Recursion Theory of Ramsey's Theorem

Università degli Studi di Siena
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Session I

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Outline

- **Session I**
  - Fragments of Peano arithmetic (PA)
  - Recursion theory on fragments of PA
  - Models of fragments of PA and the Coding Lemma
- **Session II**
  - Subsystems of second order arithmetic
  - Models of RCA_0
  - Ramsey's Theorem RT^n_k over $\mathbb{N}$
  - RT^n_k and SRT^2_2 in models of RCA_0

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- **Session III**
  - Mathias forcing
  - The Seetapun-Slaman Theorem

- **Session IV**
  - Working with models of Σ_2^0 induction
  - RT_2^2, SRT_2^2 and COH preserving Σ_2^0 induction
  - Π_1^1-conseervation

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  - The limits of double-jump construction in models of Σ^0_2 bounding

- **Session VI**
  - Π^1_1 conservation of $\mathsf{RCA}_0 + B\Sigma^0_2 + \mathsf{COH}$ over $\mathsf{RCA}_0 + B\Sigma^0_2$
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- **Session VIII**
  - Preserving definable coding in the generic extension
  - Preserving $B\Sigma_2^0$ in the genric extension
  - RT_2^2 does not imply Σ_2^0 induction

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Fragments of Peano Arithmetic

- *PA* denotes Peano axioms without mathematical induction.

- Σ_n induction $I\Sigma_n$: Mathematical induction for Σ_n formulas—If $\varphi(x)$ is Σ_n (with parameters), and $\varphi(0)$ & $\forall x(\varphi(x) \to \varphi(x+1))$, then $\forall x \varphi(x)$.

- Σ_n bounding $B\Sigma_n$: Every Σ_n definable function (with parameters) maps a "finite" set onto a "finite" set.

- Slaman [2004]: $B\Sigma_n$ is equivalent to Δ_n induction $I\Delta_n$.

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- Kirby and Paris [1978]: For $n \geq 0$,

$$\cdots \rightarrow I\Sigma_{n+1} \rightarrow B\Sigma_{n+1} \rightarrow I\Sigma_n \rightarrow B\Sigma_n \rightarrow \cdots$$

- $I\Sigma_n \leftrightarrow$ Every Σ_n definable set has a least element.
- Unless otherwise indicated, all models $\mathcal{M} = \langle M, +, \times, 0, 1 \rangle$ considered are models of *at least* $I\Sigma_1$.
- If $\mathcal{M} \models I\Sigma_1$, then for all $x \in M$, $2^x \in M$.

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Fragments of Peano Arithmetic

- Coding under $I\Sigma_1$: A set $D \subset M$ is $\mathcal{M}$-*finite* if there is an $a \in M$ such that for all x, $x \in D \leftrightarrow x|a$.

- $X \subset M$ is *bounded* if there is an $a \in M$ such that for all x, $x \in X \rightarrow x < a$.

- $\mathcal{M} \models I\Sigma_n$ if and only if every bounded Σ_n definable set of M is $\mathcal{M}$-finite.

- A *cut* $I \subset M$ satisfies: (a) I is bounded, (b) $0 \in I$, and (c) for all x, if $x \in I$ then $x + 1 \in I$. [Example: ω is a cut in a nonstandard model of $I\Sigma_n$.]

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Σ_n Cut

Fact $\mathcal{M} \models B\Sigma_n + \neg I\Sigma_n$ if and only if there is a cut I such that

- I is Σ_n definable.
- There is a Σ_n increasing, cofinal function $f : I \to M$.

I is called a Σ_n cut.

Recursion Theory on $I\Sigma_1$

- Every model $\mathcal{M}$ of $I\Sigma_1$ is a model of computation.
- A set $X \subset M$ is recursively enumerable (r.e.) if X is $\Sigma_1(\mathcal{M})$.
- X is recursive (computable) if X and $M \setminus X$ are r.e.
- All pre-Post problem results on r.e. sets hold in models of $I\Sigma_1$.
- If $X, Y \subset M$, then X is *weakly recursive* in Y, written $X \leq_w Y$, if there is an r.e. set Φ such that for all x,

$$x \in X \leftrightarrow \exists P \exists N \langle x, 0, P, N \rangle \in \Phi$$

and

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Recursion Theory on Models of $I\Sigma_1$

- X is *recursive* in Y, written $X \leq_T Y$, if x is replaced by D ($\mathcal{M}$-finite set) in the above definition.

- $\leq_w$ and $\leq_T$ are reflexive.

- In general, $\leq_w$ is *NOT* transitive, while $\leq_T$ always is. Hence $\equiv_T$ is well-defined.

- If $X \equiv_T Y$, then they have the same Turing degree.

- The collection of Turing degrees forms an upper semilattice.

