In this article, we use the theory of elliptic functions to construct theta function identities which are equivalent to Macdonald’s identities for $A_2$, $B_2$ and $G_2$. Using these identities, we express, for $d = 8$, 10 or 14, certain theta functions in the form $\eta^d(\tau)F(P, Q, R)$, where $\eta(\tau)$ is Dedekind’s eta-function, and $F(P, Q, R)$ is a polynomial in Ramanujan’s Eisenstein series $P$, $Q$ and $R$. We also derive identities in the case when $d = 26$. These lead to a new expression for $\eta^{26}(\tau)$. This work generalizes the results for $d = 1$ and $d = 3$ which were given by Ramanujan on page 369 of ‘The Lost Notebook’.

1. Introduction

Let $\text{Im}(\tau) > 0$ and put $q = \exp(2\pi i \tau)$. Dedekind’s eta-function is defined by

$$\eta(\tau) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k),$$

and Ramanujan’s Eisenstein series are

$$P = P(q) = 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1 - q^k},$$
$$Q = Q(q) = 1 + 240 \sum_{k=1}^{\infty} \frac{k^3q^k}{1 - q^k}$$

and

$$R = R(q) = 1 - 504 \sum_{k=1}^{\infty} \frac{k^5q^k}{1 - q^k}.$$  

On page 369 of The Lost Notebook [28], Ramanujan gave the following results.

**Theorem 1.1 (Ramanujan).** Let

$$S_1(m) = \sum_{\alpha \equiv 1 \pmod{6}} (-1)^{(\alpha-1)/6} \alpha^m q^{\alpha^2/24},$$

$$S_3(m) = \sum_{\alpha \equiv 1 \pmod{4}} \alpha^m q^{\alpha^2/8}.$$  

Then

$$S_1(0) = \eta(\tau),$$

$$S_1(2) = \eta(\tau)P,$$

$$S_1(4) = \eta(\tau)(3P^2 - 2Q),$$

$$S_1(6) = \eta(\tau)(15P^3 - 30PQ + 16R),$$

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and in general
\[
S_1(2m) = \eta(\tau) \sum_{i+2j+3k=m} a_{ijk}P^iQ^jR^k,
\]
where \(a_{ijk}\) are integers and \(i, j\) and \(k\) are non-negative integers. Also
\[
S_3(1) = \eta^3(\tau),
\]
\[
S_3(3) = \eta^3(\tau)P,
\]
\[
S_3(5) = \eta^3(\tau)\frac{(5P^2 - 2Q)}{3},
\]
\[
S_3(7) = \eta^3(\tau)\frac{(35P^3 - 42PQ + 16R)}{9},
\]
and in general
\[
S_3(2m + 1) = \eta^3(\tau) \sum_{i+2j+3k=m} b_{ijk}P^iQ^jR^k,
\]
where \(b_{ijk}\) are rational numbers and \(i, j\) and \(k\) are non-negative integers.

The results for \(S_1(0)\) and \(S_3(1)\) are well-known consequences of the Jacobi triple product identity [1, p. 500]. Ramanujan also listed the values of \(S_1(8)\), \(S_1(10)\), \(S_3(9)\) and \(S_3(11)\). He indicated that these results may be proved by induction, using differentiation and the Ramanujan differential equations [26, equation (30)]
\[
q \frac{dP}{dq} = \frac{P^2 - Q}{12}, \quad q \frac{dQ}{dq} = \frac{PQ - R}{3}, \quad q \frac{dR}{dq} = \frac{PR - Q^2}{2}.
\]

Theorem 1.1 has been studied by Venkatachaliengar [36, pp. 31–32] (where both \(S_1\) and \(S_3\) are studied), Berndt and Yee [5] (where \(S_1\) is studied) and Berndt, Chan, Liu and Yesilyurt [6] (where \(S_3\) is studied). For a different approach to these identities, see Ramanujan [27, Chapter 16, Entry 35(i)] (for \(S_3\), Berndt [4, p. 61] (for \(S_3\)) and Liu [22] (for \(S_1\)).

The first purpose of this article is to prove analogous results corresponding to the 2nd, 4th, 6th, 8th, 10th, 14th and 26th powers of \(\eta(\tau)\), these being the even powers of \(\eta(\tau)\) that are lacunary [33, Theorem 1]. For example, the result for the 14th power is as follows. For non-negative integers \(m, n, \ell\), let
\[
S_{14}(m, n, \ell) = \sum_{\substack{\alpha \equiv 2 \pmod{6} \\ \beta \equiv 1 \pmod{4}}} \frac{(-1)^{(\alpha^2-2)/6} (\beta(\alpha^2 - \beta^2))^m}{12} (\alpha(\alpha^2 - 9\beta^2))^n \times (\alpha^2 + 3\beta^2)^\ell q^{(\alpha^2 + 3\beta^2)/12}.
\]

Then
\[
S_{14}(2m + 1, 2n + 1, \ell) = \eta^{14}(\tau) \sum_{i+2j+3k=3m+3n+\ell} c_{ijk}P^iQ^jR^k, \quad m, n, \ell \geq 0, \quad (1.1)
\]
where \(c_{ijk}\) are rational numbers and \(i, j\) and \(k\) are non-negative integers. The first few instances of (1.1) are
\[
S_{14}(1, 1, 0) = -30\eta^{14}(\tau),
\]
\[
S_{14}(1, 1, 1) = -210\eta^{14}(\tau)P,
\]
\[
S_{14}(1, 1, 2) = -210\eta^{14}(\tau)(8P^2 - Q),
\]
\[
S_{14}(3, 1, 0) = -5\eta^{14}(\tau)(56P^3 - 21PQ + 19R),
\]
\[
S_{14}(1, 3, 0) = -15\eta^{14}(\tau)(504P^3 - 189PQ - 115R).
\]
An equation equivalent to the one for \( S_{14}(1, 1, 0) \) was stated without proof by Winquist [38]. Since
\[
\beta(\alpha^2 - \beta^2)\alpha(\alpha^2 - 9\beta^2) = \alpha^5 \beta - 10\alpha^3 \beta^3 + 9\alpha \beta^5 = \frac{1}{6\sqrt{3}} \text{Im} \left( (\alpha + i\beta\sqrt{3})^6 \right),
\]
the result for \( S_{14}(1, 1, 0) \) may be written as
\[
\sum_{\substack{\alpha \equiv 2 \pmod{6} \\ \beta \equiv 1 \pmod{4}}} (-1)^{(\alpha-2)/6} \text{Im} \left( (\alpha + i\beta\sqrt{3})^6 \right) q^{(\alpha^2 + 3\beta^2)/12} = -180\sqrt{3} \eta^{14}(\tau).
\]

The second purpose of this article is to prove results of the type
\[
\sum_{\substack{\alpha \equiv 2 \pmod{6} \\ \beta \equiv 1 \pmod{4}}} (-1)^{(\alpha-2)/6} \text{Im} \left( (\alpha + i\beta\sqrt{3})^6 \right) q^{(\alpha^2 + 3\beta^2)/12} = \sqrt{3} \eta^{14}(\tau) \sum_{2j+3k=3(n-1)} d_{jk} Q^j R^k,
\]
where \( d_{jk} \) are rational numbers and \( j \) and \( k \) are non-negative integers. We shall state analogues of this result for the 2nd, 4th, 6th, 8th, 10th and 26th powers of \( \eta(\tau) \) and give a detailed proof for the 10th power.

