RAMANUJAN’S CUBIC CONTINUED FRACTION REVISITED

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Abstract. In this article, we derive a sequence of numbers which converge to $1/\pi$. We will also derive a new series for $1/\pi$. These new results are motivated by the study of Ramanujan’s cubic continued fraction.

1. Introduction

Let $q = e^{2\pi i \tau}$ and

$$G(q) = \frac{q^{1/3}}{1} + \frac{q + q^2}{1} + \frac{q^2 + q^4}{1} + \frac{q^3 + q^6}{1} \cdots.$$ 

In 1996, inspired by page 366 of Ramanujan’s Lost Notebook [9], H.H. Chan [4] derived several new results satisfied by $G(q)$. For example, he showed that

$$G^3(q) = G(q^3) \frac{1 - G(q^3) + G^2(q^3)}{1 + 2G(q^3) + 4G^2(q^3)}.$$ 

From (1.1), Chan constructed an algorithm for computing $e^{\pi}$. This iteration prompted F.G. Garvan to ask if there were any iteration to $\pi$ which can be derived from the study of $G(q)$. In this paper, we will show that such an iteration exists. We will also derive the following series for $1/\pi$:

$$\frac{2\sqrt{3}(3 + 2\sqrt{2})}{9\pi} = \sum_{k=0}^{\infty} C_k \left( k + 1 - \frac{2}{3} \sqrt{2} \right) \left( -1 + \frac{3}{4} \sqrt{2} \right)^k$$

where

$$C_k = \sum_{m=0}^{k} \left\{ \sum_{j=0}^{m} \binom{m}{j} \sum_{i=0}^{k-m} \binom{k-m}{i} \right\}.$$ 

The proof of (1.2) involves the identity

$$G^3(e^{-2\pi/\sqrt{6}}) = -1 + \frac{3}{4} \sqrt{2}.$$
2. A TRIPLETION FORMULA FOR \( G(q) \) AND A NEW ITERATION TO 1/\( \pi \)

In [1], C. Adiga, T. Kim, M.S.M. Naika and H.S. Madhusudhan gave a new proof of (1.1) by first proving the identity

\[
1 - 3 \frac{G(q^3)}{1 + G(q^3)} = \left( 1 - 9 \frac{G(q)}{1 + G(q)} \right)^{1/3}.
\]

This identity allows one to write \( G(q^3) \) in terms of \( G(q) \), namely,

\[
G(q^3) = \frac{1 - H(q)}{2 + H(q)},
\]

with

\[
H(q) = \left( \frac{1 - 8G^3(q)}{1 + G^3(q)} \right)^{1/3}.
\]

The above triplication formula for \( G(q) \) is analogous to the Borweins-Ramanujan triplication formula for the cubic singular modulus defined by

\[
\frac{1}{\alpha(q)} = 1 + \frac{1}{27} \left( \frac{\eta(\tau)}{\eta(3\tau)} \right)^{12},
\]

where \( q = e^{2\pi i \tau} \) and

\[
\eta(\tau) = q^{1/24} \prod_{k=1}^{\infty} (1 - q^k).
\]

In the case for \( \alpha(q) \), the triplication formula is given by

\[
\alpha(q^3) = \left( \frac{1 - \sqrt[3]{1 - \alpha(q)}}{1 + 2\sqrt[3]{1 - \alpha(q)}} \right)^3.
\]

Two rapidly convergent sequences for \( \pi \) can be constructed from (2.4). These iterations are given as follow:

**The Borweins Iteration** [8]. Let \( t_0 = \frac{1}{3}, s_0 = \frac{\sqrt[3]{3} - 1}{2}, \)

\[
s_n = \frac{1 - (1 - s_{n-1}^2)^{1/3}}{1 + 2(1 - s_{n-1}^3)^{1/3}} \quad \text{and} \quad t_n = (1 + 2s_n)^2t_{n-1} - 3^{n-1}(1 + 2s_n)^2 - 1.
\]

Then \( t_n^{-1} \) converges cubically to \( \pi \).

**Chan’s iteration** [6]. Let \( k_0 = 0, s_0 = \frac{1}{2^{1/3}}, \)

\[
s_n = \frac{1 - (1 - s_{n-1}^3)^{1/3}}{1 + 2(1 - s_{n-1}^3)^{1/3}} \quad \text{and} \quad k_n = (1 + 2s_n)^2k_{n-1} + 8 \cdot 3^{n-2} \sqrt[3]{3s_n} \frac{1 - s_n^3}{1 + 2s_n}.
\]

Then \( k_n^{-1} \) converges cubically to \( \pi \).

