Logical Analysis of Ramsey's Theorem

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Ramsey's Theorem

- F. P. Ramsey (1931)

  RT^n_k: Let $n \geq 2$. If $f : [\mathbb{N}]^n \rightarrow k$, then there is an infinite set H_f such that f is a constant on $[H_f]^n$.

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- What is the strength of RT^n_k, in Peano arithmetic and in subsystems of second order arithmetic?

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- Every recursive $f : [\mathbb{N}]^n \to k$ has an H_f that is Π_n.

First Order Analysis

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- Mathematical induction

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- Paris and Kirby (1978): For $n \geq 0$,

$$\cdots \to B\Sigma_{n+1} \to I\Sigma_n \to B\Sigma_n \to \cdots$$

Models of Fragments of PA

- Definition. Let $\mathcal{M} = \langle M, +, \times, 0, 1\rangle$ and $\mathcal{M} \models P^- + B\Sigma_n$ ($n \geq 0$). $X \subset M$ is $\mathcal{M}$-finite if it is coded in M.

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- Definition. $X \subset M$ is *regular* if $X \upharpoonright a$ is $\mathcal{M}$-finite for all $a \in M$.

- Definition. $X \subset M$ has *unbounded size* if for all $a \in M$, there is an $\mathcal{M}$-finite $D \subset X$ such that $|D| \geq a$ (in $\mathcal{M}$).

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- (1) Every recursive partition $f : [M]^n \longrightarrow 2$ has an H_f that is regular, of unbounded size, and Π_n definable.
- (2) $\mathcal{M} \models \Sigma_n$ induction

  Same conclusion for Δ_{n+1} sets.

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- Recursive comprehension axiom:

$$\exists X \forall x [x \in X \leftrightarrow \varphi(x)],$$

where φ is Δ^0_1 with number and set parameters.

Second Order Analysis

- Imbedded in the proof of Definable RT: If $\mathcal{M} \models \mathrm{RCA}_0 + \mathrm{RT}^n_2$, then $\mathcal{M} \models B\Sigma^0_n$.

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- For $n \geq 3$, $\mathrm{RCA}_0 + \mathrm{RT}^n_2 \rightarrow I\Sigma^0_n$

- Does $\mathrm{RCA}_0 + \mathrm{RT}^2_2 \rightarrow I\Sigma^0_2$?

Second Order Analysis of RT^n_2

- Slaman and Seetapun (1995): Over RCA_0, $\mathrm{RT}^2_2 \not\to \mathrm{RT}^3_2$ ($\leftrightarrow \mathrm{RT}^n_2$, for $n > 2$).

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- Investigate the strength of RT^2_2.

Cohesiveness

- If $\mathcal{R} = \{R_i\}$ is a sequence of sets, then C is $\mathcal{R}$-cohesive if for each i, either $C \cap R_i$ is $\mathcal{M}$-finite or $C \cap \bar{R}_i$ is $\mathcal{M}$-finite.

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- COH: Every sequence $\mathcal{R}$ in $\mathcal{M}$ has an $\mathcal{R}$-cohesive set in $\mathcal{M}$.

Stable Coloring

- $f : [M]^2 \longrightarrow 2$ is *stable on* C if for all x, $\lim_{s \in C} f(x, s)$ exists (eventually RED or BLUE on C).

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- If $f : [M]^2 \longrightarrow 2$ and C is $\mathcal{R}$-cohesive, where $\mathcal{R} = \{R_i\}$ and

$$R_i = \{s | f(i, s) = \text{RED}\},$$

then f is stable on C.

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- *Question.* Does $\mathrm{SRT}^2_2 \longrightarrow \mathrm{RT}^2_2$?

Analysis of $B\Sigma^0_2$ Models

- $\mathcal{M} \models \mathrm{RCA}_0 + \varphi + B\Sigma^0_2$ is a double-jump basis for φ if there is a solution for φ in $\mathcal{M}$ recursive in the double jump of the parameters in φ.

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- Theorem. There is no $\mathcal{M}$ that is a double-jump basis for $\mathrm{RCA}_0 + \mathrm{RT}^2_2 + B\Sigma^0_2$.

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- Theorem. There is no $\mathcal{M}$ that is a double-jump basis for $\mathrm{RCA}_0 + \mathrm{RT}^2_2 + B\Sigma^0_2$.
- *Proof.* Suppose $\mathcal{M}$ is a double-jump basis for RT^2_2. Proof of Definable RT^2_2 on $\mathcal{M}$ implies there exist X and $f \leq_T X \in \mathcal{M}$ such that $f : [M]^2 \to 2$ has no $\Delta^0_2(X)$ H_f that is regular and of unbounded size.

Analysis of $B\Sigma_2^0$ Models

- Assume there is an $H_f \in \mathcal{M}$ and $H_f \leq_T X''$ with $\mathcal{M} \models B\Sigma_2^0(H_f \oplus X)$.

