The Seetapun-Slaman Theorem

**Theorem (3.1)**

(Seetapun and Slaman [1995]) Given \( \{C_i\}_{i<\omega} \) such that \( C_i \geq_T \emptyset \), and \( f : [\mathbb{N}]^2 \to 2 \) where \( f \) is recursive, there is an \( H_f \) such that \( C_i \not\leq_T H_f \) for all \( i \).

**Corollary**

There is an \( \omega \)-model of \( \text{RCA}_0 + \text{WKL}_0 + \text{RT}_2^2 \) which is not a model of \( \text{RT}_2^n \) for \( n \geq 3 \). Hence \( \text{RCA}_0 + \text{RT}_2^2 + \text{WKL}_0 \not\rightarrow \text{RT}_2^n \) for \( n \geq 3 \).
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Let $Z \subset \omega$ and $f : [\mathbb{N}]^2 \rightarrow 2$ be $Z$-recursive.

**Definition**

If $f(x, y) = 0$, then $\{x, y\}$ is colored red. Otherwise it is colored blue.

**Definition**

Given two (finite) binary strings $\sigma$ and $\tau$, a set $X$ is suitable for $\langle \sigma, \tau \rangle$ if $\{x, z\}$ is red for all $x \in \sigma$ and $z \in X$, and $\{y, z\}$ is blue for all $y \in \tau$ and $z \in X$. 
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Let $Z \subseteq \omega$ and $f : [\mathbb{N}]^2 \to 2$ be $Z$-recursive.

**Definition**

If $f(x, y) = 0$, then $\{x, y\}$ is colored *red*. Otherwise it is colored *blue*.

**Definition**

Given two (finite) binary strings $\sigma$ and $\tau$, a set $X$ is *suitable* for $\langle \sigma, \tau \rangle$ if $\{x, z\}$ is red for all $x \in \sigma$ and $z \in X$, and $\{y, z\}$ is blue for all $y \in \tau$ and $z \in X$. 
Mathias Forcing Avoiding $\emptyset'$

**Definition**

A forcing condition $p = \langle \sigma, \tau, X \rangle$ is a triple of the form

- $\sigma, \tau$ are finite strings;
- $\sigma$ is red: $\{x, z\}$ is red for any $x, z \in \sigma$;
- $\tau$ is blue: $\{y, z\}$ is blue for any $y, z \in \tau$;
- $X$ is infinite, $X \oplus Z \not\geq_T \emptyset'$, and is suitable for $\sigma, \tau$;
- $\text{Max } \sigma \cup \tau < \text{Min } X$.

**Definition**

$q = \langle \sigma_q, \tau_q, X_q \rangle$ is **stronger** than $p = \langle \sigma_p, \tau_p, X_p \rangle$ ($p \geq q$) if

- $\sigma_p \leq \sigma_q$ and $\tau_p \leq \tau_q$;
- $X_p \supset X_q$;
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**Definition**

- If $\sigma \cup \sigma_0$ and $\sigma \cup \sigma_1$ are red, and $\sigma \leq \sigma_0$ and $\sigma \leq \sigma_1$ are incompatible, then $\sigma_0$ and $\sigma_1$ *Φ-split* for red over $\sigma$ if there is an $x \leq s = \text{Max} \ \sigma_0 \cup \sigma_1$ such that $\Phi^{\sigma_0 \oplus Z}_s(x) \downarrow \neq \Phi^{\sigma_1 \oplus Z}_s(x) \downarrow$;
- Define Φ-split for blue over $\tau$ similarly.

Let $G_R$ and $G_B$ be generic: For all $\varphi$ in the language of second order arithmetic with set constants $G_R$ and $G_B$ and constant $Z$, there is a condition $p = \langle \sigma, \tau, X \rangle$ such that $\sigma < G_R$ and $\tau < G_B$ ("<" means initial segment) and $p \models \varphi$ or $\neg \varphi$. 
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Mathias Forcing Avoiding $\emptyset'$

Lemma (3.1)

Given $\langle \sigma, \tau, X \rangle$, if no incompatible $\sigma_0, \sigma_1 \geq \sigma$ with $\sigma_0 \setminus \sigma, \sigma_1 \setminus \sigma \subset X$ $\Phi$-split for red over $\sigma$ and if $\Phi^{GR \oplus Z}$ is total, then it is recursive in $X \oplus Z$. Same conclusion for $\tau$.

