CONGRUENCES SATISFIED BY APÉRY-LIKE NUMBERS

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In this article, we investigate congruences satisfied by Apéry-like numbers.

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1. Introduction: Apéry Numbers

In his proof of the irrationality of $\zeta(3)$, R. Apéry introduced the numbers

$$\alpha_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{n+j}{j}^2, \quad n \in \mathbb{N}.$$  

These numbers are now known as the Apéry numbers. Since the appearance of Apéry’s work, properties of $\alpha_n$ were gradually discovered. One of these is the observation that for primes $p \geq 5$,

$$\alpha_p \equiv \alpha_1 \pmod{p^3}. \quad (1.1)$$

The congruence (1.1) was conjectured by Chowla et al. [6] and proved by Gessel [7], who established the stronger result

$$\alpha_{pn} \equiv \alpha_n \pmod{p^3}. \quad (1.2)$$

In this article, we investigate other sequences of integers $\{f_n\}_{n=1}^{\infty}$ that satisfy relations similar to (1.2).
Let
\[ \eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \]
where \( q = \exp(2\pi i \tau) \) and \( \Im(\tau) > 0 \). It can be shown [10] that if
\[ t_1(\tau) = \left( \frac{\eta(6\tau)\eta(\tau)}{\eta(2\tau)\eta(3\tau)} \right)^{12} \]
and \( F_1(\tau) = \frac{\eta^7(2\tau)\eta^7(3\tau)}{\eta^5(\tau)\eta^5(6\tau)} \),
then
\[ F_1(\tau) = \sum_{n=0}^{\infty} \alpha_n t_1^n(\tau) \quad (1.3) \]
for suitably small \( |t_1(\tau)| \). The identification of \( \alpha_n \) as the coefficients of certain power series serves as a starting point for us in our search of other sequences \( \{f_n\}_{n=1}^{\infty} \) satisfying congruences similar to (1.2).

2. The Domb Numbers

Consider the functions
\[ t_2(\tau) = \left( \frac{\eta(2\tau)\eta(6\tau)}{\eta(\tau)\eta(3\tau)} \right)^6 \]
and \( F_2(\tau) = \frac{(\eta(\tau)\eta(3\tau))^4}{(\eta(2\tau)\eta(6\tau))^2} \).

It can be shown [2, (4.14)] that when \( |t_2(\tau)| \) is sufficiently small, we have
\[ F_2(\tau) = \sum_{n=0}^{\infty} (-1)^n \beta_n t_2^n(\tau) \quad (2.1) \]
where
\[ \beta_n = \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j} \binom{2(n-j)}{n-j}. \]

The sequence \( \{\beta_n\}_{n=1}^{\infty} \) turns out to satisfy the congruence

**Theorem 2.1.** For primes \( p \geq 5 \),
\[ \beta_{pn} \equiv \beta_n \quad (\text{mod } p^3). \]

**Proof.** The method of proof given here is due to Gessel [7]. For a prime \( p \geq 5 \), we find that
\[ \beta_{pn} = \sum_{j=0}^{pn} \binom{pn}{j}^2 \binom{2j}{j} \binom{2(pm-j)}{pm-j} \]
\[ = S_1 + S_2, \quad (2.2) \]
where
\[ S_1 = \sum_{j=0}^{n} \binom{pn}{pj}^2 \binom{2pj}{pj} \binom{2(pn-j)}{p(n-j)} \]
and

\[ S_2 = \sum_{k=1}^{p-1} \sum_{m=0}^{n-1} \binom{pn}{k + pm}^2 \binom{2(k + pm)}{k + pm} \binom{2(pm - k - pm)}{pm - k - pm}. \]

Now,

\[ S_1 \equiv \sum_{j=0}^{n} \binom{n}{j}^2 \binom{2j}{j} \binom{2(n - j)}{n - j} \pmod{p^3} \]

since [8]

\[ \binom{pa}{pb} \equiv \binom{a}{b} \pmod{p^3} \quad \text{for primes } p \geq 5. \] (2.3)

Therefore,

\[ S_1 \equiv \beta_n \pmod{p^3}. \] (2.4)

For \(0 < k < p\), we have [7]

\[ \binom{pm}{k + pm} \equiv (-1)^k \binom{pm}{k} \binom{n - 1}{m} \pmod{p^3}. \]

