Recursion Theory of Ramsey's Theorem

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Session II

Chitat CHONG

Department of Mathematics
National University of Singapore

chongct@math.nus.edu.sg

Subsystems of Second Order Arithmetic

Language of second order arithmetic:

- First order variables $x, y, z, \ldots$
- Set variables $X, Y, Z, \ldots$
- Constants: $0, 1$
- Relation symbol: $\in$
- Function symbols: $+, \times$

A structure $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ is a model of a subsystem of second order arithmetic T if all axioms in T are true in $\mathcal{M}$, where in particular $\mathbb{X} \subset 2^M$ and all set variables are interpreted as members of $\mathbb{X}$.

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Subsystems of Second Order Arithmetic

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An ω-model $\mathcal{M}$ is a structure such that $M = \omega$.

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If T is a subsystem of second order arithmetic, and $\mathcal{M}, \mathcal{N} \models T$, then $\mathcal{M} \subset \mathcal{N}$ if and only if $M \subset N$, and $\mathbb{X}_{\mathcal{M}} \subset \mathbb{X}_{\mathcal{N}}$, where $\mathcal{M} = \langle M, \mathcal{X}_{\mathcal{M}}, +, \times, 0, 1 \rangle$ and $\mathcal{N} = \langle N, \mathbb{X}_{\mathcal{N}}, +, \times, 0, 1 \rangle$.

Definition

If $\mathcal{M}, \mathcal{N} \models T$, then $\mathcal{M}$ is an *M-submodel* of $\mathcal{N}$ if $M = N$. Hence only second order elements are added to $\mathcal{M}$ to obtain $\mathcal{N}$.

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RCA_0

$\mathcal{M}$ is a model of the system RCA_0 (*Recursive Comprehension Axiom*) if it satisfies the following:

- P^-, the Peano axioms without mathematical induction
- Σ^0_1 induction of the form

  $$[(\varphi(0) \,\&\, \forall x(\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x \varphi(x)],$$

  where φ is $\Sigma^0_1(\mathcal{M})$, with number and set constants.

- (Recursive comprehension)

  $$\exists X \forall x(x \in X \leftrightarrow \varphi(x)),$$

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Models of RCA_0

- If $\mathcal{M} \models \mathrm{RCA}_0$, then $\mathbb{X}$ is closed under Turing reducibility and joint, since these operations are $\Delta^0_1(\mathcal{M})$ definable. Thus in any model of RCA_0, the second order objects form *an ideal*.
- $\mathcal{M} = \langle \omega, 2^\omega, +, \times, 0, 1 \rangle$ is a model of RCA_0 (in fact, of any comprehension scheme).
- If $\mathcal{M} = \langle M, +, \times, 0, 1 \rangle$ is a first order model of $I\Sigma^0_1$, then it can be expanded into a model $\mathcal{M}^* = \langle M, \mathbb{X}, +, \times, 0, 1 \rangle$ of RCA_0 by letting $\mathbb{X} = \{X | X \text{is recursive}\}$.

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Weak König's Lemma WKL_0

Definition

Let $\mathcal{M} \models \mathsf{RCA}_0$. $\mathcal{M} \models \mathsf{WKL}_0$ if every $\mathcal{M}$-infinite binary tree T coded in $\mathcal{M}$ has an $\mathcal{M}$-infinite path X. Thus if $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ and $T \in \mathbb{X}$, then $X \in \mathbb{X}$.

Theorem (2.1)

$\mathrm{WKL}_0 \rightarrow \mathrm{RCA}_0$, *but not conversely.*

Proof. Suppose $\mathcal{M} \models \mathsf{WKL}_0$ and let φ be $\Delta^0_1(\mathcal{M})$. Let

$$T = \{\sigma | \exists a \in M[\sigma \in 2^a \;\&\; \forall x(\sigma(x) = 1 \leftrightarrow \varphi(x))\}.$$

Assume $\{x|\varphi(x)\}$ $\mathcal{M}$-infinite. Then T is an $\mathcal{M}$-infinite tree with unique path $X \in \mathbb{X}$ by WKL_0. This $X =$ characteristic function of a recursive set (relative to the parameters defining φ).

