger timestep (the Courant–Friedrichs–Lewy (CFL) number) can be used. Otherwise, the CFL number must be reduced by as much as four times for stability. In this study, we define $\Delta t = CFL/N^2$.

The traditional way of imposing the exact boundary condition at $x = 1$ can be described as the following.

For $j = 0, \ldots, N$ and $v_j^{(0)} = g(0)$,

$$
\begin{aligned}
v_j^{(1)} &= v_j^{(n)} + \frac{\Delta t}{3} D v_j^{(n)}, \\
v_0^{(1)} &= g\left(t_n + \frac{\Delta t}{3}\right), \\
v_j^{(2)} &= v_j^{(n)} + \frac{2\Delta t}{3} D v_j^{(1)}, \\
v_0^{(2)} &= g\left(t_n + \frac{2\Delta t}{3}\right), \\
v_j^{(n+1)} &= \frac{1}{4} v_j^{(n)} + \frac{3}{4} v_j^{(1)} + \frac{3\Delta t}{4} D v_j^{(2)}, \\
v_0^{(n+1)} &= g(t_n + \Delta t).
\end{aligned}
\tag{5.4}
$$

However, as shown in Table 1, this procedure would lead to reduction of accuracy in time as $N$ increases.

TABLE 1
*$L_2$ error and order of accuracy for* (5.4) *with* $k = 1$.

| N | Error | Rate | Error | Rate | Error | Rate |
|---|---|---|---|---|---|---|
| 16 | 0.82E−03 | | 0.10E−03 | | 0.29E−05 | |
| 32 | 0.15E−04 | 2.89 | 0.18E−05 | 2.91 | 0.28E−07 | 3.35 |
| 64 | 0.42E−06 | 2.57 | 0.49E−07 | 2.61 | 0.72E−09 | 2.64 |
| 128 | 0.17E−07 | 2.31 | 0.19E−08 | 2.33 | 0.28E−10 | 2.34 |
| CFL | 8 | | 4 | | 1 | |

Hence, the above procedure is modified as follows. (Detailed discussion and analysis will appear in a future paper.)

For $j = 0, 1, \ldots, N$ and $v_j^{(0)} = g(0)$,

$$
\begin{aligned}
v_j^{(1)} &= v_j^{(n)} + \frac{\Delta t}{3} D v_j^{(n)}, \\
v_0^{(1)} &= g(t_n) + \frac{\Delta t}{3} g'(t_n), \\
v_j^{(2)} &= v_j^{(n)} + \frac{2\Delta t}{3} D v_j^{(1)}, \\
v_0^{(2)} &= g(t_n) + \frac{2\Delta t}{3} g'(t_n) + \frac{2\Delta t^2}{9} g''(t_n), \\
v_j^{(n+1)} &= \frac{1}{4} v_j^{(n)} + \frac{3}{4} v_j^{(1)} + \frac{3\Delta t}{4} D v_j^{(2)}, \\
v_0^{(n+1)} &= g(t_{n+1}).
\end{aligned}
\tag{5.5}
$$

Table 2 indicates that the order-of-time accuracy for this procedure (5.5) is third order for all $N$.

Next, using the CL method, one gets the $L_2$ error and the order of accuracy as listed in Table 3.

From Table 4, we can see that for $\tau < 2\omega_0$ CL becomes unstable, while for $\tau \geq 2\omega_0$ the convergent of the scheme confirms the theoretical prediction.

TABLE 2
*$L_2$ error and order of accuracy for* (5.5) *with* $k = 1$.

| N | Error | Rate | Error | Rate | Error | Rate |
|---|---|---|---|---|---|---|
| 16 | 0.77E−03 | | 0.98E−04 | | 0.28E−05 | |
| 32 | 0.12E−04 | 3.00 | 0.15E−05 | 3.00 | 0.24E−07 | 3.44 |
| 64 | 0.19E−06 | 3.00 | 0.24E−07 | 3.00 | 0.37E−09 | 3.00 |
| 128 | 0.30E−08 | 3.00 | 0.37E−09 | 3.00 | 0.58E−11 | 2.99 |
| CFL | 8 | | 4 | | 1 | |

TABLE 3
*$L_2$ error and order of accuracy for* CL *method with* $k = 1, \tau = 4\omega_0$.

| N | Error | Rate | Error | Rate | Error | Rate |
|---|---|---|---|---|---|---|
| 16 | 0.47E−03 | | 0.60E−04 | | 0.28E−05 | |
| 32 | 0.74E−05 | 2.99 | 0.93E−06 | 3.01 | 0.15E−07 | 3.81 |
| 64 | 0.12E−06 | 3.00 | 0.15E−07 | 3.00 | 0.23E−09 | 3.00 |
| 128 | 0.18E−08 | 3.00 | 0.23E−09 | 3.00 | 0.36E−11 | 2.99 |
| CFL | 8 | | 4 | | 1 | |

TABLE 4
*$L_2$ error of* CL *method for different choices of* $\tau = 2\omega_0\alpha$ *with* $k = 1, CFL = 1$.

| N | $\alpha = 8$ | $\alpha = 2$ | $\alpha = 1$ | $\alpha = 0.9$ | $\alpha = 0.5$ |
|---|---|---|---|---|---|
| 16 | 0.31E−05 | 0.28E−05 | 0.65E−05 | 0.83E−05 | 0.32E−03 |
| 32 | 0.15E−07 | 0.15E−07 | 0.15E−07 | 0.15E−07 | 0.18E−01 |
| 64 | 0.23E−09 | 0.23E−09 | 0.23E−09 | 0.23E−06 | unstable |
| 128 | 0.36E−11 | 0.36E−11 | 0.36E−11 | unstable | unstable |

TABLE 5
*$L_2$ error of* CL *method for different choices of* $\tau = 2\omega_0\alpha$ *with* $k = 1, CFL = 1$.

| N | $\alpha = 8$ | $\alpha = 4$ | $\alpha = 3$ | $\alpha = 2.5$ | Exact BC |
|---|---|---|---|---|---|
| 16 | 0.86E−02 | 0.10E−01 | 0.18E−01 | 0.27E−01 | 0.72E−02 |
| 32 | 0.40E−07 | 0.40E−07 | 0.40E−07 | 0.11E−04 | 0.39E−07 |
| 64 | 0.68E−09 | 0.68E−09 | 0.68E−09 | unstable | 0.67E−09 |
| 128 | 0.11E−10 | 0.11E−10 | 0.11E−10 | unstable | 0.11E−10 |

*Case* 2 (Nonlinear scalar PDE). Consider the scalar nonlinear initial boundary value hyperbolic equation

$$U_t = \tfrac{1}{2}U_x^2 - 2\pi k\cos(2\pi k(x+t))(1+\sin(2\pi k(x+t))) \qquad (5.6)$$
$$-1 \le x \le 1, \qquad t > 0$$

with initial condition

$$U(x,0) = 2 + \sin(2\pi kx)$$

and boundary condition (BC) at $x = 1$,

$$U(1,t) = g(t) = 2 + \sin(2\pi k(1+t))\,.$$

This PDE has an exact solution given as $U(x,t) = 2 + \sin(2\pi k(x+t))$.

Table 5 gives the $L_2$ error of the CL method for different choices of $\tau = 2\omega_0\alpha$ with $k = 1, CFL = 1$. Different values of $k$ are also tested; similar results are obtained.

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