Partition identities and congruences associated with the Fourier coefficients of the Euler products

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Dedicated to Professor Srinivasa Rao on the occasion of his 60th birthday

Abstract

In this article, we discuss two applications of the operator $U(m)$ (see (1.1)) defined on the product of two power series.

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1. Introduction

Let $m$ be a positive integer and define the operator $U(m)$ on a formal power series $\sum_{n=0}^{\infty} a_n q^n$ by

$$\left. \sum_{n=0}^{\infty} a_n q^n \right|_{U(m)} = \sum_{n=0}^{\infty} a_{mn} q^n.$$  

The operator $U(m)$ acts on the product of two power series as follows:

$$\left( \sum_{n=0}^{\infty} b_n q^m \sum_{n=0}^{\infty} a_n q^n \right) \left|_{U(m)} \right. = \sum_{n=0}^{\infty} b_n q^n \sum_{n=0}^{\infty} a_{mn} q^n. \quad (1.1)$$

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Relation (1.1) shows that under $U(m)$, we may “shift” the “$m$” from the power of $q$ in the first series to the subscript of the coefficients of the second series. This fact was known to Atkin and O’Brien [1, (28)].

In Section 2, we prove, with the aid of (1.1), Ramanujan’s famous congruences [7]

$$p(5n + 4) \equiv 0 \pmod{5},$$  
(1.2)

$$p(7n + 5) \equiv 0 \pmod{7}$$
(1.3)

and

$$p(11n + 6) \equiv 0 \pmod{11},$$
(1.4)

where $p(n)$ denotes the number of unrestricted partitions of the nonnegative integer $n$.

It is obvious that (1.2) and (1.3) follows from Ramanujan’s identities

$$\sum_{n=0}^{\infty} p(5n + 4)q^n = 5 \prod_{n=1}^{\infty} \frac{(1-q^{5n})^5}{(1-q^n)^6},$$
(1.5)

$$\sum_{n=0}^{\infty} p(7n + 5)q^n = 7 \prod_{n=1}^{\infty} \frac{(1-q^{7n})^3}{(1-q^n)^4} + 49q \prod_{n=1}^{\infty} \frac{(1-q^{7n})^7}{(1-q^n)^8}.$$  
(1.6)

Identities such as (1.5) and (1.6) are more difficult to establish than congruences (1.2) and (1.3). In [10, (1.15)], Zuckerman obtained the following analogue of (1.5) and (1.6):

$$\sum_{n=0}^{\infty} p(13n + 6)q^n = 11 \prod_{n=1}^{\infty} \frac{(1-q^{13n})^2}{(1-q^n)^2} + 468q \prod_{n=1}^{\infty} \frac{(1-q^{13n})^3}{(1-q^n)^4} + 6422q^2 \prod_{n=1}^{\infty} \frac{(1-q^{13n})^5}{(1-q^n)^6}$$

$$+ 43940q^3 \prod_{n=1}^{\infty} \frac{(1-q^{13n})^7}{(1-q^n)^8} + 171366q^4 \prod_{n=1}^{\infty} \frac{(1-q^{13n})^9}{(1-q^n)^{10}}$$

$$+ 371293q^5 \prod_{n=1}^{\infty} \frac{(1-q^{13n})^{11}}{(1-q^n)^{12}} + 371293q^6 \prod_{n=1}^{\infty} \frac{(1-q^{13n})^{13}}{(1-q^n)^{14}}.$$
(1.7)

In Section 3, we use (1.1) and results in [4] to establish identities associated with

$$\sum_{n=0}^{\infty} p_{-r}(l^k n + \delta_{l,k,r})q^n, \quad l = 5, 7 \text{ and } 13,$$

where

$$\delta_{l,k,r} = \begin{cases} 
\frac{r(1-l^k)}{24} & \text{if } k \text{ is even}, \\
\frac{r(1-l^{k+1})}{24} & \text{if } k \text{ is odd}
\end{cases}$$
(1.8)
and

\[ \prod_{n=1}^{\infty} (1 - q^n)^r = \sum_{n=0}^{\infty} p_r(n) q^n. \]  

(1.9)

When \((l,k,r) = (5,1,-1), (7,1,-1),\) and \((13,1,-1)\) we obtain \((1.5)-(1.7)\) and when \((l,k,r) = (5,1,-2)\) and \((l,k,r) = (5,1,-3),\) we find that

\[ \sum_{n=0}^{\infty} p_{-2}(5n - 2) q^n = 10q \prod_{n=1}^{\infty} \frac{(1 - q^{5n})^4}{(1 - q^n)^6} + 125q^2 \prod_{n=1}^{\infty} \frac{(1 - q^{5n})^{10}}{(1 - q^n)^{12}} \]  

