Recursion Theory of Ramsey’s Theorem

Università degli Studi di Siena
April 2006
Session II

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Subsystems of Second Order Arithmetic

Language of second order arithmetic:

- First order variables $x, y, z, \ldots$
- Set variables $X, Y, Z, \ldots$
- Constants: 0, 1
- Relation symbol: $\in$
- Function symbols: $+, \times$

A structure $\mathcal{M} = \langle M, X, +, \times, 0, 1 \rangle$ is a model of a subsystem of second order arithmetic $T$ if all axioms in $T$ are true in $\mathcal{M}$, where in particular $X \subset 2^M$ and all set variables are interpreted as members of $X$. 
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Subsystems of Second Order Arithmetic

Definition
An $\omega$-model $M$ is a structure such that $M = \omega$.

Definition
If $T$ is a subsystem of second order arithmetic, and $M, N \models T$, then $M \subseteq N$ if and only if $M \subseteq N$, and $X_M \subseteq X_N$, where $M = \langle M, X_M, +, \times, 0, 1 \rangle$ and $N = \langle N, X_N, +, \times, 0, 1 \rangle$.

Definition
If $M, N \models T$, then $M$ is an $M$-submodel of $N$ if $M = N$. Hence only second order elements are added to $M$ to obtain $N$. 
An $\omega$-model $\mathcal{M}$ is a structure such that $M = \omega$.

If $T$ is a subsystem of second order arithmetic, and $\mathcal{M}, \mathcal{N} \models T$, then $\mathcal{M} \subset \mathcal{N}$ if and only if $M \subset N$, and $X_M \subset X_N$, where $\mathcal{M} = \langle M, X_M, +, \times, 0, 1 \rangle$ and $\mathcal{N} = \langle N, X_N, +, \times, 0, 1 \rangle$.

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If $\mathcal{M}, \mathcal{N} \models T$, then $\mathcal{M}$ is an $M$-submodel of $\mathcal{N}$ if $M = N$. Hence only second order elements are added to $\mathcal{M}$ to obtain $\mathcal{N}$.
$\mathcal{M}$ is a model of the system RCA$_0$ (**Recursive Comprehension Axiom**) if it satisfies the following:

- $P^-$, the Peano axioms without mathematical induction
- $\Sigma^0_1$ induction of the form

$$[(\varphi(0) \& \forall x(\varphi(x) \rightarrow \varphi(x + 1))) \rightarrow \forall x \varphi(x)],$$

where $\varphi$ is $\Sigma^0_1(\mathcal{M})$, with number and set constants.

- (Recursive comprehension)

$$\exists X \forall x(x \in X \leftrightarrow \varphi(x)),$$

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If $\mathcal{M} \models \text{RCA}_0$, then $X$ is closed under Turing reducibility and joint, since these operations are $\Delta^0_1(\mathcal{M})$ definable. Thus in any model of RCA$_0$, the second order objects form an ideal.

$\mathcal{M} = \langle \omega, 2^\omega, +, \times, 0, 1 \rangle$ is a model of RCA$_0$ (in fact, of any comprehension scheme).

If $\mathcal{M} = \langle M, +, \times, 0, 1 \rangle$ is a first order model of $I\Sigma^0_1$, then it can be expanded into a model $\mathcal{M}^* = \langle M, X, +, \times, 0, 1 \rangle$ of RCA$_0$ by letting $X = \{X | X \text{ is recursive}\}$.
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Models of RCA₀

- If $\mathcal{M} \models RCA₀$, then $X$ is closed under Turing reducibility and joint, since these operations are $\Delta^0_1(\mathcal{M})$ definable. Thus in any model of RCA₀, the second order objects form an ideal.

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Weak König’s Lemma $\text{WKL}_0$

**Definition**

Let $\mathcal{M} \models \text{RCA}_0$. $\mathcal{M} \models \text{WKL}_0$ if every $\mathcal{M}$-infinite binary tree $T$ coded in $\mathcal{M}$ has an $\mathcal{M}$-infinite path $X$. Thus if $\mathcal{M} = \langle M, X, +, \times, 0, 1 \rangle$ and $T \in X$, then $X \in X$.