Hence,

\[ S_2 \equiv \sum_{k=1}^{p-1} \sum_{m=0}^{n-1} \binom{n-1}{m}^2 \binom{2k + 2pm}{k + pm} \binom{2(pm - k - pm)}{pm - k - pm} \pmod{p^3}. \] (2.5)

In order to prove that

\[ S_2 \equiv 0 \pmod{p^3}, \]

it suffices to show that

\[ \sum_{k=1}^{p-1} \sum_{m=0}^{n-1} \binom{n-1}{m}^2 \binom{2k + 2pm}{k + pm} \binom{2(pm - k - pm)}{pm - k - pm} \equiv 0 \pmod{p}. \] (2.6)

By Lucas’ congruence [9],

\[ \binom{a + pb}{c + pd} \equiv \binom{a}{c} \binom{b}{d} \pmod{p}. \] (2.7)

Hence, we deduce that

\[ \sum_{k=1}^{p-1} \sum_{m=0}^{n-1} \binom{n-1}{m}^2 \binom{2k + 2pm}{k + pm} \binom{2(pm - k - pm)}{pm - k - pm} \]

\[ = \sum_{k=1}^{p-1} \sum_{m=0}^{n} \binom{n-1}{m-1}^2 \binom{2k + 2p(m - 1)}{k + p(m - 1)} \binom{2(pm - k - p(m - 1))}{pm - k - p(m - 1)} \]

\[ \equiv \sum_{k=1}^{p-1} \sum_{m=1}^{n} \binom{n-1}{m-1}^2 \binom{2k}{k} \binom{2(m - 1)}{m - 1} \binom{2(n - m)}{n - m} \binom{2(p - k)}{p - k} \pmod{p}. \]
But for $1 \leq k \leq p - 1$,

$$p \mid \binom{2k}{k} \quad \text{or} \quad p \mid \binom{2(p - k)}{p - k}.$$ 

Hence,

$$\binom{2k}{k} \binom{2(p - k)}{p - k} \equiv 0 \pmod{p},$$

and we deduce (2.6).

A simple corollary of Theorem 2.1 is that

$$\beta_p \equiv \beta_1 \equiv 4 \pmod{p^3}$$

for all prime numbers $p > 3$.

### 3. Almkvist–Zudilin Sequence

The study of the sequence $\{\beta_n\}_{n=1}^{\infty}$ is inspired by the fact that $\alpha_n$ appears as the coefficients of the power series given by (1.3). As we have seen above, $\beta_n$ are coefficients of the power series given by (2.1). There is a third sequence that behaves similarly to both $\alpha_n$ and $\beta_n$. To motivate our discovery of this third sequence, we observe that $F_1$ and $F_2$ are modular forms associated with $\Gamma_0(6)_{+6}$ and $\Gamma_0(6)_{+3}$ respectively. Naturally, one would expect to have a third sequence arising from $\Gamma_0(6)_{+2}$. Indeed, in [5] it was shown that if

$$t_3(\tau) = \left( \frac{\eta(3\tau)\eta(6\tau)}{\eta(\tau)\eta(2\tau)} \right)^4 \quad \text{and} \quad F_3(\tau) = \frac{(\eta(\tau)\eta(2\tau))^3}{\eta(3\tau)\eta(6\tau)},$$

and $|t_3(\tau)|$ is sufficiently small, then

$$F_3(\tau) = \sum_{n=0}^{\infty} (-1)^n \gamma_n t_3^n(\tau),$$

where $\gamma_n$ are the Almkvist–Zudilin numbers [1], given by

$$\gamma_n = \sum_{j=0}^{\lfloor n/3 \rfloor} (-1)^j 3^{n-3j}(3j)! \binom{n}{j} \binom{n+j}{j}. \quad (3.1)$$

The numbers $\gamma_n$ appear to satisfy the congruence

**Conjecture 3.1.**

$$\gamma_{pn} \equiv \gamma_n \pmod{p^3}$$

for all primes $p > 3$.