(1.10)

and

\[ \sum_{n=0}^{\infty} p_{-3}(5n - 3) q^n = 9q \prod_{n=1}^{\infty} \frac{(1 - q^{5n})^3}{(1 - q^n)^6} + 375q^2 \prod_{n=1}^{\infty} \frac{(1 - q^{5n})^9}{(1 - q^n)^{12}} + 3125q^3 \prod_{n=1}^{\infty} \frac{(1 - q^{5n})^{15}}{(1 - q^n)^{18}} \]  

(1.11)

Identities \((1.10)\) and \((1.11)\) appear to be new.

2. Ramanujan’s congruences

Congruence properties of \(p_r(n)\) (see \((1.9)\)) were studied by Ramanujan, who deduced \((1.2)\) and \((1.3)\) from

\[ p_4(5n + 4) \equiv 0 \pmod{5} \quad \text{and} \quad p_6(7n + 5) \equiv 0 \pmod{7}, \]

respectively. In [9], Winquist showed \((1.4)\) by proving that

\[ p_{10}(11n + 6) \equiv 0 \pmod{11}. \]

Since then, many congruences have been discovered for \(p_r(n)\) (see for example [2,5]). In this section, we show that in order to obtain congruences for \(p_r(n)\) of the type

\[ p_r(ln - N) \equiv 0 \pmod{l}, \quad n \geq 1, \]

it suffices to check if \(l\) divides \(\tau_N(lf),\) \(1 \leq j \leq N,\) where

\[ \Delta^N(z) := q^N \prod_{n=1}^{\infty} (1 - q^n)^{24N} = \sum_{n=0}^{\infty} \tau_N(n) q^n, \quad q = e^{2\pi i z}. \]

Note that \(\tau_1(n)\) is the famous Ramanujan’s \(\tau\)-function.

**Proof of \((1.2)\):** It is known that \(\Delta(z)\) is an eigenform in \(\mathcal{S}_{12}(SL_2(\mathbb{Z})),\) where \(\mathcal{S}_k(SL_2(\mathbb{Z}))\) denotes the space of weight \(k\) cusp forms invariant under \(SL_2(\mathbb{Z}).\) Hence,

\[ \Delta(z) \big|_{T_p} = \tau(p) \Delta(z), \]

where \(T_p\) is the Hecke operator defined by

\[ \sum_{n=0}^{\infty} a_n q^n \bigg|_{T_p} = \sum_{n=0}^{\infty} (a(pn) + p^{k-1} a(n/p)) q^n \]
with \( k \) being the weight of the modular form \( \sum_{n=0}^{\infty} a_n q^n \) invariant under \( SL_2(\mathbb{Z}) \). Note that since the coefficient of \( q^5 \) in \( \Delta(z) \) is \( \tau(5) = 4830 \), we conclude that
\[
\Delta(z)|_{T_5} = \tau(5) \Delta(z) \equiv 0 \pmod{5}.
\] (2.1)

We now write
\[
\Delta(z) = q \prod_{n=1}^{\infty} (1 - q^n)^{5} \prod_{n=1}^{\infty} (1 - q^{5n}) \equiv \prod_{n=1}^{\infty} (1 - q^{5n})^r \sum_{n=0}^{\infty} p_k(n-1)q^n \pmod{5},
\] (2.2)

where \( r \) and \( s \) are integers. Since
\[
\sum_{n=0}^{\infty} a_n q^n \bigg|_{U(p)} \equiv \sum_{n=0}^{\infty} a_n q^n \bigg|_{T_p} \pmod{5},
\]
we find by (1.1), (2.2) and (2.1) that
\[
\prod_{n=1}^{\infty} (1 - q^{5n})^r \sum_{n=0}^{\infty} p_k(n-1)q^n \equiv \prod_{n=1}^{\infty} (1 - q^{5n})^s \sum_{n=0}^{\infty} p_k(5n-1)q^n
\]
\[
\equiv \Delta(z)|_{T_5} \equiv 0 \pmod{5}.
\] (2.3)

This implies that \( p_k(5n-1) \equiv 0 \pmod{5} \) for all \( r \) satisfying the equation
\[
24 = 5s + r
\]
or
\[
p_{24-5s}(5n-1) \equiv 0 \pmod{5}, \quad s \in \mathbb{Z},
\]
which immediately yields Ramanujan’s congruences for \( p(5n+4) \) and \( p_4(5n+4) \).

