Maximal Chains and Antichains in the Turing Degrees
(Joint Work with Liang YU)

C T Chong

National University of Singapore

chongct@math.nus.edu.sg

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Chains and Antichains

- $\langle \mathcal{D}, \leq \rangle$ denotes the structure of Turing degrees.

- $\mathcal{A} \subset \mathcal{D}$ is a chain if for any two elements $a, b \in \mathcal{A}$, either $a < b$ or $b < a$.

- $\mathcal{A} \subset \mathcal{D}$ is an antichain if no two $a, b$ are comparable.

- Every maximal chain has size $\aleph_1$. Every maximal antichain has size $2^{\aleph_0}$.

- AC implies existence of maximal chains and antichains.

*Is there a definable maximal chain?*
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Existence of a $\Pi^1_1$ Maximal Chain

**Fact.** There is no $\Sigma^1_1$ subset of $2^\omega$ whose Turing degrees form a maximal chain: Any uncountable $\Sigma^1_1$ set contains a perfect subset. Any perfect set contains two Turing incomparable members.

**Theorem**

(Chong and Yu) *Let $L[a]$ denote the relative constructible universe with $a \subset \omega$. Then $\omega_1^{L[a]} = \omega_1$ if and only if there exists a $\Pi^1_1[a]$ subset of $2^\omega$ whose Turing degrees form a maximal chain.*
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When $\omega_1^L = \omega_1$

Lemma

For $A \subset D$ countable, the set of minimal upper bounds of $A$ has double jumps cofinal in $D$.

Corollary. Borel Determinacy implies that there is a cone of degrees which are double jumps of minimal upper bounds of $A$.

$V = L$ allows a sequence $A^* = \{a_\gamma | \gamma < \omega_1^L\}$ to be constructed effectively and uniformly, so that

1. $a_\gamma$ is a minimal upper bound of $\{a_\delta | \delta < \gamma\}$
2. $a_\gamma''$ is Turing equivalent to a master code in the sense of fine structure theory of $L$ (Jensen, Boolos and Putnam).
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2. $a''_\gamma$ is Turing equivalent to a master code in the sense of fine structure theory of $L$ (Jensen, Boolos and Putnam).
Gandy-Spector analysis shows that $A^*$ is $\Pi^1_1$ if and only if there is a $\Sigma_1$ ("effective") definition uniformly over $L_{\omega_1^x}[x]$ for each $x \in A^*$.

$\{a_\gamma | \gamma < \omega_1^L\}$ forms a $\Pi^1_1$ maximal chain in $L$.

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Conversely, any uncountable $\Pi^1_1$ subset of $2^\omega$ whose Turing degrees form a maximal chain contains no perfect subset, hence constructible (Mansfield-Solovay). So $\omega_1 = \omega_1^L$.

**Corollary.** The following are equiconsistent:

(i) ZFC+ “There is an inaccessible cardinal”.

(ii) There is no (bold face) $\Pi^1_1$ subset of $2^\omega$ whose Turing degrees form a maximal chain.
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Definable Maximal Antichains

- $\mathcal{A}^* \subset 2^\omega$ is thin if it contains no perfect subset.

- ZFC implies there is a thin set whose Turing degrees form a maximal antichain.

- There is no uncountable $\Sigma^1_1$ thin set of reals to form a maximal antichain of Turing degrees.

Theorem (Chong and Yu) There is a $\Pi^1_1[a]$ thin set whose Turing degrees form a maximal antichain if and only if $(2^\omega)^{L[a]} = 2^\omega$. 
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When \((2^\omega)^L = 2^\omega\)

- For \(\mathcal{A}\) a countable antichain, the double jumps of Turing degrees \(a\) where \(\mathcal{A} \cup \{a\}\) forms an antichain is cofinal in \(\mathcal{D}\).

- Borel determinacy implies that there is a cone in which every member is the double jump of some \(a\) where \(\mathcal{A} \cup \{a\}\) is an antichain.

- \((2^\omega)^L = 2^\omega\) implies that there is an effective constructible sequence \(\mathcal{A}^* = \{a_\gamma | \gamma < \omega_1^L\}\) whose degrees form a maximal antichain in \(L\), such that \(a_\gamma''\) is a master code for each \(\gamma\).
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- This ensures that \(a_\gamma \in L_{\omega_1^{\omega_1}}\), so that \(A^*\) is thin (Mansfield-Solovay).

- Gandy-Spector analysis guarantees that \(A^*\) is \(\Pi^1_1\). Then \(\mathcal{A} = \{\deg(a_\gamma) | \gamma < \omega_1^L\}\) is a thin \(\Pi^1_1\) maximal antichain.

- Conversely, if \(A^*\) is \(\Pi^1_1\) and thin, then \(A^* \subset L\) by Mansfield-Solovay. For any real \(x\), there is a \(y\) of minimal degree such that \(x \leq_T y'\) (Cooper). Then \(y \leq_T z\) for some \(z \in A^*\), so that \(x \in L\). Thus \((2^\omega)^L = 2^\omega\).
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