We have been unable to give a proof of Conjecture 3.1 as Gessel’s method does not seem to work in this case.
4. Yang–Zudilin Sequence

For positive integers $k$ and $n$, let

$$y_{k,n} = \sum_{j=0}^{n} \binom{n}{j}^k.$$  

Around 2003, Zudilin realized that $y_{4,n}$ is associated with a certain modular form and modular function as in the case for the Apéry numbers, Domb numbers and the Almkvist–Zudilin numbers. This form and function were eventually obtained by Yang [11] (see [4] for the explicit forms of the form and function).

In this section, we will deduce that for primes $p \geq 7$,

$$y_{4,p} \equiv y_{4,1} \equiv 2 \pmod{p^5}$$

by showing the following more general result:

**Theorem 4.1.** Suppose $k$ is even, and $p > 3$ is a prime number for which $p-1 \nmid k$. Then

$$y_{k,p} \equiv 2 \pmod{p^{k+1}}.$$

**Proof.** Observe that

$$p \bigg| \binom{p}{j} \quad \text{for} \quad 1 \leq j \leq p-1.$$  

Hence it suffices to show that

$$p \bigg| \sum_{j=1}^{p-1} \left( \frac{(p-1)!}{j!(p-j)!} \right)^k. \quad (4.1)$$

Now

$$\frac{(p-1)!}{j!(p-j)!} = \frac{1}{j} \prod_{i=1}^{p-j} \frac{p-i}{i} \equiv \frac{1}{j} (-1)^{p-j} \pmod{p}. $$

Thus, since $k$ is even,

$$\sum_{j=1}^{p-1} \left( \frac{(p-1)!}{j!(p-j)!} \right)^k \equiv \sum_{j=1}^{p-1} \frac{1}{j^k} \equiv \sum_{j=1}^{p-1} j^k \pmod{p}. \quad (4.2)$$

But

$$\sum_{j=1}^{p-1} j^k \equiv \begin{cases} 0 & \text{if } p-1 \nmid k, \\ -1 & \text{if } p-1 \mid k. \end{cases} \quad (4.3)$$

By hypothesis $p-1 \mid k$, therefore (4.1) follows from (4.2) and (4.3). This completes the proof.

Theorem 4.1 does not have a generalization modulo $p^{k+1}$ similar to Theorem 2.1. However, we have the following result:

**Theorem 4.2.** Let $p > 3$ be prime and let $k > 1$ be an integer. Then

$$y_{k,pn} \equiv y_{k,n} \pmod{p^3}.$$
Proof. When \( k = 2 \) we have

\[
y_{2,n} = \sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n},
\]

so

\[
y_{2,pn} = \binom{2pn}{pn} \equiv \binom{2n}{n} \equiv y_{2,n} \pmod{p^3}
\]

by (2.3). This establishes the result for \( k = 2 \). For the remainder of the proof, suppose \( k \geq 3 \) and write

\[
y_{pn} = \sum_{j=0}^{pn} \binom{pn}{j}^k = T_1 + T_2,
\]

where

\[
T_1 = \sum_{j=0}^{n} \binom{pn}{jp}^k,
\]

and

\[
T_2 = \sum_{j=1}^{p-1} \sum_{m=0}^{n-1} \binom{pn}{j + pm}^k.
\]

Using (2.3), we deduce

\[
T_1 \equiv y_n \pmod{p^3}.
\]

Next, we rewrite \( T_2 \) as

\[
T_2 = \sum_{j=1}^{p-1} \sum_{m=0}^{n-1} \left( \binom{pn}{j + pm}^k \right) = \sum_{j=1}^{p-1} \sum_{m=0}^{n-1} \left( \binom{p + p(n - 1)}{j + pm}^k \right).
\]

By (2.7), we find that

\[
\binom{p + p(n - 1)}{j + pm} \equiv \binom{n - 1}{m} \binom{p}{j} \equiv 0 \pmod{p}.
\]

This implies that for \( k \geq 3 \) and \( 1 \leq j \leq p - 1 \),

\[
\binom{p + p(n - 1)}{j + pm}^k \equiv 0 \pmod{p^3}.
\]

Substituting (4.5) into (4.4), we conclude that

\[
T_2 \equiv 0 \pmod{p^3}
\]

and this completes the proof of Theorem 4.2.