Our computation shows that one only needs to know \( \tau(5) \) in \( \Delta(z) \) in order to deduce the above congruences. In general, we always obtain a collection of congruences of the form
\[
p_24 - ls(5n-1) \equiv 0 \pmod{l}
\]
for each \( l \) satisfying
\[
\tau(l) \equiv 0 \pmod{l}.
\] (2.4)

Questions involving primes satisfying (2.4) can be found in [8, 5.2(b)].

**Proof of (1.3):** To prove Ramanujan’s congruences for \( p(7n+5) \), we express \( \Delta^2(z)|_{T_7} \) in terms of \( \Delta^2(z) \) and \( \Delta(z)Q^3(q) \), where
\[
Q(q) = 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1-q^n}.
\]
This turns out to be
\[
\Delta^2(z)|_{T_7} = -985 824 \Delta(z)Q^3(q) - 525 803 656 \Delta^2(z).
\] (2.5)

Note that the coefficients of \( \Delta(z)Q^3(q) \) and \( \Delta^2(z) \) in the above identities are both divisible by 7. Hence we conclude that
\[
p_{48-7s}(7n-2) \equiv 0 \pmod{7}, \quad s \in \mathbb{Z}.
\]
In particular, we obtain (1.3), as well as the congruence for \( p_6(7n+5) \).
It is clear from the above calculations that to obtain congruences such as
\[ p_{24N-l}(ln-N) \equiv 0 \pmod{l}, \]
it suffices to compute the image of \( A^N(z) \) under \( T_l \). If
\[ A^N(z)|_{T_l} = a_1B_1 + a_2B_2 + \cdots + a_NB_N, \]
where \( N = \) dimension of \( \mathcal{H}_k(\text{SL}_2(\mathbb{Z})) \), then each \( a_i \) is a \( \mathbb{Z} \)-linear combination of \( \tau_N(lj) \) for \( N \) values of \( j, 1 \leq j \leq N \). For example, in order to verify that
\[ \tau_N(lj) \equiv 0 \pmod{l} \]
holds, it suffices to verify it for \( 1 \leq j \leq N \). Therefore, to prove (1.4), it suffices to check that
11 divides \( \tau_5(11j), 1 \leq j \leq 5 \).

3. Partition identities

In this section, we give proofs of (1.5)–(1.7) and their generalizations.

We begin this section with the proof of (1.5). It is known that \( \eta(25z)/\eta(z) \) is a modular function on \( \Gamma_0(25) \) [6], where
\[ \eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \]
Since
\[ \frac{\eta(25z)}{\eta(z)} = \prod_{n=1}^{\infty} (1 - q^{25n}) \sum_{n=0}^{\infty} p(n-1)q^n, \]
we conclude by (1.1) that
\[ \prod_{n=1}^{\infty} (1 - q^{5n}) \sum_{n=0}^{\infty} p(5n-1)q^n = \frac{\eta(25z)}{\eta(z)} \bigg|_{U(5)}. \]
Following the method illustrated in [4, Theorem 4], we find that \( \frac{\eta(25z)}{\eta(z)} \bigg|_{U(5)} \) is an entire modular function on \( \Gamma_0(5) \). It is known that these functions are polynomials in \( h_5(z) := \eta^6(5z)/\eta^6(z) \) [3]. Hence, we conclude immediately that
\[ \prod_{n=1}^{\infty} (1 - q^{5n}) \sum_{n=0}^{\infty} p(5n-1)q^n = \frac{\eta^6(5z)}{\eta^6(z)} \]
which is (1.5).

The proof of (1.6) and (1.7) is similar since \( \eta(l^2z)/\eta(z) \) is an entire modular function on \( \Gamma_0(l^2) \) and entire modular functions on \( \Gamma_0(7) \) and \( \Gamma_0(13) \) are polynomials in \( \eta^4(7z)/\eta^4(z) \) and \( \eta^2(13z)/\eta^2(z) \) [3], respectively.

The method of proof illustrated above yields the following:
Theorem 3.1 (Lehner [4, Theorem 4]). Let \( l > 3 \) be an odd prime. Then
\[
\prod_{n=1}^{\infty} (1 - q^n) \sum_{n=0}^{\infty} p(n + (1 - l^2)/24)q^n
\]
is an entire modular function on \( \Gamma_0(l) \).