Since the above iterations are constructed from (2.4), it is therefore natural to construct new cubic iteration to \( \pi \) from (2.2). In the following two sections, we will establish the following result:
Theorem 2.1. Let \( k_0 = 0 \) and \( s_0 = \sqrt[3]{\frac{3\sqrt{2}}{4}} - 1 \). Set
\[
s_n = \frac{(1 + s_{n-1}^3)^{1/3} - (1 - 8s_{n-1}^3)^{1/3}}{2(1 + s_{n-1}^3)^{1/3} + (1 - 8s_{n-1}^3)^{1/3}}.
\]
If
\[
k_n = \frac{(1 + 2s_n + 4s_n^2)(1 + s_n)^2}{(1 - s_n + s_n^2)}k_{n-1}
+ \frac{2 \cdot 3^{n-1} s_n(1 - 2s_n)(8s_n^4 - 10s_n^3 + 6s_n^2 + 11s_n + 5)}{1 + s_n^3},
\]
then \( k_n^{-1} \) converges cubically to \( \pi \).

3. New identities satisfied by \( G(q) \)

We first relate \( G(q) \) with the Borweins’ cubic singular modulus \( \alpha(q) \) (see (2.3)) and deduce results associated with \( G(q) \) using Ramanujan-Borweins’
theory of elliptic functions to the cubic base.

Lemma 3.1. Let
\[
\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2},
\]
\[
a(q) = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},
\]
\[
X = G^3(q)
\]
and
\[
z = \frac{\varphi^3(-q^3)}{\varphi(-q)}.
\]

Then
\[
a(q) = z(1 + 4X)
\]
and
\[
\alpha(q) = 27 \frac{X}{(1 + 4X)^3}.
\]

Proof. From [2, p. 460, Entry 3(ii)], we find that
\[
a(q^2) = \frac{\varphi^4(-q) + 3\varphi^4(-q^3)}{\varphi(-q)\varphi(-q^3)}
= z \left( \frac{1}{4} \frac{\varphi^4(-q)}{\varphi^4(-q^3)} + \frac{3}{4} \right).
\]

(3.3)
Since [2, p. 347]
\[
\frac{\varphi^4(-q)}{\varphi^4(-q^3)} = 1 - 8X,
\]
we deduce that
\[
a(q^2) = z (1 - 2X). \tag{3.5}
\]
On the other hand, we know that [3, p. 4189]
\[
a(q) = 3\frac{\varphi^3(-q^3)}{\varphi(-q)} - 2a(q^2).
\]
Hence, by (3.5), we find that
\[
a(q) = z (1 + 4X),
\]
which yields (3.1).

To prove (3.2), we recall the identity [2, p. 345, Entry 1 (iv)]
\[
1 + \frac{1}{27} \left( \frac{\eta(\tau)}{\eta(3\tau)} \right)^{12} = \frac{(1 + 4X)^3}{27X}.
\]
Using (2.3), we immediately deduce (3.2).

\[\square\]

**Corollary 3.2.** The functions $z$ and $X$ satisfy the following differential equations:
\[
\frac{dX}{dq} = z^2(X - 7X^2 - 8X^3) \tag{3.6}
\]

**Proof.** We recall the differential equation satisfied by $a := a(q)$ and $\alpha := \alpha(q)$ [5, (4.7)]:
\[
\frac{d\alpha}{dq} = a^2\alpha(1 - \alpha). \tag{3.7}
\]
Differentiating (3.2) with respect to $q$ and using (3.7) and (3.1), we immediately deduce (3.6).

\[\square\]