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- *Proof.* (Continued)
  On the other hand, there is an infinite $T \subset 2^{<\omega}$ recursive tree with no recursive infinite path: Let A and B be disjoint, r.e. and recursively inseparable enumerated by f_A and f_B. σ of length k is on T if and only if for all $m, n \leq \mathrm{lth}(\sigma)$, $f_A(m) = n \to \sigma(n) = 1$ and $g_A(m) = n \to \sigma(n) = 0$. Then every infinite path on T separates A from B, hence is not recursive. Thus if $\mathcal{M} = \langle \omega, \mathrm{REC}, +, \times, 0 \rangle$, where REC denotes the class of recursive sets in ω, then $\mathcal{M} \models \mathrm{RCA}_0 + \neg \mathrm{WKL}_0$.
- RCA_0 is equivalent to the Intermediate Value Theorem. WKL_0 is equivalent over RCA_0 to the Heine-Borel Theorem, the Brouwer Fixed Point Theorem, the local existence theorem for solutions of ordinary differential equations etc.

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Arithmetic Comprehension Axiom ACA$_o$

Definition

$\mathcal{M} \models$ ACA$_0$ if and only if it satisfies the following sentence:

$$\exists X[\forall x(x \in X \leftrightarrow \varphi(x))],$$

where φ is $\Sigma^0_n(\mathcal{M})$ for $n \in \omega$.

Note. Restricting n to 1 in the definition is equivalent to letting n range over ω, since if all $\Sigma^0_n(\mathcal{M})$ sets are in $\mathbb{X}$, then so are all $\Sigma^0_{n+1}(\mathcal{M})$ sets by iteration. Thus $\mathcal{M} \models$ ACA$_0$ if and only if $\mathbb{X}$ is closed under the Turing jump operation.

Theorem (2.2)

$\mathrm{ACA}_0 \rightarrow \mathrm{WKL}_0$ *but not conversely.*

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- *Proof.* If $\mathcal{M} \models \mathsf{ACA}_0$ then $\mathcal{M} \models I\Sigma^0_n$ for all $n \in \omega$. This is true since every bounded definable set is $\mathcal{M}$-finite. Now there is a definable solution to weak König's Lemma: For every $\mathcal{M}$-infinite $T \subset 2^{<M}$ tree, there is an $\mathcal{M}$-infinite path X that is recursive in T''. This path is not bounded in M since it will otherwise be $\mathcal{M}$-finite. But then $X \in \mathbb{X}$ and so ACA_0 implies WKL_0.

  On the other hand, the Low Basis Theorem (Jockusch and Soare [1972]) states that every infinite, recursively bounded tree T has an infinite path in T not computing $\emptyset'$. Hence there is a model $\mathcal{M}$ of $\mathsf{WKL}_0 + \neg\mathsf{ACA}_0$ where $\mathbb{X}_M$ is a class of low sets and $\emptyset' \notin \mathbb{X}_M$.

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Stronger Subsystems

- ATR_0: Arithmetic Transfinite Recursion. The Turing jump exists along every countable well-ordering.

- Over RCA_0, ATR_0 is equivalent to the mathematical statement that "every uncountable set contains a perfect subset."

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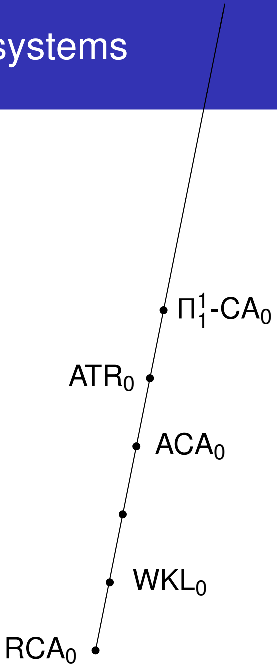
A Hierarchy of Subsystems



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ATR$_0$

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Ramsey's Theorem

Theorem (2.3)

(F. P. Ramsey [1931]) RT^n_k: *Let* $n \geq 2$. *If* $f : [\mathbb{N}]^n \to k$, *then there is an infinite set* H_f *such that* f *is a constant on* $[H_f]^n$.

- H_f is *homogeneous* for f.
- $\mathrm{RT}^n_k \to \mathrm{RT}^n_{k+1}$ and $\mathrm{RT}^{n+1}_k \to \mathrm{RT}^n_k$.
- What is the complexity of H_f? Is there a "basis theorem" for RT^n_k?
- What is the strength of RT^n_k, in Peano arithmetic and in subsystems of second order arithmetic?

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Definable Solutions of Ramsey's Theorem

- Specker [1971]: There is a recursive $f : [\mathbb{N}]^2 \to 2$ with no recursive H_f.