**Theorem (2.1)**

$\text{WKL}_0 \rightarrow \text{RCA}_0$, but not conversely.

*Proof.* Suppose $\mathcal{M} \models \text{WKL}_0$ and let $\varphi$ be $\Delta^0_1(\mathcal{M})$. Let

$$T = \{ \sigma | \exists a \in M[\sigma \in 2^a \& \forall x(\sigma(x) = 1 \leftrightarrow \varphi(x)) \}.$$

Assume $\{ x | \varphi(x) \} \mathcal{M}$-infinite. Then $T$ is an $\mathcal{M}$-infinite tree with unique path $X \in X$ by $\text{WKL}_0$. This $X$ = characteristic function of a recursive set (relative to the parameters defining $\varphi$).
Weak König’s Lemma WKL₀

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Proof. (Continued)
On the other hand, there is an infinite $T \subset 2^{<\omega}$ recursive
tree with no recursive infinite path: Let $A$ and $B$ be disjoint,
r.e. and recursively inseparable enumerated by $f_A$ and $f_B$. $\sigma$
of length $k$ is on $T$ if and only if for all $m, n \leq \text{lth}(\sigma)$,
\[ f_A(m) = n \rightarrow \sigma(n) = 1 \] and \[ g_A(m) = n \rightarrow \sigma(n) = 0. \] Then
every infinite path on $T$ separates $A$ from $B$, hence is not recursive. Thus if $\mathcal{M} = \langle \omega, \text{REC}, +, \times, 0 \rangle$, where REC
denotes the class of recursive sets in $\omega$, then
$\mathcal{M} \models \text{RCA}_0 + \neg \text{WKL}_0$.

$\text{RCA}_0$ is equivalent to the Intermediate Value Theorem.
$\text{WKL}_0$ is equivalent over $\text{RCA}_0$ to the Heine-Borel
Theorem, the Brouwer Fixed Point Theorem, the local
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$\text{RCA}_0$ is equivalent to the Intermediate Value Theorem. $\text{WKL}_0$ is equivalent over $\text{RCA}_0$ to the Heine-Borel Theorem, the Brouwer Fixed Point Theorem, the local existence theorem for solutions of ordinary differential equations etc.
Arithmetic Comprehension Axiom ACA₀

**Definition**

\( \mathcal{M} \models ACA₀ \) if and only if it satisfies the following sentence:

\[ \exists X \left[ \forall x (x \in X \leftrightarrow \varphi(x)) \right], \]

where \( \varphi \) is \( \Sigma^0_n(\mathcal{M}) \) for \( n \in \omega \).

*Note.* Restricting \( n \) to 1 in the definition is equivalent to letting \( n \) range over \( \omega \), since if all \( \Sigma^0_n(\mathcal{M}) \) sets are in \( X \), then so are all \( \Sigma^0_{n+1}(\mathcal{M}) \) sets by iteration. Thus \( \mathcal{M} \models ACA₀ \) if and only if \( X \) is closed under the Turing jump operation.

**Theorem (2.2)**

\( ACA₀ \rightarrow WKL₀ \) but not conversely.
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Arithmetic Comprehension Axiom ACA$_0$

**Definition**

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**Note.** Restricting $n$ to 1 in the definition is equivalent to letting $n$ range over $\omega$, since if all $\Sigma^0_n(\mathcal{M})$ sets are in $X$, then so are all $\Sigma^0_{n+1}(\mathcal{M})$ sets by iteration. Thus $\mathcal{M} \models ACA_0$ if and only if $X$ is closed under the Turing jump operation.