### 4. Proof of Theorem 2.1

We begin our proof with the following transformation formula:
\[
(1 + X(e^{-2\pi/\sqrt{6}}))(1 + X(e^{-2\pi/\sqrt{6}})) = \frac{9}{8}. \tag{4.1}
\]
This identity can be proved by rearranging the identity [1]
\[
\left(1 + \frac{1}{X(e^{-2\pi/\sqrt{6}})}\right) \left(1 - 8X(e^{-2\pi/\sqrt{6}})\right) = 9. \tag{4.2}
\]
Differentiating (4.1) with respect to $t$ and using (3.6), we find that
\begin{align*}
(4.3) \quad tZ(e^{-2\pi\sqrt{t/6}})X(e^{-2\pi\sqrt{t/6}})(1 - 8X(e^{-2\pi\sqrt{t/6}})) &= Z(e^{-2\pi\sqrt{t/6}})X(e^{-2\pi\sqrt{t/6}})(1 - 8X(e^{-2\pi\sqrt{t/6}})),
\end{align*}
where
\begin{align*}
Z(q) &= z^2.
\end{align*}
From (4.2), we have
\begin{align*}
(4.4) \quad X(e^{-2\pi/\sqrt{6t}}) &= \frac{1}{9} \left(1 + X(e^{-2\pi/\sqrt{6t}})\right) \left(1 - 8X(e^{-2\pi/\sqrt{6t}})\right)
\end{align*}
and
\begin{align*}
(4.5) \quad X(e^{-2\pi\sqrt{t/6}}) &= \frac{1}{9} \left(1 + X(e^{-2\pi\sqrt{t/6}})\right) \left(1 - 8X(e^{-2\pi\sqrt{t/6}})\right).
\end{align*}
Substituting (4.4) and (4.5) into (4.3), we find that
\begin{align*}
(4.6) \quad tZ(e^{-2\pi\sqrt{t/6}})(1 + X(e^{-2\pi/\sqrt{6t}})) &= Z(e^{-2\pi/\sqrt{6t}})(1 + X(e^{-2\pi/\sqrt{6t}})).
\end{align*}
The above transformation formula motivates us to set
\begin{align*}
A(q) &= Z(q)(1 + X(q)).
\end{align*}
Consequently, we can express (4.6) as
\begin{align*}
(4.7) \quad tA(e^{-2\pi\sqrt{t/6}}) &= A(e^{-2\pi\sqrt{t/6}}).
\end{align*}
Define
\begin{align*}
(4.8) \quad \kappa(t) &= \frac{1}{\pi A(e^{-2\pi\sqrt{t/6}})} - 2 \sqrt{\frac{t}{6}} \frac{\bar{A}}{A^2} (e^{-2\pi\sqrt{t/6}}),
\end{align*}
where
\begin{align*}
\bar{f} := \frac{df}{dq}.
\end{align*}
Differentiating both sides of (4.7) with respect to $t$, we find that
\begin{align*}
(4.9) \quad \sqrt{\frac{1}{6t}} \frac{\bar{A}}{A} (e^{-2\pi\sqrt{t/6}}) + \sqrt{\frac{1}{6}} \frac{\bar{A}}{A} (e^{-2\pi/\sqrt{6t}}) &= \frac{1}{\pi}.
\end{align*}
Rewriting (4.9) in terms of $\kappa(t)$ yields
\begin{align*}
(4.10) \quad \kappa(t) + t\kappa\left(\frac{1}{t}\right) &= 0.
\end{align*}
When $t = 1$, (4.10) implies that
\begin{align*}
(4.11) \quad \kappa(1) &= 0.
\end{align*}
Next, let
\begin{align*}
(4.12) \quad M_N(q) &= \frac{A(q)}{A(q^N)}.
\end{align*}
Setting $q = e^{2\pi \sqrt{t/6}}$ and differentiating (4.12) with respect to $t$, we find using (4.8) that

\[(4.13) \quad \kappa(N^2 t) = 2 \sqrt{\frac{t}{6} M_N(e^{-2\pi \sqrt{t/6}})} \frac{1}{A(e^{-2\pi \sqrt{N^2 t/6}})} - M_N(e^{-2\pi \sqrt{t/6}}) \kappa(t) .\]

Note that $\kappa(N^2 t)$ tends to $\frac{1}{\pi}$ at the rate of order $N$ as $N$ tends to $\infty$.

In order to obtain a cubic iteration to $1/\pi$ from (4.13), let $N = 3$. If $y = G(q^3)$ then from [4, (2.9)], we have

\[(4.14) \quad \varphi(q^9) = \frac{1}{1 - 2y} .\]

Using (3.4) and (4.14), we deduce that

\[
\frac{Z(q)}{Z(q^3)} = \frac{\varphi^6(-q^3) \varphi^2(-q^9)}{\varphi^2(-q) \varphi^6(-q^9)} = \frac{\varphi^8(-q^3) \varphi^2(-q^9)}{\varphi^8(-q^9) \varphi^2(-q)} = \left(\frac{1 - 8y^3}{1 - 2y}\right)^2 = (1 + 2y + 4y^2)^2 .
\]

Hence,

\[(4.15) \quad M_3 = (1 + 2y + 4y^2)^2 \frac{(1 + X)}{(1 + y^3)} = \frac{(1 + 2y + 4y^2)(1 + y^2)}{1 - y + y^2} ,
\]

by (1.1).