This work is organized as follows.

Notation and properties of theta functions are established in Section 2.

Sections 3, 4 and 5 are devoted to the 8th, 10th and 14th powers of \( \eta(\tau) \), respectively. Each section begins with a multivariate theta function identity which is then used to prove the analogues of \( (1.1) \) for the 8th, 10th or 14th power of \( \eta(\tau) \).

Section 6 is concerned with the analogues of \( (1.1) \) for 2nd, 4th and 6th powers of \( \eta(\tau) \). These follow from Ramanujan’s Theorem 1.1.

In Section 7, we prove results analogous to \( (1.2) \) for the 2nd, 4th, 6th, 8th, 10th and 14th powers of \( \eta(\tau) \). Since Ramanujan’s Eisenstein series \( P \) does not occur in these results, the modular transformation for multiple theta series given by Schoeneberg [32] can be used to prove them.

In Section 8, we give a simple proof of a series expansion for \( \eta^{26}(\tau) \), as well as analogues of \( (1.1) \) and \( (1.2) \) for the 26th power of \( \eta(\tau) \) which are new. The proofs rely on two different analogues of \( (1.2) \) for \( \eta^2(\tau) \).

Finally, in Section 9, we make some remarks about lacunary series and the Hecke operator, and a new formula for \( \eta^{24}(\tau) \) is presented.

2. Preliminaries

In the classical theory of theta functions [37], the notation \( q = \exp(\pi it) \) is used, whereas in the theory of modular forms \( q = \exp(2\pi i\tau) \). Because we will use both theories, we let \( t = 2\tau \) and define
\[
q = \exp(\pi it) = \exp(2\pi i\tau).
\]

We will use \( t \) when working with theta functions and \( \tau \) for modular forms and Dedekind’s eta-function.

The Jacobi theta functions [1, p. 509; 37, Chapter 21], are defined by
\[
\theta_1(z|t) = 2 \sum_{k=0}^{\infty} (-1)^k q^{(k+1/2)^2} \sin(2k + 1)z,
\]
\[
\theta_2(z|t) = 2 \sum_{k=0}^{\infty} q^{(k+1/2)^2} \cos(2k + 1)z,
\]
\[
\theta_3(z|t) = 1 + 2 \sum_{k=1}^{\infty} q^{k^2} \cos 2kz
\]
and

\[ \theta_4(z|t) = 1 + 2 \sum_{k=1}^{\infty} (-1)^k q^k \cos 2kz. \]

Let

\[ G_2(z|t) = 2 \sum_{\alpha \equiv 1 \pmod{6}} q^{\alpha^2/12} \sin(\alpha z), \]
\[ G_3(z|t) = 2 \sum_{\alpha \equiv 4 \pmod{6}} q^{\alpha^2/12} \sin(\alpha z), \]
\[ H(z|t) = G_2(4z|4t) - G_3(4z|4t) \]
\[ = 2 \sum_{\alpha \equiv 2 \pmod{6}} (-1)^{(\alpha-2)/6} q^{\alpha^2/12} \sin(2\alpha z) \]

and

\[ T(z|t) = \theta_4(2z|t). \]

These functions satisfy the transformation properties

\[ \theta_1(z + \pi|t) = -\theta_1(z|t), \quad \theta_1(z + \pi t|t) = -q^{-1} e^{-2iz} \theta_1(z|t), \]
\[ \theta_2(z + \pi|t) = -\theta_2(z|t), \quad \theta_2(z + \pi t|t) = q^{-1} e^{-2iz} \theta_2(z|t), \]
\[ \theta_3(z + \pi|t) = \theta_3(z|t), \quad \theta_3(z + \pi t|t) = q^{-1} e^{-2iz} \theta_3(z|t), \]
\[ \theta_4(z + \pi|t) = \theta_4(z|t), \quad \theta_4(z + \pi t|t) = -q^{-1} e^{-2iz} \theta_4(z|t), \]
\[ G_2(z + \pi|t) = -G_2(z|t), \quad G_2(z + \pi t|t) = q^{-3} e^{-6iz} G_2(z|t), \]
\[ G_3(z + \pi|t) = G_3(z|t), \quad G_3(z + \pi t|t) = q^{-3} e^{-6iz} G_3(z|t), \]
\[ H\left(z + \frac{\pi}{2}\right) = H(z|t), \quad H\left(z + \frac{\pi t}{2}\right) = -q^{-3} e^{-12iz} H(z|t), \]
\[ T\left(z + \frac{\pi}{2}\right) = -T(z|t), \quad T\left(z + \frac{\pi t}{2}\right) = -q^{-1} e^{-4iz} T(z|t). \]

By the Jacobi triple product identity [1, p. 497],

\[ \theta_1(z|t) = 2q^{1/4} \sin z \prod_{k=1}^{\infty} (1 - q^{2k} e^{2iz})(1 - q^{2k} e^{-2iz})(1 - q^{2k}). \]

Therefore, \( \theta_1(z|t) \) has simple zeros at \( z = \pi m + \pi tn, \ m, n \in \mathbb{Z}, \) and no other zeros.

We will also need the results

\[ \theta_2(z|t) G_2(z|t) = \eta(2\tau) \theta_1(2z|t), \quad (2.1) \]
\[ \theta_3(z|t) G_3(z|t) = -\eta(2\tau) \theta_1(2z|t). \quad (2.2) \]

These are equivalent to the quintuple product identity. For example, see [34, Proposition 2.1], where these and two other similar equations are given. Equations (2.1) and (2.2), together with the Jacobi triple product identity, imply that \( G_2(z|t) \) has simple zeros when \( z = \pi m/2 + \pi tn/2, \) where \( m \) and \( n \) are integers and \( (m, n) \neq (1, 0) \pmod{2}, \) and no other zeros. Similarly, \( G_3(z|t) \) has simple zeros when \( z = \pi m/2 + \pi tn/2, \) where \( m \) and \( n \) are integers and \( (m, n) \neq (1, 1) \pmod{2}, \) and no other zeros. Equations (2.1) and (2.2) also imply that

\[ \theta_2(z|t) G_2(z|t) + \theta_3(z|t) G_3(z|t) = 0. \]
The following lemma is of fundamental importance and will be used several times in the proofs in the subsequent sections. Let $f^{(\ell)}(z|t)$ denote the $\ell$th derivative of $f(z|t)$ with respect to $z$.

**Lemma 2.1.**

$$
\theta_1^{(2\ell_1+1)} \left( \frac{t}{2} \right) \theta_1^{(2\ell_2+1)} \left( \frac{t}{2} \right) \cdots \theta_1^{(2\ell_m+1)} \left( \frac{t}{2} \right) = (\eta(\tau))^{3m} \sum_{i+j+3k=\ell_1+\ell_2+\cdots+\ell_m} a_{ijk} P^i Q^j R^k
$$

for some rational numbers $a_{ijk}$, where $i$, $j$ and $k$ are non-negative integers.