A finite set $D$ is red if $\{x, z\}$ is red for all $x, z \in D$.

Definition

Let $p = \langle \sigma, \tau, X \rangle$ be a condition. $X$ is blue-hyperimmune if for any infinite $X \oplus Z$-recursive array of red finite sets $\langle D_i | i \in \omega \rangle$, there is a $D_i$ and an infinite $Y \subset X$ such that $Y \leq_T X$, $\text{Max } D_i < \text{Min } Y$ and $\{x, z\}$ is red for all $x \in D_i$ and $z \in Y$.

Note. In this case $\langle \sigma, \tau, X \rangle \geq \langle \sigma_{D_i}, \tau, Y \rangle$, where $\sigma_{D_i}(x) = \sigma(x)$ for $x \leq \text{lth}(\sigma)$, and equal to $D_i(x)$ otherwise.
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Definition

Let \( p = \langle \sigma, \tau, X \rangle \) be a condition. \( X \) is \textit{blue-hyperimmune} if for any infinite \( X \oplus Z \)-recursive array of red finite sets \( \langle D_i | i \in \omega \rangle \), there is a \( D_i \) and an infinite \( Y \subset X \) such that \( Y \leq_T X \), Max \( D_i < \text{Min } Y \) and \( \{x, z\} \) is red for all \( x \in D_i \) and \( z \in Y \).

Note. In this case \( \langle \sigma, \tau, X \rangle \geq \langle \sigma_{D_i}, \tau, Y \rangle \), where \( \sigma_{D_i}(x) = \sigma(x) \) for \( x \leq \text{lth}(\sigma) \), and equal to \( D_i(x) \) otherwise.
Mathias Forcing Avoiding $\emptyset'$

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Given $\langle \sigma, \tau, X \rangle$, if no incompatible $\sigma_0, \sigma_1 \geq \sigma$ with $\sigma_0 \setminus \sigma, \sigma_1 \setminus \sigma \subset X$ $\Phi$-split for red over $\sigma$ and if $\Phi^{G_{R \oplus Z}}$ is total, then it is recursive in $X \oplus Z$. Same conclusion for $\tau$.

A finite set $D$ is red if $\{x, z\}$ is red for all $x, z \in D$.

Definition

Let $p = \langle \sigma, \tau, X \rangle$ be a condition. $X$ is blue-hyperimmune if for any infinite $X \oplus Z$-recursive array of red finite sets $\langle D_i | i \in \omega \rangle$, there is a $D_i$ and an infinite $Y \subset X$ such that $Y \leq_T X$, $\text{Max } D_i < \text{Min } Y$ and $\{x, z\}$ is red for all $x \in D_i$ and $z \in Y$.

Note. In this case $\langle \sigma, \tau, X \rangle \geq \langle \sigma_{D_i}, \tau, Y \rangle$, where $\sigma_{D_i}(x) = \sigma(x)$ for $x \leq \text{lth}(\sigma)$, and equal to $D_i(x)$ otherwise.
Avoiding $\emptyset'$

Lemma (3.2)

*If the set of conditions $\langle \sigma, \tau, X \rangle$ with $X$ blue-hyperimmune is dense, then any $G_R$ is an $H_f$ satisfying $G_R \oplus Z \not\geq_T \emptyset'$.***

*Proof.* Suppose $\Phi^{G_R \oplus Z} = \emptyset'$. Then there is a $p = \langle \sigma, \tau, X \rangle$ such that $p \models \Phi^{G_R \oplus Z} = \emptyset'$, $\sigma < G_R$, and $X$ is blue-hyperimmune. Enumerate $X \oplus Z$-recursively pairwise disjoint red finite subsets $\langle D_{i,0}, D_{i,1} \rangle$ of $X$ such that $\langle \sigma_{D_{i,0}}, \sigma_{D_{i,1}} \rangle \Phi$-split for red over $\sigma$.