Using (3.6) with $q$ replaced by $q^3$, we have

\[\bar{y} = A(q^3) y(1 - 8y^3) .\]

This allows us to differentiate both sides of (4.15) and conclude that

\[(4.16) \quad \frac{1}{M_3(q) A(q^3)} \tilde{M}_3(q) = \frac{(1 - 2y) y(8y^4 - 10y^3 + 6y^2 + 11y + 5)}{(y + 1)(1 - y + y^2)} .
\]

We are now ready to construct our sequence $k_n$. Let $s_n = G(e^{-2\pi \sqrt{3n/6}})$ and $k_n = \kappa(3^n)$. Writing (4.13) in terms of $s_n$ and $k_n$, we find that

\[(4.17) \quad k_n = \frac{(1 + 2s_n + 4s_n^2)(1 + s_n)^2}{(1 - s_n + s_n^2) k_{n-1}} \frac{1}{1 - 2s_n + s_n^2} + \frac{2 \cdot 3^{n-1} s_n (1 - 2s_n)(8s_n^4 - 10s_n^3 + 6s_n^2 + 11s_n + 5)}{1 + s_n^4} ,
\]

From (4.11), we know that the initial value of $k_n$ is

\[k_0 = 0 .\]
By letting \( t = 1 \) in (4.1), we find that the initial value of \( s_0 \) is
\[
(4.18) \quad s_0 = G(e^{-2\pi/\sqrt{6}}) = \left( \frac{3\sqrt{2}}{4} - 1 \right)^{1/3}.
\]
We can then evaluate \( s_n \) from \( s_{n-1} \) using (2.2). Substituting \( s_n \) into (4.17), we construct the sequence \( \{k_n\} \) which converges cubically to \( 1/\pi \) and this completes the proof of Theorem 2.1.

5. A SERIES FOR \( \frac{1}{\pi} \)

Set \( t = 1 \) in (4.9). We find that
\[
(5.1) \quad \frac{\tilde{A}}{A}(e^{-2\pi/\sqrt{6}}) = \frac{\sqrt{6}}{2\pi}.
\]

Using the relation (3.1) and (3.2) in the differential equation
\[
\alpha(1-\alpha) \frac{d^2 a}{d\alpha^2} + (1-2\alpha) \frac{da}{d\alpha} - \frac{2}{9} a = 0,
\]
we deduce that
\[
(5.2) \quad X(8X-1)(1+X) \frac{d^2 z}{dX^2} + (24X^2+14X-1) \frac{dz}{dX} + 2(1+4X)z = 0.
\]
If
\[
z = \sum_{k=0}^{\infty} c_k X^k,
\]
then from (5.2), we know that \( a_k \) satisfies the recurrence
\[
k^2 c_k - (7k^2 - 7k + 2)c_{k-1} - 8(k-1)^2 c_{k-2} = 0.
\]
The solution of the above recurrence with \( c_0 = 1, c_1 = 2 \) is given by [10, Table 2] \(^1\)
\[
c_k = \sum_{j=0}^{k} \binom{k}{j}^3.
\]
Hence,
\[
z = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{k}{j}^3 X^k.
\]
Therefore,
\[
Z = z^2 = \sum_{k=0}^{\infty} C_k X^k
\]
where \( C_k \) is given by (1.3), or
\[
(5.3) \quad A = \sum_{k=0}^{\infty} C_k X^k(1 + X).
\]

\(^1\)According to H.A. Verrill, the solution to the recurrence is due to D. Zagier.
From (5.3), we deduce that
\[
\frac{\tilde{A}}{A} = \frac{1}{A} \frac{dA}{dX} \tilde{X} = (1 - 8X) \sum_{k=0}^{\infty} C_k X^k (k(1 + X) + X),
\]
by (3.6).

Set \( q = e^{-2\pi/\sqrt{6}} \) in (5.4). From (4.18), we know that
\[
X(e^{-2\pi/\sqrt{6}}) = x_1 = -1 + \frac{3\sqrt{2}}{4}.
\]

Hence, we have
\[
(1 - 8x_1) \sum_{k=0}^{\infty} C_k x_1^k (k(1 + x_1) + x_1) = \frac{\sqrt{6}}{2\pi}.
\]
Simplifying the above yields (1.2).

6. Conclusions

1. We have seen here that (4.1) plays an important role for our determination of \( A(q) \). In general, if we have a modular function (i.e. a Hauptmodul) associated to congruence subgroup \( \Gamma \) of \( SL_2(\mathbb{Z}) \) with genus zero, we need to determine a “nice” modular form of weight 2 on \( \Gamma \) in order to derive new series for \( 1/\pi \). It is therefore possible to derive new series for \( 1/\pi \) associated with the Rogers-Ramanujan continued fraction.

2. We can also obtain another cubic iteration to \( 1/\pi \) if we use the alternative formula [1]
\[
\left( 1 + \frac{1}{G^3(-e^{-\pi t})} \right) \left( 1 + \frac{1}{G^3(-e^{-\pi/t})} \right) = 9.
\]

We leave this as an exercise for the readers.

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References