**Theorem (2.2)**

$ACA_0 \rightarrow WKL_0$ *but not conversely.*
ACA₀

**Proof.** If \( \mathcal{M} \models ACA₀ \) then \( \mathcal{M} \models IΣ^0_n \) for all \( n \in \omega \). This is true since every bounded definable set is \( \mathcal{M} \)-finite. Now there is a definable solution to weak König’s Lemma: For every \( \mathcal{M} \)-infinite \( T \subset 2^{<M} \) tree, there is an \( \mathcal{M} \)-infinite path \( X \) that is recursive in \( T'' \). This path is not bounded in \( M \) since it will otherwise be \( \mathcal{M} \)-finite. But then \( X \in X \) and so \( ACA₀ \) implies \( WKL₀ \).

On the other hand, the Low Basis Theorem (Jockusch and Soare [1972]) states that every infinite, recursively bounded tree \( T \) has an infinite path in \( T \) not computing \( \emptyset' \). Hence there is a model \( \mathcal{M} \) of \( WKL₀ + \neg ACA₀ \) where \( \Xi_M \) is a class of low sets and \( \emptyset' \notin \Xi_M \).

**ACA₀** is equivalent over \( RCA₀ \) to the Bolzano-Weierstrass Theorem as well as König’s Lemma.
ACA₀

Proof. If \( M \models ACA₀ \) then \( M \models IΣ^0_n \) for all \( n \in \omega \). This is true since every bounded definable set is \( M \)-finite. Now there is a definable solution to weak König’s Lemma: For every \( M \)-infinite \( T \subseteq 2^{<M} \) tree, there is an \( M \)-infinite path \( X \) that is recursive in \( T'' \). This path is not bounded in \( M \) since it will otherwise be \( M \)-finite. But then \( X \in X \) and so \( ACA₀ \) implies \( WKL₀ \).

On the other hand, the Low Basis Theorem (Jockusch and Soare [1972]) states that every infinite, recursively bounded tree \( T \) has an infinite path in \( T \) not computing \( ∅' \). Hence there is a model \( M \) of \( WKL₀ + ¬ACA₀ \) where \( X_M \) is a class of low sets and \( ∅' \notin X_M \).

\( ACA₀ \) is equivalent over \( RCA₀ \) to the Bolzano-Weierstrass Theorem as well as König’s Lemma.
\textbf{ACA}_0

- \textit{Proof.} If $\mathcal{M} \models \text{ACA}_0$ then $\mathcal{M} \models I\Sigma^0_n$ for all $n \in \omega$. This is true since every bounded definable set is $\mathcal{M}$-finite. Now there is a definable solution to weak König’s Lemma: For every $\mathcal{M}$-infinite $T \subset 2^{<\mathcal{M}}$ tree, there is an $\mathcal{M}$-infinite path $X$ that is recursive in $T''$. This path is not bounded in $\mathcal{M}$ since it will otherwise be $\mathcal{M}$-finite. But then $X \in X$ and so $\text{ACA}_0$ implies $\text{WKL}_0$.

On the other hand, the Low Basis Theorem (Jockusch and Soare [1972]) states that every infinite, recursively bounded tree $T$ has an infinite path in $T$ not computing $\emptyset'$. Hence there is a model $\mathcal{M}$ of $\text{WKL}_0 + \neg \text{ACA}_0$ where $\overline{X}_M$ is a class of low sets and $\emptyset' \notin \overline{X}_M$.

- $\text{ACA}_0$ is equivalent over $\text{RCA}_0$ to the Bolzano-Weierstrass Theorem as well as König’s Lemma.
Stronger Subsystems

- **ATR₀**: Arithmetic Transfinite Recursion. The Turing jump exists along every countable well-ordering.

- Over RCA₀, ATR₀ is equivalent to the mathematical statement that “every uncountable set contains a perfect subset.”