- **Case 1.** There are only finitely many such $i$'s. Then there is an $s$ such that if $Y = X \cap \{x | x > s\}$, no pair of strings that $\Phi$-split for red over $\sigma$ exists by choosing them from finite subsets of $Y$. Now $q = \langle \sigma, \tau, Y \rangle \leq p$ and so $q \models \Phi^{G_R \oplus Z} = \emptyset'$. However, Lemma 3.1 says $q \models \Phi^{G_R \oplus Z} \leq_T Y \oplus Z$. Since $Y \oplus Z \not\geq_T \emptyset'$, we have a contradiction.
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Case 2. There are infinitely many $i$'s. Then since $X$ is blue-hyperimmune, there exist $i$ and infinite $Y \subset X$ with $Y \leq_T X$ such that $\{x, z\}$ is red for all $x \in D_{i,0} \cup D_{i,1}$ and $z \in Y$, where $\text{Max } D_{i,0} \cup D_{i,1} < \text{Min } Y$. Then $\langle \sigma_{D_{i,0}}, \tau, Y \rangle$ and $\langle \sigma_{D_{i,1}}, \tau, Y \rangle \leq p$ and $\langle \sigma_{D_{i,0}}, \sigma_{D_{i,1}} \rangle \Phi$-split for red over $\sigma$, contradicting the assumption that $p \models \Phi^{G_R \oplus Z} = \emptyset'$.

**Definition**

Let $X$ be infinite. A *Seetapun tree* for $X$ is an infinite $X$-recursively bounded $X$-recursive tree $T$ such that for each $n$, there is a string $\nu \in T$ of length $n$ satisfying: for all $i \leq n$ and all infinite $Y \subset X$ with $Y \leq_T X$, $\{z \in Y | \{\nu(i), z\} \text{ is blue}\}$ is infinite.
Lemma (3.3)

Assume that \( \{ \langle \sigma, \tau, X \rangle \mid X \text{ is blue-hyperimmune} \} \) is not dense. Then there is a \( G_B \) that is an \( H_f \) and \( G_B \oplus Z \not\leq_T \emptyset' \).

Proof Fix \( p = \langle \sigma_p, \tau_p, X_p \rangle \) so that if \( q = \langle \sigma_q, \tau_q, X_q \rangle \leq p \), then \( X_q \) is not blue-hyperimmune. Let \( G_B \) be such that \( \tau_q < G_B \). Suppose \( \Phi_{G_B \oplus Z} = \emptyset' \) and let \( q \vDash \Phi_{G_B \oplus Z} = \emptyset' \) with \( q \leq p \).

Let \( \langle D_i \rangle \) be an infinite \( X_q \)-recursive pairwise disjoint array of finite subsets of \( X_q \) that witnesses the non-blue-hyperimmunity of \( X_q \). Define \( T \) to be the collection of all strings \( \nu \) such that if \( n = \text{ith}(\nu) \), then \( \nu(i) \in D_i \) for \( i \leq n \). We may assume that \( \text{Max } D_i < \text{Min } D_{i+1} \).

\( T \) is a Seetapun tree.
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Assume that \( \{ \langle \sigma, \tau, X \rangle \mid X \text{ is blue-hyperimmune} \} \) is not dense. Then there is a \( G_B \) that is an \( H_f \) and \( G_B \oplus Z \nleq_T \emptyset' \).

**Proof** Fix \( p = \langle \sigma_p, \tau_p, X_p \rangle \) so that if \( q = \langle \sigma_q, \tau_q, X_q \rangle \leq p \), then \( X_q \) is not blue-hyperimmune. Let \( G_B \) be such that \( \tau_q < G_B \). Suppose \( \Phi^{G_B \oplus Z} = \emptyset' \) and let \( q \models \Phi^{G_B \oplus Z} = \emptyset' \) with \( q \leq p \).