**Proof.** Let us first consider the case $m = 1$. From the definition of $\theta_1$, we have

$$
\theta_1^{(2\ell_1+1)}(z|t) = 2(-1)^\ell \sum_{k=0}^{\infty} (-1)^k (2k + 1)^{2\ell_1+1} q^{(k+(1/2))^2/2} \cos(2k+1)z.
$$

Therefore

$$
\theta_1^{(2\ell_1+1)} \left( \frac{t}{2} \right) = 2(-1)^\ell \sum_{k=0}^{\infty} (-1)^k (2k + 1)^{2\ell_1+1} q^{(k+(1/2))^2/2}
$$

$$
= 2(-1)^\ell \sum_{k=-\infty}^{\infty} (4k + 1)^{2\ell_1+1} q^{(4k+1)^2/8}
$$

$$
= 2(-1)^\ell S_3(2\ell + 1)
$$

$$
= \eta^3(\tau) \sum_{i+j+3k=\ell} a_{ijk} P^i Q^j R^k
$$

by Theorem 1.1. The general case $m \geq 1$ now follows by multiplying $m$ copies of this result together.

Finally, we define the standard notation for products:

$$
(x; q)_\infty = \prod_{k=0}^{\infty} (1 - xq^k)
$$

and

$$
(x_1, x_2, \ldots, x_m; q)_{\infty} = (x_1; q)_\infty (x_2; q)_\infty \cdots (x_m; q)_\infty.
$$

3. **The eighth power of $\eta(\tau)$**

The main tool used in this section is the following.

**Theorem 3.1.**

$$
G_2(x|t)\theta_2(y|t) + G_3(x|t)\theta_3(y|t) = \frac{1}{\eta(\tau)} \theta_1 \left( \frac{t}{2} \right) \theta_1 \left( \frac{x+y}{2} \right) \theta_1 \left( \frac{x-y}{2} \right).
$$

**Proof.** Let

$$
M_8(x, y|t) = G_2(x|t)\theta_2(y|t) + G_3(x|t)\theta_3(y|t)
$$

and

$$
N_8(x, y|t) = \theta_1 \left( \frac{t}{2} \right) \theta_1 \left( \frac{x+y}{2} \right) \theta_1 \left( \frac{x-y}{2} \right).
$$
Then the formulae listed in Section 2 imply that $M_8$ and $N_8$ satisfy the transformation properties

\begin{align*}
    f(x + 2\pi, y|t) &= f(x, y|t),
    f(x + \pi t, y|t) = q^{-3}e^{-6ix}f(x, y|t),
    f(x, y + 2\pi|t) &= f(x, y|t),
    f(x, y + \pi|t) = q^{-1}e^{-2iy}f(x, y|t).
\end{align*}

Fix $y$ and consider $M_8$ and $N_8$ as functions of $x$; $N_8$ has simple zeros at $x = \pi m + \pi t n / 2$, $\pm y + 2\pi m + \pi t n$, $m$, $n \in \mathbb{Z}$, and no other zeros. By the results in Section 2, we see that $M_8$ also has zeros at these points, and possibly at other points too. Therefore, $M_8(x, y|t)/N_8(x, y|t)$ is an elliptic function of $x$ with no poles and thus is a constant independent of $x$.

Now fix $x$ and consider $M_8$ and $N_8$ as functions of $y$; $N_8$ has simple zeros at $y = \pm x + 2\pi m + \pi t n$ and no other zeros. It is easy to check that $M_8$ also has zeros at these points, and possibly at other points too. Therefore, $M_8/N_8$ is an elliptic function of $y$ with no poles and thus is a constant independent of $y$.

It follows that

\[ \frac{M_8(x, y|t)}{N_8(x, y|t)} = C(q) \]

for some $C(q)$ independent of $x$ and $y$. To calculate $C(q)$, let $x = \pi /2$ and $y = \pi$. Since $G_3(\pi /2|t) = 0$, we have

\[
    M_8\left(\frac{\pi}{2}|\pi t\right) = G_2\left(\frac{\pi}{2}|t\right) \theta_2(\pi|t)
    = -2 \sum_{k=-\infty}^{\infty} (-1)^k q^{(6k+1)/12} \sum_{j=-\infty}^{\infty} q^{(j+(1/2))^2}
    = -4\eta(2\tau) q^{1/4}(-q^2, -q^2, q^2, q^2)_{\infty}
    = -4\eta^2(4\tau).
\]

On the other hand

\[
    N_8\left(\frac{\pi}{2}|\pi t\right) = \theta_1\left(-\frac{\pi}{4}|\frac{t}{2}\right) \theta_1\left(\frac{\pi}{2}|\frac{t}{2}\right) \theta_1\left(\frac{3\pi}{4}|\frac{t}{2}\right)
    = -\left(2q^{1/8}\right)^3 \sin \frac{\pi}{4} \sin \frac{\pi}{2} \sin \frac{3\pi}{4} (iq, -iq, q)_{\infty}^2 (-q, -q, q)_{\infty}
    = -4\eta(\tau) \eta^2(4\tau),
\]

after simplifying. Therefore

\[
    C(q) = \frac{M_8(\pi/2, \pi|t)}{N_8(\pi/2, \pi|t)} = \frac{1}{\eta(\tau)}.
\]

This completes the proof of Theorem 3.1. \hfill \Box

**Theorem 3.2.** Let $m$ and $n$ be non-negative integers and define

\[
    S_8(m, n) = \sum_{\alpha \equiv 1 \pmod{3}, \alpha + \beta \equiv 0 \pmod{2}} \alpha^m \beta^n q^{(\alpha^2 + 3\beta^2)/12}.
\]

Then $S_8(1, 0) = 0$ and

\[
    S_8(2m + 1, 2n) = \eta^8(\tau) \sum_{i + 2j + 3k = m + n - 1} a_{ijk} P^i Q^j R^k,
\]

provided $m + n \geq 1$. Here, $a_{ijk}$ are rational numbers and $i$, $j$ and $k$ are non-negative integers.
Proof. Apply \( \partial (2m+2n+1)/(\partial x^{2m+1} \partial y^{2n}) \) to the identity in Theorem 3.1 and let \( x = y = 0 \). The left-hand side is

\[
\begin{align*}
G_2^{(2m+1)}(0|t) \theta_2^{(2n)}(0|\tau) + G_3^{(2m+1)}(0|t) \theta_3^{(2n)}(0|\tau) \\
= 2(-1)^{m+n} \alpha^{2m+1} q^{\alpha^2/12} \beta^{2n} q^{\beta^2/4} \\
+ 2(-1)^{m+n} \alpha^{2m+1} q^{\alpha^2/12} \beta^{2n} q^{\beta^2/4} \\
= 2(-1)^{m+n} \sum_{\alpha \equiv 1 \text{ (mod 3)}} \alpha^{2m+1} \beta^{2n} q^{(\alpha^2+3\beta^2)/12}.
\end{align*}
\]

Since \( \theta_1(z|t) \) is an odd function, the right-hand side is a linear combination of terms of the form

\[
\frac{1}{\eta(\tau)} \theta_1^{(2\ell_1+1)}(0|t/2) \theta_1^{(2\ell_2+1)}(0|t/2) \theta_1^{(2\ell_3+1)}(0|t/2),
\]

where \( (2\ell_1 + 1) + (2\ell_2 + 1) + (2\ell_3 + 1) = 2m + 2n + 1 \). By Lemma 2.1, the right-hand side is therefore of the form

\[
\eta^8(\tau) \sum_{i+2j+3k=m+n-1} a_{ijk} P^i Q^j R^k.
\]

