Π¹₁ Conservation of COH Over $BΣ₂$

(Joint work with Ted Slaman and Yue Yang)

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Fix $\mathcal{M} = \langle M, X, +, \cdot, 0, 1 \rangle$ to be a structure in the language of second order arithmetic. $X \subset M$ is $M$-finite if it is coded in $M$. Fix $n \geq 1$.

- $\mathcal{M} \models I\Sigma_n$ ($\Sigma_n$ induction) if it satisfies every $\Sigma_n$ instance (with parameters in $\mathcal{M}$) of the induction scheme.

- $\mathcal{M} \models B\Sigma_n$ ($\Sigma_n$ bounding) if every $\Sigma_n$ definable function maps an $M$-finite set onto an $M$-finite set.

Kirby-Paris: $\cdots \rightarrow I\Sigma_{n+1} \rightarrow B\Sigma_{n+1} \rightarrow I\Sigma_n \rightarrow \cdots$

- We take as base theory RCA$_0$ (Recursive Comprehension Axiom plus $I\Sigma_1$).
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Hierarchy of the Induction Scheme

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The Combinatorial Principle COH

**Definition**

Let $R \in \mathbb{X}$ and $R_s = \{ t | (s, t) \in R \}$. $C \subset M$ is cohesive for $R$ if for all $s$, either $C \cap R_s$ is $M$-finite or $C \cap \bar{R}_s$ is $M$-finite.

COH: $M \models \text{COH}$ if for all $R \in \mathbb{X}$, there is a $C \in \mathbb{X}$ that is cohesive for $R$.

An $M$-extension of $M$ is a structure $M^* = \langle M^*, \mathbb{X}^*, +, \cdot, 0, 1 \rangle$ such that $M = M^*$ and $\mathbb{X} \subseteq \mathbb{X}^*$.

**Theorem**

(Cholak, Jockusch and Slaman) Let $n = 1, 2$. Every countable $M \models \text{RCA}_0 + I\Sigma_n$ has an $M$-extension $M^* \models \text{RCA}_0 + \text{COH} + I\Sigma_n$. 
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COH and $B\Sigma_2$

Corollary

$COH + RCA_0 + I\Sigma_n$ is $\Pi^1_1$ conservative over $RCA_0 + I\Sigma_n$, i.e. if $\varphi$ is $\Pi^1_1$ and $RCA_0 + COH + I\Sigma_n \vdash \varphi$, then $RCA_0 + I\Sigma_n \vdash \varphi$.

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An $M$-extension Theorem

Let $\mathcal{M} \models RCA_0 + B\Sigma_2$ be countable. If $R \in \mathbb{X}$, then $\mathcal{M}$ has an $M$-extension $\mathcal{M}^* = \mathcal{M}[G] \models RCA_0 + B\Sigma_2$ such that $G$ is cohesive for $R$.

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This is established using a two stage forcing construction.
Stage 1. Build an $R'$-recursive tree $T$ for which every unbounded path $X$ on $T$ is cohesive for $R$ and GL$_1$ relative to $R$, i.e. $X \oplus R' \equiv_T X'$.

Let $I$ be a $\Sigma_2$ cut in $M$ and $g : I \rightarrow M$ be $\Sigma_2$, increasing and cofinal.

Build a uniformly $R'$-recursive nested sequence $\{C_i | i \in I\}$ of $M$-infinite $R$-recursive trees such that for all $i \in I$:

(i) $C_i \supset C_{i+1}$
(ii) Every unbounded path on $C_i$ is cohesive for $R_s$, $s < g(i)$
(iii) Every unbounded path on $C_i$ is 1-generic on $C_i$ for $\exists x \varphi_s$, $s < g(i)$, where $\varphi_s$ is $\Delta_0$
(iv) $T = \bigcap C_i$. 

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For each $i \in I$, need to argue that there is a condition forcing $\exists x \varphi_s$ for all $s < g(i)$.

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Define countable sequences $\{T_n\}$ and $\{\sigma_n\}, n < \omega$, such that for each $n$,

- $T_n \supset T_{n+1}$ are recursive in $R'$
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Stage 2. Define a path $G$ (from the *outside*) on $T$ such that $\mathcal{M}[G] \models B\Sigma_2$.