5. Other Sequences

We hope that we have illustrated that sequences arising from the study of modular forms serve as a good source of numbers satisfying interesting congruences modulo
certain power of primes. We end this article with a series of conjectures associated with various modular forms. The letter \( p \) always denotes a prime number.

**Conjecture 5.1.** If

\[
z_2 = \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} q^{m^2+n^2} \quad \text{and} \quad x_2 = \frac{\eta^{12}(2\tau)}{x_2^6}
\]

and

\[
z_2 = \sum_{n=0}^{\infty} f_{2,n} x_2^n,
\]

then

\[f_{2,pn} \equiv f_{2,n} \pmod{p^2} \quad \text{when} \quad p \equiv 1 \pmod{4}.
\]

**Conjecture 5.2.** If

\[
z_3 = \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} q^{m^2+mn+n^2} \quad \text{and} \quad x_3 = \frac{\eta^6(\tau)\eta^6(3\tau)}{z_3^6}
\]

and

\[
z_3 = \sum_{n=0}^{\infty} f_{3,n} x_3^n,
\]

then

\[f_{3,pn} \equiv f_{3,n} \pmod{p^2} \quad \text{when} \quad \left(\frac{p}{3}\right) = 1.
\]

**Conjecture 5.3.** If

\[
z_5 = \frac{\eta^5(\tau)}{\eta(5\tau)} \quad \text{and} \quad x_5 = \frac{\eta^5(5\tau)}{\eta^5(\tau)}
\]

and

\[
z_5 = \sum_{n=0}^{\infty} f_{5,n} x_5^n,
\]

then

\[f_{5,pn} \equiv f_{5,n} \pmod{p^3} \quad \text{for all primes} \quad p, \quad \text{including} \quad p = 2.
\]

**Conjecture 5.4.** If

\[
z_7 = \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} q^{m^2+mn+2n^2} \quad \text{and} \quad x_7 = \frac{\eta^3(\tau)\eta^3(7\tau)}{z_7^3}
\]

and

\[
z_7 = \sum_{n=0}^{\infty} f_{7,n} x_7^n,
\]

then

\[f_{7,pn} \equiv f_{7,n} \pmod{p^3} \quad \text{when} \quad \left(\frac{p}{7}\right) = 1.
\]
Conjecture 5.5. If
\[ z_{11} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+3n^2} \quad \text{and} \quad x_{11} = \frac{\eta^2(\tau)\eta^2(11\tau)}{z_{11}^2} \]
and
\[ z_{11} = \sum_{n=0}^{\infty} f_{11,n} x_{11}^n, \]
then
\[ f_{11,pn} \equiv f_{11,n} \pmod{p^2} \quad \text{when} \quad \left( \frac{p}{11} \right) = 1. \]

Conjecture 5.6. If
\[ z_{23} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{m^2+mn+6n^2} \quad \text{and} \quad x_{23} = \frac{\eta(\tau)\eta(23\tau)}{z_{23}} \]
and
\[ z_{23} = \sum_{n=0}^{\infty} f_{23,n} x_{23}^n, \]
then
\[ f_{23,pn} \equiv f_{23,n} \pmod{p} \quad \text{when} \quad \left( \frac{p}{23} \right) = 1. \]

Conjecture 5.7. If
\[ Z_{23} = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} q^{2m^2+mn+3n^2} \quad \text{and} \quad X_{23} = \frac{\eta(\tau)\eta(23\tau)}{Z_{23}} \]
and
\[ Z_{23} = \sum_{n=0}^{\infty} F_{23,n} X_{23}^n, \]
then
\[ F_{23,pn} \equiv F_{23,n} \pmod{p} \quad \text{when} \quad \left( \frac{p}{23} \right) = 1. \]

Remarks. One can verify that
\[ f_{2,n} = 64^n \frac{(\frac{1}{3})_n^2}{(n!)^2} \quad \text{and} \quad f_{3,n} = 108^n \frac{(\frac{1}{7})_n (\frac{1}{7})_n}{(n!)^2}, \]
where \( (a)_k = a(a+1)(a+2) \cdots (a+k-1) \). There are no known closed forms for \( f_{r,n} \) for \( r = 5, 7, 11 \) and 23 but they satisfy certain recurrence relations. The functions \( z_r \) and \( x_r \), for \( r = 3, 7, 11 \) and 23, were studied in [3].
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References