If we combine (3.2) and (3.3), then we complete the proof of the theorem for the case \( m + n \geq 1 \). The result for \( S_8(1, 0) \) is obtained similarly. \( \square \)

The following identities are consequences of Theorem 3.2:

\[
\begin{align*}
S_8(1, 0) &= 0, \\
S_8(3, 0) &= -6\eta^8(\tau), \\
S_8(5, 0) &= -30\eta^8(\tau) P, \\
S_8(7, 0) &= -63/2 \eta^8(\tau) (5P^2 - Q), \\
S_8(7, 2) &= 2\eta^8(\tau) R, \\
S_8(5, 4) &= \eta^8(\tau) (5P^3 - 3PQ).
\end{align*}
\]

We also have

\[
\begin{align*}
S_8(3, 0) : S_8(1, 2) &= -3 : 1, \\
S_8(5, 0) : S_8(3, 2) : S_8(1, 4) &= -15 : 1 : 1, \\
S_8(7, 0) : S_8(5, 2) : S_8(3, 4) : S_8(1, 6) &= -63 : 1 : 1 : 1, \\
\begin{pmatrix}
S_8(9, 0) \\
S_8(3, 6) \\
S_8(1, 8)
\end{pmatrix} &= 
\begin{pmatrix}
-66 & -189 \\
1/3 & 2/3 \\
2/9 & 7/9
\end{pmatrix} 
\begin{pmatrix}
S_8(7, 2) \\
S_8(5, 4)
\end{pmatrix}.
\end{align*}
\]

An identity equivalent to \( S_8(1, 2) = 2\eta^8(\tau) \) was stated without proof by Winquist [38]. The formula for \( \eta^8(\tau) \) given by Klein and Fricke [19, p. 373] can be shown to be equivalent to
$S_8(3,0) + 27S_8(1,2) = 48q^{8}(\tau)$. Schoeneberg [31, equation (11)] gave the attractive form

$$\eta^{8}(\tau) = \frac{1}{6} \sum_{\mu \in \mathbb{Z}[\exp(2\pi i/3)]} \chi(\mu) \mu^{3} \exp \left( \frac{2\pi i \tau |\mu|^{2}}{3} \right),$$

where

$$\chi(\mu) = \begin{cases} 1 & \text{if } \mu \equiv 1 \pmod{\sqrt{-3}}, \\ -1 & \text{if } \mu \equiv -1 \pmod{\sqrt{-3}}. \end{cases}$$

(The sum over the terms satisfying $\mu \equiv 0(\mod \sqrt{-3})$ is zero.) Schoeneberg’s formula can be deduced from the formulae for $S_8(3,0)$ and $S_8(1,2)$.

Theorem 3.1 is equivalent to Macdonald’s identity for $A_2$ (see [10; 11, Theorem 2.1; 23], or [35, p. 146]) in the form

$$(u, qu^{-1}, v, qv^{-1}, uv, qu^{-1}v^{-1}, q, q; q)_{\infty} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{3m^{2}} q^{-3mn+3n^{2}+m+n} h_{m,n}(u, v),$$

where $u = e^{i(x+y)}$, $v = e^{i(x-y)}$ and

$$h_{m,n}(u, v) = uv \left\{ (u^{3m-1}v^{-3n-1} - u^{3m+1}v^{3n+1}) + (u^{3n-3m}v^{-3n} - u^{3m-3n}v^{3m-3n}) \right\}.$$ 

4. The tenth power of $\eta(\tau)$

The main tool used in this section is the following.

**Theorem 4.1.**

$$G_{3}(x|t)G_{2}(y|t) - G_{2}(x|t)G_{3}(y|t) = \frac{1}{\eta^{10}(\tau)} \theta_{1}\left( x \left\lfloor \frac{t}{2} \right\rfloor \theta_{1}\left( y \left\lfloor \frac{t}{2} \right\rfloor \theta_{1}\left( \frac{x+y}{2} \left\lfloor \frac{t}{2} \right\rfloor \theta_{1}\left( \frac{x-y}{2} \left\lfloor \frac{t}{2} \right. \right) \right) \right) \right).$$

**Proof.** Apply the technique used in the proof of Theorem 3.1. Let

$$M_{10}(x, y|t) := G_{3}(x|t)G_{2}(y|t) - G_{2}(x|t)G_{3}(y|t)$$

and

$$N_{10}(x, y|t) := \theta_{1}\left( x \left\lfloor \frac{t}{2} \right\rfloor \theta_{1}\left( y \left\lfloor \frac{t}{2} \right\rfloor \theta_{1}\left( \frac{x+y}{2} \left\lfloor \frac{t}{2} \right\rfloor \theta_{1}\left( \frac{x-y}{2} \left\lfloor \frac{t}{2} \right) \right) \right) \right).$$

Then $M_{10}$ and $N_{10}$ satisfy the transformation formulae

$$f(x + 2\pi, y|t) = f(x, y|t), \quad f(x + \pi t, y|t) = q^{-3}e^{-6i\pi}f(x, y|t),$$

$$f(x, y + 2\pi|t) = f(x, y|t), \quad f(x, y + \pi|t) = q^{-3}e^{-6i\pi}f(x, y|t).$$

Let $y$ be fixed. Then $N_{10}$ has simple zeros at $x = \pi m + \pi n/2, \pm y + 2\pi m + \pi n$, $m, n \in \mathbb{Z}$, and no other zeros. The results in Section 2 imply that $M_{10}$ also has zeros at the same points as $N_{10}$, and possibly at other points too. Thus $M_{10}(x, y|t)/N_{10}(x, y|t)$ is an elliptic function of $x$ with no poles and, therefore, is a constant which is independent of $x$.

By the symmetry in $x$ and $y$, we find that $M_{10}(x, y|t)/N_{10}(x, y|t)$ is also independent of $y$ and, therefore, depends only on $q$. Let us denote the constant by $D(q)$. To determine its value,
let $x = \pi/2$ and $y = \pi/6$. Since $G_3(\pi/2|t) = 0$ we have
\[
M_{10}(\frac{\pi}{2}, \frac{\pi}{6}|t) = -G_2(\frac{\pi}{2}|t)G_3(\frac{\pi}{6}|t)
\]
\[
= -4 \sum_{j=-\infty}^{\infty} q^{(6j+1)^2/12} \sin((3j + \frac{1}{2})\pi) \sum_{k=-\infty}^{\infty} q^{(6k-2)^2/12} \sin((k - \frac{1}{3})\pi)
\]
\[
= 2\sqrt{3} \left( q^{1/12} \sum_{j=-\infty}^{\infty} (-1)^j q^{3j^2+j} \right) \left( q^{1/3} \sum_{k=-\infty}^{\infty} (-1)^k q^{3k^2-2k} \right)
\]
\[
= 2\sqrt{3} \eta(2\pi)q^{1/3}(q, q^3, q^6; q)^{\infty}
\]
\[
= 2\sqrt{3} \frac{\eta(3\pi)}{\eta(3\pi)}
\]