Define countable sequences $\{T_n\}$ and $\{\sigma_n\}$, $n < \omega$, such that for each $n$,

- $T_n \supset T_{n+1}$ are recursive in $R'$
- $\sigma_n \in T_n$, $\sigma_n \leq \sigma_{n+1}$
- $\sigma_n \oplus R'$ forces $B\Sigma_1(G')$ for the $n$th $\Sigma_1(G \oplus R')$ sentence.
- $T_n$ above $\sigma_n$ is $\mathcal{M}$-infinite.

Put $G = \bigcup_n \sigma_n$. 
Ramsey’s Theorem For Pairs

Let $\mathcal{M} \models \text{RCA}_0$.

$\text{RT}^2_2$: Every two coloring of $[M]^2$ (pairs of elements of $M$) has a homogeneous set in $\mathcal{M}$.

$\text{SRT}^2_2$: Every stable two coloring of $[M]^2$ has a homogeneous set in $\mathcal{M}$ ($f : [M]^2 \to 2$ is stable if for all $x$, $\lim_y f(x, y)$ exists).

Hirst: Over $\text{RCA}_0$, $\text{RT}^2_2 \rightarrow B\Sigma_2$

Cholak, Jockusch and Slaman: Over $\text{RCA}_0$, $\text{RT}^2_2 \leftrightarrow \text{COH} + \text{SRT}^2_2$.

Question: Over $\text{RCA}_0$, does $\text{RT}^2_2 \rightarrow I\Sigma_2$? Does $\text{SRT}^2_2 \rightarrow \text{RT}^2_2$?
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For $A \Delta_2$, call any infinite $X \subset A$ or $\bar{A}$ a solution for $A$.

Interpreting these $\Delta_2$ solutions in $\text{RCA}_0 + B\Sigma_2$: 
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Nonstandard Methods in $RT^2_2$

- Chong and Yag, Mytilinaios and Slaman: Let $\mathcal{M} \models RCA_0 + B\Sigma_2$. If $G$ is $\Delta_2(\mathcal{M})$, then $\mathcal{M}[G]$ satisfies $RCA_0 \setminus I\Sigma_1$ plus either $B\Sigma_1$, $I\Sigma_1$ or $B\Sigma_2$. Furthermore
  - There is an $\mathcal{M}$ in which each of the three possibilities occurs;
  - There is an $\mathcal{M}$ in which every $\Delta_2(\mathcal{M})$ $G$ satisfies either $\mathcal{M}[G] \models RCA_0 \setminus I\Sigma_1$ plus $B\Sigma_1$ or $B\Sigma_2$ (and each possibility occurs).

$P$: For every $\mathcal{M} \models RCA_0 + B\Sigma_2$, there is a $\Delta_2 A \subset M$ for which no $\Delta_2$ solution $G$ exists with an $M$-extension $\mathcal{M}[G] \models RCA_0 + B\Sigma_2$

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Nonstandard Methods in $\text{RT}_2^2$

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Either $P$ or $Q$ is false.

Conjecture 1: There is a countable $\mathcal{M} \models \text{RCA}_0 + B\Sigma_2$ with an $M$-extension for the same theory in which every $\Delta_2$ set has a solution.

Corollary (to Conjecture 1): $\text{RT}_2^2$ does not imply $I\Sigma_2$.

Jockusch: There is a recursive two coloring of $[\mathbb{N}]^2$ with no $\Delta_2$ homogeneous set.

**Theorem**

There is a (first order) $\mathcal{M} \models B\Sigma_2$ with a recursive two coloring of $[M]^2$ having no regular $\emptyset''$-recursive homogeneous set.
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**Theorem**

There is a (first order) $\mathcal{M} \models B\Sigma_2$ with a recursive two coloring of $[\mathcal{M}]^2$ having no regular $\emptyset''$-recursive homogeneous set.
Conjecture 2: There is a countable $\mathcal{M} \models \text{RCA}_0 + B\Sigma_2$ with an $M$-extension for the same theory in which every $\Delta_2$ set has a solution, and in which there is a recursive 2-coloring of $[M]^2$ with no homogeneous set.

Corollary (to Conjecture 2): $\text{RT}_2^2$ does not imply $\text{SRT}_2^2$. 
Conjecture 2: There is a countable $\mathcal{M} \models \text{RCA}_0 + B\Sigma_2$ with an $\mathcal{M}$-extension for the same theory in which every $\Delta_2$ set has a solution, and in which there is a recursive 2-coloring of $[\mathcal{M}]^2$ with no homogeneous set.

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