- **Π¹₁-CA₀**: Π¹₁ Comprehension Axiom. 
  \[ \exists X[\forall x(x \in X \leftrightarrow \varphi(x))], \text{ where } \varphi \text{ is } \Pi^1_1(M). \]

- Over RCA₀, Π¹₁-CA₀ is equivalent to the Cantor-Bendixson Theorem: Every uncountable set is the union of a countable set and a perfect set.
Stronger Subsystems

- **ATR\(_0\):** Arithmetic Transfinite Recursion. The Turing jump exists along every countable well-ordering.

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- **\(\Pi^1_1\)-CA\(_0\):** \(\Pi^1_1\) Comprehension Axiom.
  \[ \exists X[\forall x(x \in X \leftrightarrow \varphi(x))], \text{ where } \varphi \text{ is } \Pi^1_1(M). \]

- Over RCA\(_0\), \(\Pi^1_1\)-CA\(_0\) is equivalent to the Cantor-Bendixson Theorem: Every uncountable set is the union of a countable set and a perfect set.
ATR<sub>0</sub>: Arithmetic Transfinite Recursion. The Turing jump exists along every countable well-ordering.

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Π<sub>1</sub><sup>1</sup>-CA<sub>0</sub>: Π<sub>1</sub><sup>1</sup> Comprehension Axiom.

\[ \exists X[\forall x (x \in X \leftrightarrow \varphi(x))], \text{ where } \varphi \text{ is } \Pi_1^1(M). \]

Over RCA<sub>0</sub>, Π<sub>1</sub><sup>1</sup>-CA<sub>0</sub> is equivalent to the Cantor-Bendixson Theorem: Every uncountable set is the union of a countable set and a perfect set.
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\[ \exists X [\forall x (x \in X \iff \varphi(x))] \], where \( \varphi \) is Π<sub>1</sub>\( (\mathcal{M}) \).

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A Hierarchy of Subsystems

- RCA₀
- WKL₀
- ACA₀
- ATR₀
- \( \Pi^1_1\)-CA₀
Ramsey’s Theorem

Theorem (2.3)

(F. P. Ramsey [1931]) $RT^n_k$: Let $n \geq 2$. If $f : \mathbb{N}^n \rightarrow k$, then there is an infinite set $H_f$ such that $f$ is a constant on $[H_f]^n$.

- $H_f$ is homogeneous for $f$.

- $RT^n_k \rightarrow RT^n_{k+1}$ and $RT^{n+1}_k \rightarrow RT^n_k$.

- What is the complexity of $H_f$? Is there a “basis theorem” for $RT^n_k$?

- What is the strength of $RT^n_k$, in Peano arithmetic and in subsystems of second order arithmetic?
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- What is the strength of $RT^n_k$, in Peano arithmetic and in subsystems of second order arithmetic?
■ Specker [1971]: There is a recursive $f : [\mathbb{N}]^2 \rightarrow 2$ with no recursive $H_f$.

Theorem (2.4)

(Jockusch [1972]).

(i) There is a recursive $f : [\mathbb{N}]^2 \rightarrow 2$ with no $\Delta^0_2 H_f$.

(ii) Every recursive $f : [\mathbb{N}]^n \rightarrow k$ has an $H_f$ that is $\Pi^0_n$.

(iii) For $n \geq 2$, there is a recursive $f : [\mathbb{N}]^n \rightarrow 2$ such that every $H_f \geq_T \emptyset^{(n-2)}$. 
Specker [1971]: There is a recursive $f : [\mathbb{N}]^2 \to 2$ with no recursive $H_f$.

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(iii) *For* \( n \geq 2 \), *there is a recursive* \( f : [\mathbb{N}]^n \rightarrow 2 \) *such that every* \( H_f \geq_T \emptyset^{(n-2)} \).
Definable Solutions of Ramsey’s Theorem

**Proof of (iii).** Consider \( n = 3 \). Let \( K \) be a complete r.e. set and \( f(s, t, u) = 1 \) if \( K_t \upharpoonright s = K_u \upharpoonright s \), and 0 otherwise. Then \( f([H_f]^3) = 1 \), since for any \( s \), \( f(s, t, u) = 1 \) for all sufficiently large \( t \) and \( u \). But then for all \( s, t, u \in H_f \), \( K_t \upharpoonright s = K_u \upharpoonright s = K \upharpoonright s \). Thus \( H_f \geq_T \emptyset' \).