On the other hand, writing $\gamma = \exp(i\pi/3)$, we have
\[
N_{10}(\frac{\pi}{2}, \frac{\pi}{6} | t) = \theta_1(\frac{\pi}{6}, \frac{1}{2})^2 \theta_1(\frac{\pi}{3}, \frac{1}{2})^2 \theta_1(\frac{\pi}{2}, \frac{1}{2})^2
\]
\[
= \left( 2q^{1/8} \right) \sin^{\pi/3} \sin^{\pi/3} \sin^{\pi/3} \gamma q, \gamma^3 q, q; q)^{\infty} (\gamma^3 q, \gamma^3 q, q; q)^{\infty}
\]
\[
= 2\sqrt{3} \frac{\eta^3(\pi)q^2(6\pi)}{\eta(3\pi)}
\]
after simplifying the infinite products. So,
\[
D(q) = \frac{M_{10}(\pi/3, (\pi/6)|t)}{N_{10}(\pi/3, (\pi/6)|t)} = \frac{1}{\eta^2(\tau)}.
\]

**Theorem 4.2.** Let
\[
S_{10}(m, n) = \sum_{\alpha \equiv 1 \pmod{6}, \beta \equiv 4 \pmod{6}} (\alpha^m \beta^n - \alpha^n \beta^m) q^{(\alpha^2 + \beta^2)/12}.
\]

Then
\[
S_{10}(2m + 1, 2n + 1) = \eta^{10}(\tau) \sum_{i+2j+3k=m+n-1} a_{ijk} P^i Q^j R^k,
\]
where $a_{ijk}$ are rational numbers and $i, j$ and $k$ are non-negative integers.

**Proof.** Apply $\partial^2m+2n+2/(\partial x^{m+1}\partial y^{n+1})$ to both sides of Theorem 4.1; then let $x = y = 0$. We omit the details as they are similar to those in the proof of Theorem 3.2. 

The first few examples of Theorem 4.2 are
\[
S_{10}(3, 1) = 6\eta^{10}(\tau),
\]
\[
S_{10}(5, 1) = 30\eta^{10}(\tau) P,
\]
\[
S_{10}(7, 1) = \frac{63}{2} \eta^{10}(\tau)(5P^2 - Q),
\]
\[
S_{10}(5, 3) = \frac{3}{2} \eta^{10}(\tau)(15P^2 + Q),
\]
\[
S_{10}(9, 1) = 3\eta^{10}(\tau)(315P^3 - 189PQ + 44R),
\]
\[
S_{10}(7, 3) = \frac{3}{2} \eta^{10}(\tau)(105P^3 - 21PQ - 4R).
\]

Theorem 4.1 is equivalent to Winquist’s identity [38, Theorem 1.1]: put $a = e^{i(x+y)}$, $b = e^{i(x-y)}$ in Theorem 4.1 to get [38, Theorem 1.1]. Observe that the left-hand side of
Theorem 4.1 is a difference of two terms, and each term is a product of two series that can be summed by the quintuple product identity. This was first noticed by Kang [18]. More information on Winquist’s identity can be found in [6, 7, 9, 14, 17, 20, 21].

5. The fourteenth power of $\eta(\tau)$

The main tool used in this section is the following.

**Theorem 5.1.**

$$H(x|t)T(y|t) + H\left(\frac{x-y}{2}\right)T\left(\frac{3x+y}{2}\right) + H\left(\frac{x+y}{2}\right)T\left(\frac{-3x+y}{2}\right)$$

$$= \frac{1}{\eta^4(\tau)} \theta_1\left(x, \frac{t}{2}\right) \theta_1(y, \frac{t}{2}) \theta_1\left(x+y, \frac{t}{2}\right) \theta_1\left(x+y, \frac{t}{2}\right) \theta_1\left(-3x+y, \frac{t}{2}\right).$$

**Proof.** Apply the elliptic function method used in the previous two sections. By the results in Section 2, it may be checked that both sides satisfy the transformation formulae

$$f(x + 2\pi, y|t) = f(x, y|t), \quad f(x + \pi t, y|t) = q^{-12}e^{-24ix}f(x, y|t),$$

$$f(x, y + 2\pi|t) = f(x, y|t), \quad f(x, y + \pi t|t) = q^{-4}e^{-4iy}f(x, y|t).$$

It is straightforward to check that for a fixed value of $x$ or $y$, the left-hand side is zero whenever the right-hand side is zero. Finally, the constant may be evaluated by letting $x = -\pi/8$, $y = 7\pi/8$.

Because the left-hand side of Theorem 5.1 is more complicated than the left-hand sides of Theorems 3.1 and 4.1, some extra analysis is needed before differentiating. We will need the following.

**Lemma 5.2.** Let $D_x = \partial/\partial x$ and $D_y = \partial/\partial y$. Let $f(z)$ and $g(z)$ be analytic functions. Let

$$\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \left\{\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right), \left(\begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array}\right), \left(\begin{array}{cc} 1 & 1 \\ 3 & 2 \end{array}\right)\right\}.$$

Then

$$D_xD_y(D_x^2 - D_y^2)(D_x^2 - 9D_y^2) (f(ax+by)g(cx+dy))$$

$$= f^{(5)}(ax+by)g'(cx+dy) - 10f''(ax+by)g''(cx+dy)$$

$$+ 9f'(ax+by)g^{(5)}(cx+dy). \quad (5.1)$$

More generally, for non-negative integers $m$, $n$ and $\ell$, define an operator $D_{x,y}(m, n, \ell)$ and coefficients $c_{i,j}(m, n, \ell)$ by

$$D_{x,y}(m, n, \ell) = (D_y(D_x^2 - D_y^2))^m (D_x(D_x^2 - 9D_y^2))^n (D_x^2 + 3D_y^2)^\ell$$

$$= \sum_{i+j=3m+3n+2\ell} c_{i,j}(m, n, \ell) D_x^i D_y^j.$$

Then

$$D_{x,y}(2m+1, 2n+1, \ell) (f(ax+by)g(cx+dy))$$

$$= \sum_{i+j=6(m+n+1)+2\ell} c_{i,j}(2m+1, 2n+1, \ell) (f^{(i)}(ax+by)g^{(j)}(cx+dy)). \quad (5.2)$$
Lemma 2.1.

Let $x$ it follows that

Therefore, without loss of generality, we may assume that $0 = 0$.

If we combine (5.3) and (5.4), then we obtain (5.1), which is the case $m = n = \ell = 0$ of (5.2).

The general result (5.2) now follows by induction on $m$, $n$ and $\ell$, using (5.3)–(5.5).

Theorem 5.3.

Let

$$S_{14}(m, n, \ell) = \sum_{\alpha \equiv 2 \pmod{6}, \beta \equiv 1 \pmod{4}} (-1)^{(\alpha-2)/6} (\beta(\alpha^2 - \beta^2))^m \left( \alpha(\alpha^2 - 9\beta^2) \right)^n \left( \alpha^2 + 3\beta^2 \right)^\ell q^{(\alpha^2+3\beta^2)/12}.$$ 

Then

$$S_{14}(2m + 1, 2n + 1, \ell) = \eta^{14}(\tau) \sum_{i+2j+3k=3m+3n+\ell} a_{ijk} P^i Q^j R^k,$$

where $a_{ijk}$ are rational numbers and $i$, $j$ and $k$ are non-negative integers.

Proof. Apply the operator $D_{x,y}(2m + 1, 2n + 1, \ell)$ to the identity in Theorem 5.1, then let $x = y = 0$. For the left-hand side use Lemma 5.2, and for the right-hand side use Lemma 2.1.

