A Π^1_1 Uniformization Principle for Reals

(Joint Work With Liang Yu)

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Cofinal Π^1_1 Predicates

A Π^1_1 predicate $P(x, y)$ is *cofinal* if for each real x, the set $\{y|P(x, y)\}$ is cofinal in the hyperdegrees.

Examples:

- $P(x, y) \Leftrightarrow x \leq_h y$
- $P(x, y) \Leftrightarrow y$ is the double jump of a (Turing) minimal cover of $\{(x)_n|n < \omega\}$, where $(x)_n = n$th column of x

Fact: Assume $(2^\omega)^L = 2^\omega$. If $P(x, y)$ is cofinal, then for each x, there are cofinally many y such that $P(x, y)$ and $y \in L_{\omega_1^y}$.

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The Π^1_1 Uniformization Principle $\mathfrak{I}$

Let $P(x, y)$ be Π^1_1 and cofinal. Then there exists a Π^1_1 set of reals A such that

- $A \bigcap L$ has size ω_1;
- Every $y \in A$ satisfies $y \in L_{\omega_1^y}$ [*Note:* "$y \in L_{\omega_1^y}$" is Π^1_1];
- For each $y \in A$, there is an $x \in L_{\omega_1^y}$ so that $P(x, y)$ and

$$\{(x)_n | n < \omega\} = \{z \in A | z <_L y\}.$$

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The Principle $\mathfrak{I}$

Theorem

$\omega_1^L = \omega_1$ if and only if $\mathfrak{I}$ holds.

Relativize above to any real x.

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Boldface $\mathfrak{I}$ fails if and only if for all x, $\omega_1^{L[x]} < \omega_1$.

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The Strength of Boldface $\mathfrak{I}$

Corollary

The following are equiconsistent:
(i) ZFC + Boldface $\mathfrak{I}$ fails;
(ii) ZFC + "There is an inaccessible cardinal".

Proof. (ii) $\Leftrightarrow$ (i): If there is a model with an inaccessible cardinal, then there is an $\mathfrak{M} \models \text{ZFC} + $ "ω_1 is inaccessible". In $\mathfrak{M}$, $\omega_1^{L[x]} < \omega_1$ for all x. Hence boldface $\mathfrak{I}$ fails in $\mathfrak{M}$.

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Some Applications

Theorem

$\omega_1^{L[x]} = \omega_1$ if and only if there is a $\Pi^1_1(x)$ maximal chain in the Turing degrees.

Proof. Let $P(x, y)$ hold if and only if y is the double jump of a minimal cover of $\{(x)_n | n < \omega\}$. Apply $\mathfrak{I}[x]$ to get the set A, which is a maximal $\Pi^1_1(x)$ chain.

A Π^1_1 set is *thin* if it contains no perfect subset.

Theorem

$(2^\omega)^{L[x]} = 2^\omega$ if and only if there is a thin $\Pi^1_1(x)$ maximal antichain in the Turing degrees.

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Martin's Conjecture

- $f : 2^\omega \to 2^\omega$ is *degree invariant* if $x \equiv_T y$ implies $f(x) \equiv_T f(y)$.

- f is *uniformly* degree invariant if there is a t such that $x = \Phi_i^y \;\&\; y = \Phi_j^x \to f(x) = \Phi_{t(i,j)_0}^{f(y)} \;\&\; f(y) = \Phi_{t(i,j)_1}^{f(x)}$.

- f is *increasing on a cone* if there is an x such that for all $y \geq_T x$, $f(y) \geq_T y$.

- f is *order preserving* on a cone if there is an x such that for all $z \geq_T y \geq_T x$, $f(z) \geq_T f(y)$.
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ZF + AD + Dependent Choice implies:

- (i) Every degree invariant function that is not increasing on a cone is a constant on a cone.
- Write $f \leq_M g$ if $f(x) \leq_T g(x)$ on a cone. Then $\leq_M$ is a prewellordering on degree invariant functions which are increasing on a cibe.

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(*) If f is degree invariant and $f(x) \equiv_T x$ for cofinally many x, then $f(x) \equiv_T x$ on a cone.

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- Slaman and Steel proved (*) under AD for uniformly degree invariant functions increasing on a cone.
- Hence this is a theorem of ZFC for any such f that is Δ^1_1 (thus Σ^1_1).

Assume $(2^\omega)^L = 2^\omega$. The following are consequences of $\mathfrak{I}$.

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Slaman and Steel also showed AD $\Rightarrow$ every uniformly degree invariant function that is not increasing on a cone is a constant on a cone.

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There exist Π^1_1 uniformly degree invariant increasing functions f_n, $n < \omega$, such that $f_{n+1} <_M f_n$. Hence $\leq_M$ is not a prewellordering.

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