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Finite Injury Priority Arguments

Theorem (1.1)

(a) (Mytilinaios [1988]) *$I\Sigma_1$ proves the Friedberg-Muchnik Theorem.*
(b) (Slaman and Woodin [1987]) *$B\Sigma_1$ solves Post's problem.*
(c) (Chong and Mourad [1991]) *$B\Sigma_1$ proves the Friedberg-Muchnik Theorem.*
(d) (Mourad [unpublished]) *$I\Sigma_1$ is equivalent to Sacks splitting theorem.*

Finite Injury Priority Arguments

- Proof of Theorem 1.1 (a) (Sketch).
  Apply classical construction for a Friedberg-Muchnik pair. For each requirement R_e, "e is injured at stage s" is a recursive relation.

  Claim. Every R_e is injured at most 2^e times.

  Proof of Claim. Let

  $$X = \{e | R_e \text{ is injured more than } 2^e \text{ times}\}.$$

  Then X is r.e. and so by $I\Sigma_1$ has a least element e_0. So R_{e_0-1} is injured less than 2^{e_0-1} times. But standard argument implies that R_{e_0} is injured at most 2^{e_0} times, contradiction.

  Hence every requirement is satisfied.

Post's Problem and Sacks Splitting

- Proof of Theorem 1.1 (b) (Sketch).
  If $\mathcal{M} \models I\Sigma_1$, then result follows from Theorem 1.1 (a). Otherwise, there is a cut I in M which is r.e. This set I has incomplete r.e. degree. [*NO* priority argument!]

- Proof of Theorem 1.1 (d) (One direction).
  Suppose $\mathcal{M}$ is a model of $B\Sigma_1$ but not $I\Sigma_1$. Let I be a cut which is r.e. Then any $J \subset I$ that is not bounded in I computes I as follows: Fix a that is in $\overline{I}$. Given an x, $x \notin I$ if and only if $\{y|x \leq y \leq a\} \cap J = \emptyset$. Hence $\overline{I}$ is Σ_1 in J. Since I is r.e., we have $I \leq_T J$. In particular, there is no splitting of I into two incomparable r.e. sets.,

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$\emptyset''$-Priority Arguments

Theorem (1.2)

(Chong and Yang [1997], [2004]) *Over the base theory $B\Sigma_2$, $I\Sigma_2$ is equivalent to*
(a) *Existence of a high r.e. degree;*
(b) *Sacks jump inversion theorem.*
(c) *Existence of a low$_2$ degree.*

Proof. (Sketch) In a model $\mathcal{M}$ of $B\Sigma_2$ but not $I\Sigma_2$, the jump of an r.e. degree $\mathbf{c}$ assumes one of three forms: If $\mathbf{c} = \mathbf{0}'$ then $\mathbf{c}' = \mathbf{0}''$, and

$$\mathbf{c} < \mathbf{0}' \rightarrow \mathbf{c}' = \mathbf{0}' \text{ or } \mathbf{c}' = \mathbf{i} \vee \mathbf{0}',$$

where $\mathbf{i}$ is the degree of a cut I that is Σ_2-definable.

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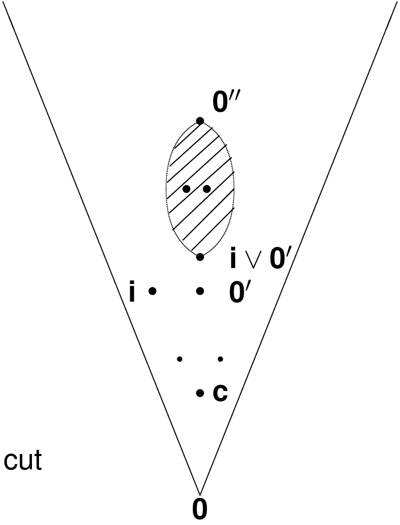
$\emptyset''$-Priority Arguments

Theorem (1.3)

(Groszek, Mytilinaios and Slaman [1996]) *$B\Sigma_2$ implies Sacks density theorem.*

Degrees Below $\emptyset''$



$I = \Sigma_2$ definable cut

The Coding Lemma

Definition

Let $\mathcal{M} \models B\Sigma_2$ and $A \subset M$. $X \subset A$ is Δ_n on X if X and $A \setminus X$ are Δ_n.

Definition

Let $A \subset M$ and $X \subset A$. Then X is *coded on A* if there is an $\mathcal{M}$-finite set $\hat{X}$ such that $\hat{X} \cap A = X$.

Theorem (1.4)

(Chong and Mourad [1991]) *Let $\mathcal{M} \models B\Sigma_n$. Let $A \subset M$. Then every $X \subset A$ that is Δ_n on A is coded on A.*

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