Since

$$(\alpha^2 + 3\beta^2)^3 = 27\beta^2(\alpha^2 - \beta^2)^2 + \alpha^2(\alpha^2 - 9\beta^2)^2,$$

it follows that

$$S_{14}(2m + 1, 2n + 1, \ell + 3) = 27S_{14}(2m + 3, 2n + 1, \ell) + S_{14}(2m + 1, 2n + 3, \ell).$$

Therefore, without loss of generality, we may assume that $0 \leq \ell \leq 2$.

The first few examples of Theorem 5.3 were given in Section 1. Theorem 5.1 is equivalent to Macdonald’s identity for $G_2$ (see [11, equation (1.8)]) written in the form

$$(u, qu^{-1}, uv, qu^{-1}u^{-1}, u^2v, qu^{-2}v^{-1}, u^3v, qu^{-3}v^{-1}, v, qu^{-1}, u^3v^2, qu^{-3}v^{-2}, q, q; q)_\infty$$

$$= \sum_m \sum_n q^{12m^2-12mn+4n^2-m-n} H_{m,n}(u, v),$$

where $u = e^{2ix}$, $v = e^{i(y-3x)}$ and

$$H_{m,n}(u, v) = u^5 v^3 \left\{ u^{12m-5} v^{4n-3} + u^{-12m+5} v^{-4n+3} \right. $$

$$- (u^{12n-12m-4} v^{4n-3} + u^{12m-12n+4} v^{-4n+3}) $$

$$+ (u^{12n-12m-4} v^{8n-12m-1} + u^{12m-12n+4} v^{12m-8n+1}) $$

$$- (u^{12n-24m+1} v^{8n-12m-1} + u^{24m-12n-1} v^{12m-8n+1}) $$

$$+ (u^{12n-24m+1} v^{4n-12m+2} + u^{24m-12n-1} v^{12m-4n-2}) $$

$$- (u^{-12m+5} v^{4n-12m+2} + u^{12m-5} v^{12m-4n-2}) \left\}.$$
6. Second, fourth and sixth powers of $\eta(\tau)$

Analogous results for $\eta^2(\tau)$, $\eta^4(\tau)$ and $\eta^6(\tau)$ can be obtained trivially by multiplying Ramanujan’s results for $S_1$ and $S_3$. Specifically, let

\[
S_2(m, n) = S_1(m)S_1(n),
\]
\[
S_4(m, n) = S_1(m)S_3(n),
\]
\[
S_6(m, n) = S_3(m)S_3(n).
\]

Then

\[
S_2(2m, 2n) = \eta^2(\tau) \sum_{i+2j+3k = m+n} a_{ijk} P^i Q^j R^k, \tag{6.1}
\]
\[
S_4(2m, 2n + 1) = \eta^4(\tau) \sum_{i+2j+3k = m+n} a_{ijk} P^i Q^j R^k, \tag{6.2}
\]
\[
S_6(2m + 1, 2n + 1) = \eta^6(\tau) \sum_{i+2j+3k = m+n} a_{ijk} P^i Q^j R^k. \tag{6.3}
\]

In each case, $a_{ijk}$ are rational numbers and $i, j$ and $k$ are non-negative integers.

Another form for $\eta^6(\tau)$ was given by Schoeneberg [31, equation (8)]:

\[
\eta^6(\tau) = \frac{1}{2} \sum_{a = -\infty}^{\infty} \sum_{b = -\infty}^{\infty} \text{Re}(a + 2ib)^2 q^{(a^2+4b^2)/4}.
\]

This formula can be shown to be equivalent to the identity for $S_6(1, 1)$ by direct series manipulations.

Results of a different type for $\eta^6(\tau)$ may be obtained using a series given by Hirschhorn [16]. Let

\[
S_6^*(m, n) = \sum_{\substack{\alpha \equiv 1 \pmod{10} \\ \beta \equiv 3 \pmod{10}}} (-1)^{(\alpha+\beta-4)/10}(\alpha^m \beta^n - \alpha^n \beta^m)q^{(\alpha^2+\beta^2)/40}.
\]

Hirschhorn’s result is

\[
S_6^*(0, 2) = 8\eta^6(\tau).
\]

Using the techniques in this paper, it can be shown that if $m + n \geq 1$, then

\[
S_6^*(2m, 2n) = \eta^6(\tau) \sum_{i+2j+3k = m+n-1} a_{ijk} P^i Q^j R^k,
\]

where $a_{ijk}$ are rational numbers and $i, j$ and $k$ are non-negative integers.

7. Identities obtained using Schoeneberg’s theta functions

In this section, we prove (1.2) and analogous results for 2nd, 4th, 6th, 8th and 10th powers of $\eta(\tau)$. Most of the results in this section are new. A few special cases can be found in Ramanujan’s The Lost Notebook, for example [28, p. 249]. Some of Ramanujan’s identities have recently been examined by Rangachari [29, 30], using Hecke’s theta functions [15].

The results we shall prove are as follows.

**Theorem 7.1.** Let

\[
C_2(n|\tau) = \sum_{\substack{\alpha \equiv 1 \pmod{6} \\ \beta \equiv 1 \pmod{6}}} (-1)^{(\alpha+\beta-2)/6}(\alpha + i\beta)^n q^{(\alpha^2+\beta^2)/24}.
\]

Then $C_2(4n|\tau)/\eta^2(\tau)$ is a modular form of weight $4n$ on $\text{SL}_2(\mathbb{Z})$. 

Theorem 7.2. Let
\[ C_2^*(n|\tau) = \sum_{\alpha \equiv 0 \atop \beta \equiv 1} (\alpha + i\beta)^n q^{(\alpha + i\beta)^{6n}/36}. \]
Then \( C_2^*(6n|\tau)/\eta^2(\tau) \) is a modular form of weight \( 6n \) on \( SL_2(\mathbb{Z}) \).

Theorem 7.3. Let
\[ C_4(n|\tau) = \sum_{\alpha \equiv 1 \atop \beta \equiv 1} (-1)^{(\alpha - 1)/6} \text{Im} \left( (\alpha + i\beta)^n \right) q^{(\alpha + i\beta)^{2n}/24}. \]
Then \( C_4(2n + 1|\tau)/\eta^2(\tau) \) is a modular form of weight \( 2n \) on \( SL_2(\mathbb{Z}) \).

Theorem 7.4. Let
\[ C_6(n|\tau) = \sum_{\alpha \equiv 1 \atop \beta \equiv 1} (\alpha + i\beta)^n q^{(\alpha + i\beta)^{4n}/8}. \]
Then \( C_6(4n + 2|\tau)/\eta^6(\tau) \) is a modular form of weight \( 4n \) on \( SL_2(\mathbb{Z}) \).

Theorem 7.5. Let
\[ C_8(n|\tau) = \sum_{\alpha \equiv 1 \atop \alpha + \beta \equiv 0} (\alpha + i\beta)^n q^{(\alpha + i\beta)^{6n}/12}. \]
Then \( C_6(6n + 3|\tau)/\eta^8(\tau) \) is a modular form of weight \( 6n \) on \( SL_2(\mathbb{Z}) \).

Theorem 7.6. Let
\[ C_{10}(n|\tau) = \sum_{\alpha \equiv 1 \atop \beta \equiv 4} \text{Im} \left( (\alpha + i\beta)^n \right) q^{(\alpha + i\beta)^{6n}/12}. \]
Then \( C_{10}(4n + 4|\tau)/\eta^{10}(\tau) \) is a modular form of weight \( 4n \) on \( SL_2(\mathbb{Z}) \).