It is also known that \( U(l) \) sends an entire modular function \( f(z) \) on \( \Gamma_0(l) \) to an entire modular function on \( \Gamma_0(l) \) if \( f \) satisfies the transformation formula [4, (2.2)]
\[
f(-1/lz) = cf(z) \quad \text{or} \quad f(-1/lz) = c/f(z). \tag{3.2}
\]
This is clearly satisfied by the functions \( \eta^6(5z)/\eta^6(z) \), \( \eta^4(7z)/\eta^4(z) \) and \( \eta^2(13z)/\eta^2(z) \), for \( l = 5, 7 \) and 13, respectively.

In the case of \( l = 5 \), we apply \( U(5) \) to the left-hand side of (3.1) to conclude that [10, (1.13)]
\[
\prod_{n=1}^{\infty} (1 - q^n) \sum_{n=0}^{\infty} p(25n - 1)q^n
= 63 \cdot 5^2 \left( \frac{\eta(5z)}{\eta(z)} \right)^6 + 52 \cdot 5^5 \left( \frac{\eta(5z)}{\eta(z)} \right)^{12}
+ 63 \cdot 5^7 \left( \frac{\eta(5z)}{\eta(z)} \right)^{18} + 6 \cdot 5^{10} \left( \frac{\eta(5z)}{\eta(z)} \right)^{24} + 5^{12} \left( \frac{\eta(5z)}{\eta(z)} \right)^{30}. \tag{3.3}
\]

To obtain identities associated with higher power of 5, we first multiply
\[
\prod_{n=1}^{\infty} (1 - q^n) \sum_{n=0}^{\infty} p(5^2n - 1)q^n
\]
by \( \eta(25z)/\eta(z) \) and note that each function on the right-hand side satisfies (3.2). Therefore, by applying \( U(5) \), we conclude that
\[
\prod_{n=1}^{\infty} (1 - q^{5n}) \sum_{n=0}^{\infty} p(5^3n - 26)q^n
\]
is an entire modular function on \( \Gamma_0(5) \) and is expressible in terms of \( h_5(z) \). It is clear that when we pass from an identity involving \( k \), where \( k \) is an odd integer, to the corresponding identity for \( k + 1 \), we only need to apply \( U(5) \) to
\[
\prod_{n=1}^{\infty} (1 - q^{5n}) \sum_{n=0}^{\infty} p(5^k n + \delta_{5,k})q^n,
\]
where \( \delta_{5,k} := \delta_{5,k,1} \), with \( \delta_{l,k,r} \) defined as in (1.8). To obtain an identity corresponding to \( k + 1 \) from an identity involving \( k \), where \( k \) is even, we have to first multiply the identity involving
\[
\prod_{n=1}^{\infty} (1 - q^n) \sum_{n=0}^{\infty} p(5^k n + \delta_{5,k})q^n
\]
by $\eta(25z)/\eta(z)$ before applying $U(5)$. In this way, we obtain an expression for

$$\prod_{n=1}^{\infty} (1 - q^{5n}) \sum_{n=0}^{\infty} p(5^k n + \delta_{5,k}) q^n$$

in terms of $h_5(z)$ for all $k \in \mathbb{N}$, with

$$\varepsilon_5 = \begin{cases} 1 & \text{if } k \text{ is odd}, \\ 1 & \text{if } k \text{ is even}. \end{cases} \quad (3.4)$$

This method can be found in [4, Theorem 7], where the case $l = 11$ is discussed.

The advantage of using (1.1) to obtain partition identities is that one does not need to know the modular behavior of the expressions such as $\sum_{n=0}^{\infty} p(5^k n + \delta_{5,k}) q^n$. The method can be modified to obtain identities for $\sum_{n=0}^{\infty} p_{-r}(5^k n + \delta_{5,k,r}) q^n$, where $p_k(n)$ and $\delta_{l,k,r}$ are defined in (1.9) and (1.8), respectively. All we have to do is to use

$$(\eta(25\tau)/\eta(\tau))^r$$

and follow the arguments illustrated as above to conclude that $\prod_{n=1}^{\infty} (1 - q^{5n})^r \sum_{n=0}^{\infty} p_{-r}(5^k n + \delta_{5,k,r}) q^n$ is a polynomial in $h_5(z)$, where $\varepsilon_5$ is defined in (3.4). For $(k,r) = (5,-2)$ and $(5,-3)$, we obtain (1.10) and (1.11), respectively.

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