Theorem (2.4)

(Jockusch [1972]).

(i) *There is a recursive* $f : [\mathbb{N}]^2 \to 2$ *with no* Δ^0_2 H_f.

(ii) *Every recursive* $f : [\mathbb{N}]^n \to k$ *has an* H_f *that is* Π^0_n.

(iii) *For* $n \geq 2$, *there is a recursive* $f : [\mathbb{N}]^n \to 2$ *such that every* $H_f \geq_T \emptyset^{(n-2)}$.

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Definable Solutions of Ramsey's Theorem

Proof of (iii). Consider $n = 3$. Let K be a complete r.e. set and $f(s, t, u) = 1$ if $K_t \restriction s = K_u \restriction s$, and 0 otherwise. Then $f([H_f]^3) = 1$, since for any s, $f(s, t, u) = 1$ for all sufficiently large t and u. But then for all $s, t, u \in H_f$, $K_t \restriction s = K_u \restriction s = K \restriction s$. Thuis $H_f \geq_T \emptyset'$.

Theorem (2.5)

(Simpson [1999]). *Let $n \geq 3$. $\mathrm{RCA}_0 \vdash RT^n_k \leftrightarrow \mathrm{ACA}_0$.*

Proof. By Theorem 2.4 (iii), every model $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ of $\mathrm{RCA}_0 + \mathrm{RT}^n_k$ contains an $X \in \mathbb{X}$ that computes $\emptyset'$. Iterating this, we see that $\mathcal{M} \models \mathrm{ACA}_0$.
On the other hand, Theorem 2.4 (ii) implies that $\mathrm{ACA}_0 \rightarrow \mathrm{RT}^n_k$ over RCA_0.

Definable Solutions of Ramsey's Theorem

Proof of (iii). Consider $n = 3$. Let K be a complete r.e. set and $f(s,t,u) = 1$ if $K_t \upharpoonright s = K_u \upharpoonright s$, and 0 otherwise. Then $f([H_f]^3) = 1$, since for any s, $f(s,t,u) = 1$ for all sufficiently large t and u. But then for all $s, t, u \in H_f$, $K_t \upharpoonright s = K_u \upharpoonright s = K \upharpoonright s$. Thuis $H_f \geq_T \emptyset'$.

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The Strength of Ramsey's Theorem

Let RT be $\forall n \forall k \ \mathrm{RT}^n_k$.

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(i) ACA_0 *does not prove* RT*.*

(ii) ATR_0 *implies* RT *but not conversely.*

Proof. Theorem 2.4 (iii) implies that any model $\mathcal{M}$ of RT must be closed under $n-2$th jump for any $n \in M$. However, there exists a nonstandrad model of ACA_0 not closed under nth jump for any $n \notin \omega$.
For (ii), any model $\mathcal{M}$ of ATR_0 is closed under nth jump for any $n \in M$, hence $\mathcal{M} \models$ RT. Conversely, $\langle \omega, \mathrm{ARTH}, +, \times, 0 \rangle$ is a model of RT but not ATR_0, where ARTH = the class of arithmetical sets.

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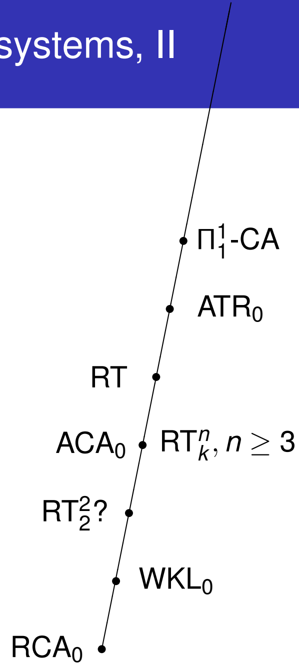
A Hierarchy of Subsystems, II


Π^1_1-CA

ATR_0

RT

ACA_0 $\mathsf{RT}^n_k, n \geq 3$

RT^2_2?

WKL_0

RCA_0

Three Questions on the Strength of RT^2_2

- Does RT^2_2 imply RT^3_2?

Definition

$f : [M]^2 \to 2$ is *stable* if for all s, $\lim_t f(s,t)$ exists.

- SRT^2_2: If $f : [M]^2 \to 2$ is stable, then there is an H_f.
- Does SRT^2_2 imply RT^2_2?
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Session II:

L’estremità