Theorem 7.7. Let
\[ C_{14}(n|\tau) = \sum_{\alpha \equiv 2 \atop \beta \equiv 1} (-1)^{(\alpha - 2)/6} \text{Im} \left( (\alpha + i\beta)^n \right) q^{(\alpha + i\beta)^{6n}/12}. \]
Then \( C_{14}(6n + 6|\tau)/\eta^{14}(\tau) \) is a modular form of weight \( 6n \) on \( SL_2(\mathbb{Z}) \).

In order to prove Theorems 7.1–7.7, we first recall some properties of a class of theta functions studied by Schoeneberg [32]. Let \( f \) be an even positive integer and \( A = (a_{\mu,\nu}) \) be a symmetric \( f \times f \) matrix such that
1. \( a_{\mu,\nu} \in \mathbb{Z} \);
2. \( a_{\mu,\nu} \) is even; and
3. \( x^t A x > 0 \) for all \( x \in \mathbb{R}^f \) such that \( x \neq 0 \).
Let \( N \) be the smallest positive integer such that \( N A^{-1} \) also satisfies conditions (1)–(3). Let
\[ P_k^A(x) := \sum_{y} c_y (y^t A x)^k, \]
where the sum is over finitely many \( y \in \mathbb{C}^f \) with the property \( y^t A y = 0 \), and \( c_y \) are arbitrary complex numbers.
When $A h \equiv 0 \pmod{N}$ and $\text{Im} \tau > 0$, we define
\[
\vartheta_{A,h,P^A}(\tau) = \sum_{n \equiv h \pmod{N}} P^A_k(n) e^{((2\pi i)\tau)/(N)(1/2)((n'A n)/N)}.
\]
The result which we need is the following \[32, Theorem 2, p. 210\].

**Theorem 7.8.** The function $\vartheta_{A,h,P^A}$ satisfies the following transformation formulae
\[
\vartheta_{A,h,P^A}(\tau + 1) = e^{((2\pi i)\tau)/(N)(1/2)((h'h'A h)/N)} \vartheta_{A,h,P^A}(\tau)
\]
and
\[
\vartheta_{A,h,P^A}\left(-\frac{1}{\tau}\right) = \frac{(-i)^{(f/2)+2k} \tau^{(f/2)+k}}{\sqrt{|\det A|}} \sum_{g \equiv 0 \pmod{N}} \sum_{g \equiv 0 \pmod{N}} e^{((2\pi i)\tau)/(N)(g'h'A h)/N)} \vartheta_{A,g,P^A}(\tau).
\]

We will also need the following.

**Lemma 7.9.** Let
\[
\varphi_{r,s}(n; \tau) = \sum_{\alpha \equiv r \pmod{12}} \sum_{\beta \equiv s \pmod{12}} (\alpha - i\beta)^n e^{((2\pi i)\tau)/(N)(1/2)((6(\alpha^2+\beta^2))/12)}.
\]
Then
\[
\varphi_{r,s}(4n; \tau + 1) = e^{6\pi i(r^2+s^2)/12} \varphi_{r,s}(4n; \tau) \tag{7.1}
\]
and
\[
\varphi_{r,s}(4n; -\frac{1}{\tau}) = \frac{(-i)^{4n+1}}{6} \sum_{(u,v) \equiv 0 \pmod{12}} \sum_{(6u,6v) \equiv 0 \pmod{12}} e^{\pi i(ru+sv)/12} \varphi_{u,v}(4n; \tau) \tag{7.2}
\]

**Proof.** These follow from Theorem 7.8 on taking
\[
A = \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}, \quad h = \begin{pmatrix} r \\ s \end{pmatrix}, \quad g = \begin{pmatrix} u \\ v \end{pmatrix}, \quad y = \begin{pmatrix} i \\ 1 \end{pmatrix},
\]
$N = 12$, $k = 4n$, and $f = 2$.

We are now ready to prove Theorems 7.1–7.7. We shall give a detailed proof of Theorem 7.6. The details for the other theorems are similar.

**Proof of Theorem 7.6.** From the first example following Theorem 4.2 and the definition of $C_{10}(4|\tau)$, it follows that
\[
C_{10}(4|\tau) = 24\eta^{10}(\tau). \quad \tag{7.3}
\]
Next, observe that
\[ C_{10}(4n|\tau) \]
\[ = \frac{1}{2i} \left( \sum_{\alpha \equiv 1 \atop \beta \equiv 4 \pmod{6}} (\alpha + i\beta)^4 q^{(\alpha^2 + \beta^2)/12} - \sum_{\alpha \equiv 1 \atop \beta \equiv 4 \pmod{6}} (\alpha - i\beta)^4 q^{(\alpha^2 + \beta^2)/12} \right) \]
\[ = \frac{1}{2^{4n+1}} \left( \sum_{\alpha \equiv 0 \atop \beta \equiv 2 \pmod{12}} (\alpha - i\beta)^4 q^{3(\alpha^2 + \beta^2)/12^2} - \sum_{\alpha \equiv 2 \atop \beta \equiv 0 \pmod{12}} (\alpha - i\beta)^4 q^{3(\alpha^2 + \beta^2)/12^2} \right) \]
\[ = \frac{1}{2^{4n+1}} (\varphi_{8,2}(4n; \tau) - \varphi_{2,8}(4n; \tau)). \] (7.4)

Equation (7.1) implies that
\[ \varphi_{8,2}(4n; \tau + 1) - \varphi_{8,2}(4n; \tau + 1) = e^{5\pi i/6} (\varphi_{8,2}(4n; \tau) - \varphi_{2,8}(4n; \tau)). \] (7.5)

Equation (7.2) gives
\[ \varphi_{8,2} \left( 4n; -\frac{1}{\tau} \right) - \varphi_{2,8} \left( 4n; -\frac{1}{\tau} \right) = -\frac{i\tau^{4n+1}}{6} \sum_{j=1}^{6} \sum_{k=1}^{6} \left( e^{\pi i(4j + k)/3} - e^{\pi i(j + 4k)/3} \right) \varphi_{2j,2k}(4n; \tau). \]

If we use the relation \( \varphi_{r,s}(4n; \tau) = \varphi_{12-r,12-s}(4n; \tau) \) and simplify, then we find that
\[ \varphi_{8,2} \left( 4n; -\frac{1}{\tau} \right) - \varphi_{2,8} \left( 4n; -\frac{1}{\tau} \right) = -\frac{i\tau^{4n+1}}{6} \left( 4(\varphi_{2,4} - \varphi_{4,2})(4n; \tau) + 2(\varphi_{8,2} - \varphi_{2,8})(4n; \tau) + 2(\varphi_{12,2} - \varphi_{2,12})(4n; \tau) + 2(\varphi_{6,12} - \varphi_{12,6})(4n; \tau) \right). \]

It is easy to check that
\[ \varphi_{2,12}(4n; \tau) = \varphi_{12,2}(4n; \tau), \]
\[ \varphi_{4,6}(4n; \tau) = \varphi_{6,4}(4n; \tau), \]
\[ \varphi_{6,12}(4n; \tau) = \varphi_{12,6}(4n; \tau), \]
\[ \varphi_{2,4}(4n; \tau) = \varphi_{8,2}(4n; \tau), \]
\[ \varphi_{4,2}(4n; \tau) = \varphi_{2,8}(4n; \tau). \]