**Theorem (2.5)**

(Simpson [1999]). Let \( n \geq 3 \). \( \text{RCA}_0 \vdash \text{RT}^n_k \leftrightarrow \text{ACA}_0 \).

**Proof.** By Theorem 2.4 (iii), every model \( \mathcal{M} = \langle M, \bar{X}, +, \times, 0, 1 \rangle \) of \( \text{RCA}_0 + \text{RT}^n_k \) contains an \( X \in \bar{X} \) that computes \( \emptyset' \). Iterating this, we see that \( \mathcal{M} \models \text{ACA}_0 \). On the other hand, Theorem 2.4 (ii) implies that \( \text{ACA}_0 \rightarrow \text{RT}^n_k \) over \( \text{RCA}_0 \).
Definable Solutions of Ramsey’s Theorem

Proof of (iii). Consider \( n = 3 \). Let \( K \) be a complete r.e. set and \( f(s, t, u) = 1 \) if \( K_t \upharpoonright s = K_u \upharpoonright s \), and 0 otherwise. Then \( f([H_f]^3) = 1 \), since for any \( s \), \( f(s, t, u) = 1 \) for all sufficiently large \( t \) and \( u \). But then for all \( s, t, u \in H_f \), \( K_t \upharpoonright s = K_u \upharpoonright s = K \upharpoonright s \). Thuis \( H_f \supseteq T \emptyset' \).

Theorem (2.5)

(Simpson [1999]). Let \( n \geq 3 \). \( \RCA_0 \vdash \RT^n_k \iff \ACA_0 \).

Proof. By Theorem 2.4 (iii), every model \( \mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1 \rangle \) of \( \RCA_0 + \RT^n_k \) contains an \( X \in \mathbb{X} \) that computes \( \emptyset' \). Iterating this, we see that \( \mathcal{M} \models \ACA_0 \).

On the other hand, Theorem 2.4 (ii) implies that \( \ACA_0 \rightarrow \RT^n_k \) over \( \RCA_0 \).
Definable Solutions of Ramsey’s Theorem

Proof of (iii). Consider $n = 3$. Let $K$ be a complete r.e. set and $f(s, t, u) = 1$ if $K_t \upharpoonright s = K_u \upharpoonright s$, and 0 otherwise. Then $f([H_f]^3) = 1$, since for any $s$, $f(s, t, u) = 1$ for all sufficiently large $t$ and $u$. But then for all $s, t, u \in H_f$, $K_t \upharpoonright s = K_u \upharpoonright s = K \upharpoonright s$. Thus $H_f \geq_T \emptyset'$.

Theorem (2.5) (Simpson [1999]). Let $n \geq 3$. $\text{RCA}_0 \vdash \text{RT}_k^n \leftrightarrow \text{ACA}_0$.

Proof. By Theorem 2.4 (iii), every model $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1 \rangle$ of $\text{RCA}_0 + \text{RT}_k^n$ contains an $X \in \mathbb{X}$ that computes $\emptyset'$. Iterating this, we see that $\mathcal{M} \models \text{ACA}_0$. On the other hand, Theorem 2.4 (ii) implies that $\text{ACA}_0 \rightarrow \text{RT}_k^n$ over $\text{RCA}_0$. 
Let $RT$ be $\forall n \forall k \, RT^n_k$.

Theorem (2.6) Simpson [1999]). Over $RCA_0$,

(i) $ACA_0$ does not prove $RT$.

(ii) $ATR_0$ implies $RT$ but not conversely.

