Maximal Chains and Antichains in the Turing Degrees

(Joint Work with Liang YU)

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June 2007

Chains and Antichains

- $\langle \mathfrak{D}, \leq \rangle$ denotes the structure of Turing degrees.
- $\mathcal{A} \subset \mathfrak{D}$ is a chain if for any two elements $\mathbf{a}, \mathbf{b} \in \mathcal{A}$, either $\mathbf{a} < \mathbf{b}$ or $\mathbf{b} < \mathbf{a}$.
- $\mathcal{A} \subset \mathfrak{D}$ is an antichain if no two $\mathbf{a}, \mathbf{b}$ are comparable.
- Every maximal chain has size $\aleph_1$. Every maximal antichain has size $2^{\aleph_0}$.
- AC implies existence of maximal chains and antichains.

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Existence of a Π^1_1 Maximal Chain

Fact. There is no Σ^1_1 subset of 2^ω whose Turing degrees form a maximal chain: Any uncountable Σ^1_1 set contains a perfect subset. Any perfect set contains two Turing incomparable members.

Theorem

(Chong and Yu) *Let $L[a]$ denote the relative constructible universe with $a \subset \omega$. Then $\omega_1^{L[a]} = \omega_1$ if and only if there exists a $\Pi^1_1[a]$ subset of 2^ω whose Turing degrees form a maximal chain.*

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When $\omega_1^L = \omega_1$

Lemma

For $\mathcal{A} \subset \mathfrak{D}$ countable, the set of minimal upper bounds of $\mathcal{A}$ has double jumps cofinal in $\mathfrak{D}$.

- **Corollary.** Borel Determinacy implies that there is a cone of degrees which are double jumps of minimal upper bounds of $\mathcal{A}$.

- $V = L$ allows a sequence $\mathcal{A}^* = \{a_\gamma | \gamma < \omega_1^L\}$ to be constructed effectively and uniformly, so that
  1. a_γ is a minimal upper bound of $\{a_\delta | \delta < \gamma\}$I
  2. a_γ'' is Turing equivalent to a master code in the sense of fine structure theory of L (Jensen, Boolos and Putnam).

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- Gandy-Spector analysis shows that $\mathcal{A}^*$ is Π^1_1 if and only if there is a Σ_1 ("effective") definition uniformly over $L_{\omega_1^x}[x]$ for each $x \in \mathcal{A}^*$.
- $\{a_\gamma | \gamma < \omega_1^L\}$ forms a Π^1_1 maximal chain in L.
- Being a chain of Turing degrees is an absolute property. This ensures that a maximal chain in $\langle \mathfrak{D}, \leq \rangle^L$ is a maximal chain in V.

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- Conversely, any uncountable Π^1_1 subset of 2^ω whose Turing degrees form a maximal chain contains no pefect subset, hence constructible (Mansfield-Solovay). So $\omega_1 = \omega_1^L$.

- **Corollary.** The following are equiconsistent:

  (i) ZFC$+$ "There is an inaccessible cardinal".

  (ii) There is no (bold face) Π^1_1 subset of 2^ω whose Turing degrees form a maximal chain.

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Definable Maximal Antichains

- $\mathcal{A}^* \subset 2^\omega$ is *thin* if it contains no perfect subset.
- ZFC implies there is a thin set whose Turing degrees form a maximal antichain.
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Theorem

(Chong and Yu) *There is a* $\Pi^1_1[a]$ *thin set whose Turing degrees form a maximal antichain if and only if* $(2^\omega)^{L[a]} = 2^\omega$.

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- For $\mathcal{A}$ a countable antichain, the double jumps of Turing degrees **a** where $\mathcal{A} \cup \{\mathbf{a}\}$ forms an antichain is cofinal in $\mathfrak{D}$.
- Borel determinacy implies that there is a cone in which every member is the double jump of some **a** where $\mathcal{A} \cup \{\mathbf{a}\}$ is an antichain.
- $(2^\omega)^L = 2^\omega$ implies that there is an effective constructible sequence $\mathcal{A}^* = \{a_\gamma | \gamma < \omega_1^L\}$ whose degrees form a maximal antichain in L, such that a_γ'' is a master code for each γ.

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- Conversely, if $\mathcal{A}^*$ is Π^1_1 and thin, then $\mathcal{A}^* \subset L$ by Mansfield-Solovay. For any real x, there is a y of minimal degree such that $x \leq_T y'$ (Cooper). Then $y \leq_T z$ for some $z \in \mathcal{A}^*$, so that $x \in L$. Thus $(2^\omega)^L = 2^\omega$.

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