Therefore,
\[ \varphi_{8,2} \left( 4n; -\frac{1}{\tau} \right) - \varphi_{2,8} \left( 4n; -\frac{1}{\tau} \right) = -i\tau^{4n+1} (\varphi_{8,2}(4n; \tau) - \varphi_{2,8}(4n; \tau)). \] (7.6)

Equations (7.3)–(7.6) imply that the function
\[ F(\tau) := \frac{C_{10}(4n|\tau)}{q^{10(\tau)}} \]
satisfies the transformation properties
\[ F(\tau + 1) = F(\tau), \quad F \left( -\frac{1}{\tau} \right) = \tau^{4n-4} F(\tau). \]

That is, \( F(\tau) \) is a modular form of weight \( 4n - 4 \) on \( \text{SL}_2(\mathbb{Z}) \). This completes the proof of Theorem 7.6. \( \square \)
The twenty-sixth power of $\eta(\tau)$

The analogue of (1.2) for the 26th power of $\eta(\tau)$ is as follows.

**Theorem 8.1.** For $n \geq 1$, the function
\[
\frac{1}{\eta^{26}(\tau)} \left( \frac{C_2(12n|\tau)}{36^n} - (-1)^n \frac{C_2(12n|\tau)}{26^n} \right)
\]
is a modular form of weight $12n - 12$ on $\text{SL}_2(\mathbb{Z})$.

**Proof.** Calculations using Theorems 7.1 and 7.2 imply that the first few terms in the $q$-expansions are
\[
C_2(12n|\tau) = (-64)^n q^{1/12} \left( 1 - ((2 + 3i)^{12n} + (2 - 3i)^{12n}) q + \cdots \right),
\]
\[
C^*_2(12n|\tau) = (729)^n q^{1/12} \left( 1 - \left( (1 + 2i\sqrt{3})^{12n} + (1 - 2i\sqrt{3})^{12n} \right) q - 5^{12n} q^2 + \cdots \right).
\]
The $q^2$ terms in the two expansions are different because $\text{Re}\left( ((4 + 3i)/5)^{12n} \right) \neq 1$ for any integer $n$ [25, Corollary 3.12]. Therefore, $C_2(12n|\tau)$ and $C^*_2(12n|\tau)$ are linearly independent. It follows that
\[
\frac{1}{\eta^{26}(\tau)} \left( \frac{C^*_2(12n|\tau)}{36^n} - (-1)^n \frac{C_2(12n|\tau)}{26^n} \right)
\]
is a cusp form of weight $12n$ on $\text{SL}_2(\mathbb{Z})$ and so must be of the form $\eta^{24}(\tau) F$, where $F$ is a modular form of weight $12n - 12$. This completes the proof.

**Corollary 8.2.**
\[
\eta^{26}(\tau) = \frac{1}{16308864} \left( \frac{C_2(12|\tau)}{64} + \frac{C^*_2(12|\tau)}{729} \right).
\]

**Proof.** Take $n = 1$ in Theorem 8.1 and observe that
\[
(2 + 3i)^{12} + (2 - 3i)^{12} - (1 + 2i\sqrt{3})^{12} - (1 - 2i\sqrt{3})^{12} = 16308864.
\]

Corollary 8.2 was discovered and proved in [8]. An equivalent form of this identity had been discovered in 1966 by Atkin [2] (unpublished), and the first published proof was given in 1985 by Serre [33]. The proof we have given here is different from those in the literature.

Here is the analogue of (1.1) for $\eta^{26}(\tau)$.

**Corollary 8.3.** Let $n$ and $\ell$ be integers satisfying $n \geq 1$, $\ell \geq 0$, and define
\[
S_{26}(n, \ell) = \left( q \frac{d}{dq} \right)^{\ell} \left( \frac{C_2^*(12n|\tau)}{36^n} - (-1)^n \frac{C_2(12n|\tau)}{26^n} \right).
\]

Then
\[
S_{26}(n, \ell) = \eta^{26}(\tau) \sum_{i+2j+3k=6(n-1)+\ell} a_{ijk} P^i Q^j R^k,
\]
where $a_{ijk}$ are rational numbers and $i, j$ and $k$ are non-negative integers.

**Proof.** This follows immediately from Theorem 8.1 and the Ramanujan differential equations.
9. Concluding remarks

9.1. Lacunarity and the Hecke operator

By a theorem of Landau [3, Theorem 10.5, p. 244], all of the series $S_2(2m,2n)$, $S_4(2m,2n+1)$, $S_6(2m+1,2n+1)$, $S_8(2m+1,2n)$, $S_{10}(2m+1,2n+1)$, $S_{14}(2m+1,2n+1,\ell)$ and $S_{26}(n,\ell)$ are lacunary. Hence the corresponding expressions on the right-hand sides of (3.1), (4.1), (5.6), (6.1)–(6.3) and (8.1) are lacunary.

Let us write

$$S_{14}(2m+1,2n+1,\ell) = A q^{7/12} \sum_{k=0}^{\infty} a(k)q^{k},$$

where $A$ is a numerical constant selected to make $a(0) = 1$. Then the technique used in [12] implies that if $p \equiv 5 \pmod{6}$ is prime, then

$$a(pk + \frac{7}{12}(p^2 - 1)) = (-1)^{(p+1)/6} p^{6(m+n+1)+2\ell} a\left(\frac{k}{p}\right).$$

Similar results for $S_2$, $S_4$, $S_6$, $S_8$, $S_{10}$ and $S_{26}$ may also be written down. These results generalize a theorem of Newman [24].

9.2. Ramanujan’s $\tau$ function

Ramanujan’s function $\tau(n)$ is defined by

$$q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n.$$

If we multiply the results for $S_{10}(3,1)$ and $S_{14}(1,1,0)$, then we obtain

$$\eta^{24}(\tau) = -\frac{1}{180} \sum_{\alpha \equiv 1 \pmod{6}, \beta \equiv 4 \pmod{6}, \gamma \equiv 2 \pmod{6}, \delta \equiv 1 \pmod{4}} (-1)^{(\gamma-2)/6} \alpha \beta (\alpha^2 - \beta^2) \gamma \delta (\gamma^2 - \delta^2) (\gamma^2 - 9\delta^2)
\times q^{(\alpha^2 + \beta^2 + \gamma^2 + 3\delta^2)/12}.$$

If we extract the coefficient of $q^n$ on both sides, then we obtain

$$\tau(n) = -\frac{1}{4320\sqrt{3}} \sum (-1)^{(\gamma-2)/6} \text{Im} \left((\alpha + i\beta)^4\right) \text{Im} \left((\gamma + i\delta\sqrt{3})^6\right),$$

where the summation is over integers satisfying

$$\alpha^2 + \beta^2 + \gamma^2 + 3\delta^2 = 12n,$$

$$\alpha \equiv 1 \pmod{6}, \quad \beta \equiv 4 \pmod{6}, \quad \gamma \equiv 2 \pmod{6}, \quad \delta \equiv 1 \pmod{4}.$$ 

This is different from the representation given by Dyson [13, p. 636].

References


27. S. Ramanujan, notebooks (2 volumes) (Tata Institute of Fundamental Research, Bombay, 1957).