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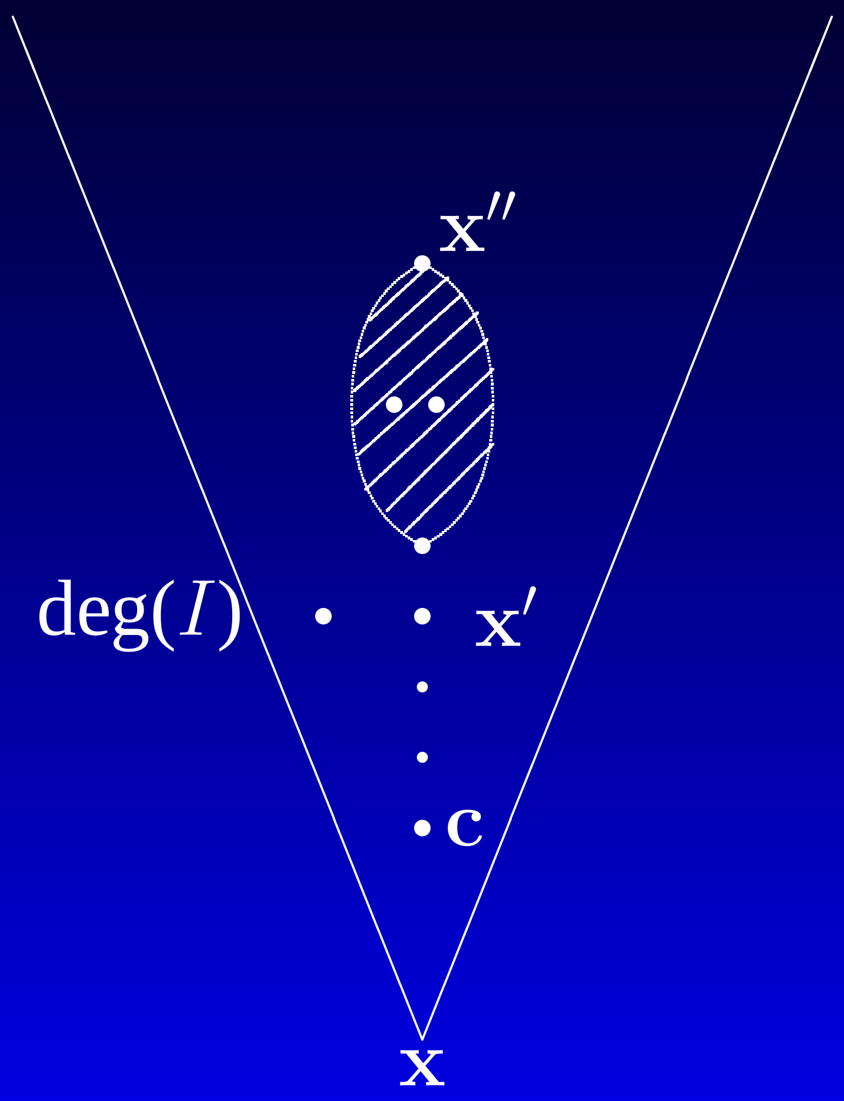
- $B\Sigma^0_2(H_f \oplus X) \Rightarrow B\Sigma^0_1((H_f \oplus X)') \Rightarrow (H_f \oplus X)'$ regular $\Rightarrow (H_f \oplus X)' \leq_T I \oplus X' \Rightarrow (H_f \oplus X)' \leq_T X' \Rightarrow H_f' \leq_T X'$.

Degrees Below $\mathbf{deg}(X'')$



$I = \Sigma_2(X)$ cut

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- Chong, Slaman and Yang (2005): There exist models of $\mathrm{RCA}_0 + \mathrm{COH} + B\Sigma^0_2$. These are not double-jump basis models.

- *Question.* Is there a double-jump basis model for $\mathrm{RCA}_0 + \mathrm{SRT}^2_2 + B\Sigma^0_2$?

Analysis of $B\Sigma^0_2$ Models

- If there is a uniform construction of a double jump basis model $\mathcal{M}^* \models \mathrm{RCA}_0 + \mathrm{SRT}^2_2 + B\Sigma^0_2$ from a given $\mathcal{M} \models \mathrm{RCA}_0 + B\Sigma^0_2$, then

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- (1) Many such $\mathcal{M}$ can be expanded to $\mathcal{M}^*$ with COH and preserving $B\Sigma^0_2$, so that $\mathrm{RT}^2_2 \not\to I\Sigma^0_2$;

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- (1) Many such $\mathcal{M}$ can be expanded to $\mathcal{M}^*$ with COH and preserving $B\Sigma^0_2$, so that $\mathrm{RT}^2_2 \not\to I\Sigma^0_2$;

- (2) Starting with a "pure" $B\Sigma^0_2$ model $\mathcal{M}_2$ (without second order objects), expand to an $\mathcal{M}^*_2 \models \mathrm{RCA}_0 + \mathrm{SRT}^2_2 + B\Sigma^0_2$. As before, every solution of SRT^2_2 in $\mathcal{M}^*_2$ is low. So $\mathcal{M}^*_2 \models \neg\mathrm{COH}$. Then $\mathrm{SRT}^2_2 \not\to \mathrm{RT}^2_2$.

Analyzing Subsets of Δ_2^0 Sets

- *Question.* If $\mathcal{M} \models B\Sigma_2^0$, and $A \subset M$ is Δ_2^0, is there $X \subset A$ or $\bar{A}$ such that $\mathcal{M}[X] \models B\Sigma_2^0$?

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- Hirschfeldt, Jockusch, Lempp, Kjos-Hanssen and Slaman (2005): For $\mathbb{N}$, there is always an $X <_T \emptyset'$ but X need not be low (Downey, Hirschfeldt, Jockusch and Lempp (2004)).

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- *General Question.* Given $\mathcal{M} \models B\Sigma_n^0$, and $A \subset M$ that is Δ_n^0, is there $X \subset A$ or $\bar{A}$ such that $\mathcal{M} \models B\Sigma_n^0(X)$?

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