Proof. Theorem 2.4 (iii) implies that any model $\mathcal{M}$ of $RT$ must be closed under $n - 2$th jump for any $n \in M$. However, there exists a nonstandard model of $ACA_0$ not closed under $n$th jump for any $n \notin \omega$.

For (ii), any model $\mathcal{M}$ of $ATR_0$ is closed under $n$th jump for any $n \in M$, hence $\mathcal{M} \models RT$. Conversely, $\langle \omega, \text{ARTH}, +, \times, 0 \rangle$ is a model of $RT$ but not $ATR_0$, where $\text{ARTH} =$ the class of arithmetical sets.
Let RT be $\forall n \forall k \text{ RT}_k^n$.

**Theorem (2.6)**

Simpson [1999]). *Over RCA$_0$,*

(i) ACA$_0$ *does not prove* RT.

(ii) ATR$_0$ *implies* RT *but not conversely.*

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The Strength of Ramsey’s Theorem

Let RT be $\forall n \forall k \ RT^n_k$.

Theorem (2.6)

Simpson [1999]). Over RCA$_0$,

(i) ACA$_0$ does not prove RT.

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A Hierarchy of Subsystems, II

\[ \text{RCA}_0 \bullet \]

\[ \text{RT}_2? \bullet \]

\[ \text{ACA}_0 \bullet \]

\[ \text{RT}_n, n \geq 3 \bullet \]

\[ \text{RT}_0 \bullet \]

\[ \Pi_1^1\text{-CA} \bullet \]

\[ \text{ATR}_0 \bullet \]

\[ \text{WKL}_0 \bullet \]
Three Questions on the Strength of RT\(^2\)_2

- Does RT\(^2\)_2 imply RT\(^3\)_2?

**Definition**

\[ f : [M]^2 \rightarrow 2 \text{ is stable if for all } s, \lim_t f(s, t) \text{ exists.} \]

- SRT\(^2\)_2: If \( f : [M]^2 \rightarrow 2 \) is stable, then there is an \( H_f \).

- Does SRT\(^2\)_2 imply RT\(^2\)_2?

- Does RT\(^2\)_2 imply \( I\Sigma_2 \)? (Hirst [1987]: RT\(^2\)_2 implies \( B\Sigma_2 \).)
Three Questions on the Strength of $\text{RT}_2^2$

- Does $\text{RT}_2^2$ imply $\text{RT}_2^3$?

**Definition**

$f : [M]^2 \to 2$ is stable if for all $s$, $\lim_t f(s, t)$ exists.

- SRT$_2^2$: If $f : [M]^2 \to 2$ is stable, then there is an $H_f$.

- Does SRT$_2^2$ imply $\text{RT}_2^2$?

- Does $\text{RT}_2^2$ imply $\text{IS}_2$? (Hirst [1987]: $\text{RT}_2^2$ implies $\text{BIS}_2$.)
Three Questions on the Strength of $RT_2^2$

- Does $RT_2^2$ imply $RT_2^3$?

**Definition**

$f : [M]^2 \rightarrow 2$ is *stable* if for all $s$, $\lim_t f(s, t)$ exists.

- $SRT_2^2$: If $f : [M]^2 \rightarrow 2$ is stable, then there is an $H_f$.

- Does $SRT_2^2$ imply $RT_2^2$?

- Does $RT_2^2$ imply $\Sigma_2$? (Hirst [1987]: $RT_2^2$ implies $B\Sigma_2$.)
Three Questions on the Strength of $RT^2_2$

- Does $RT^2_2$ imply $RT^3_2$?

**Definition**

$f : [M]^2 \rightarrow 2$ is *stable* if for all $s$, $\lim_t f(s, t)$ exists.

- $SRT^2_2$: If $f : [M]^2 \rightarrow 2$ is stable, then there is an $H_f$.

- Does $SRT^2_2$ imply $RT^2_2$?

- Does $RT^2_2$ imply $\Sigma_2$? (Hirst [1987]: $RT^2_2$ implies $B\Sigma_2$.)
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Session II:

L’estremità