Let \( \langle D_i \rangle \) be an infinite \( X_q \)-recursive pairwise disjoint array of finite subsets of \( X_q \) that witnesses the non-blue-hyperimmunity of \( X_q \). Define \( T \) to be the collection of all strings \( \nu \) such that if \( n = \text{ith}(\nu) \), then \( \nu(i) \in D_i \) for \( i \leq n \). We may assume that \( \text{Max } D_i < \text{Min } D_{i+1} \).

\( T \) is a Seetapun tree.
Let

\[ U = \{ \nu \in T | \forall E_0, E_1 \subset \nu [\langle \tau_{E_0}, \tau_{E_1} \rangle \text{ do not } \Phi\text{-split for blue over } \tau_q} \}, \]

where \( \tau_{E_i}(x) = \tau(x) \) for \( x \leq \text{lth}(\tau) \), and equal to \( E_i(x) \) otherwise. \([E \subset \nu \text{ means } \forall x(x \in E \leftrightarrow \nu(i) = x) \text{ for some } i \leq \text{lth}(\nu).]\)

- **Case 1.** \( U \) is finite. Fix \( s_0 \) such that for all \( \nu \) of length greater than \( s_0 \), there exist \( E_0, E_1 \subset \nu \) with \( \langle \tau_{E_0}, \tau_{E_1} \rangle \text{ } \Phi\text{-split for blue over } \tau \). Since \( X_q \) is not blue-hyperimmune, there is \( \langle x_0, Y_0 \rangle \) such that
  - \( x_0 \in D_0 \);
  - \( Y_0 \subset X_q \) is infinite and \( Y_0 \leq_T X_0 \);
  - \( \{x_0, z\} \) is blue for all \( z \in Y_0 \).
Seetapun Tree

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- **Case 1.** $$U$$ is finite. Fix $$s_0$$ such that for all $$\nu$$ of length greater than $$s_0$$, there exist $$E_0, E_1 \subset \nu$$ with $$\langle \tau_{E_0}, \tau_{E_1} \rangle$$ $$\Phi$$-split for blue over $$\tau$$. Since $$X_q$$ is not blue-hyperimmune, there is $$\langle x_0, Y_0 \rangle$$ such that
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Avoiding $\emptyset'$

If $\langle x_i, Y_i \rangle$ is defined so that $x_i \in D_i$, $Y_i \subset X_q$ is infinite, $Y_i \leq_T X_q$, and $\{x_i, z\}$ is blue for all $z \in Y_i$, then there exists $\langle x_{i+1}, Y_{i+1} \rangle$ such that

- $x_{i+1} \in D_{i+1}$;
- $Y_{i+1} \subset Y_i$ is infinite and $Y_{i+1} \leq_T Y_i$;
- $\{x_{i+1}, z\}$ is blue for all $z \in Y_{i+1}$.

Let $\nu^*(i) = x_i$ for $i \leq s_0 + 1$. Then there exist $E_0, E_1 \subset \nu^*$ so that $\langle \tau_{E_0}, \tau_{E_1} \rangle \Phi$-split for blue over $\tau_q$. Then $\langle \sigma, \tau_{E_0}, Y_{s_0+1} \rangle$ and $\langle \sigma, \tau_{E_1}, Y_{s_0+1} \rangle$ are two incompatible extensions of $q$ that split $\Phi$, contradiction.
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If $\langle x_i, Y_i \rangle$ is defined so that $x_i \in D_i$, $Y_i \subseteq X_q$ is infinite, $Y_i \leq_T X_q$ and $\{x_i, z\}$ is blue for all $z \in Y_i$, then there exists $\langle x_{i+1}, Y_{i+1} \rangle$ such that

- $x_{i+1} \in D_{i+1}$;
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If $\langle x_i, Y_i \rangle$ is defined so that $x_i \in D_i$, $Y_i \subseteq X_q$ is infinite, $Y_i \leq_T X_q$ and \{x_i, z\} is blue for all $z \in Y_i$, then there exists $\langle x_{i+1}, Y_{i+1} \rangle$ such that

- $x_{i+1} \in D_{i+1}$;
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Avoiding \( \emptyset' \)

If \( \langle x_i, Y_i \rangle \) is defined so that \( x_i \in D_i, Y_i \subset X_q \) is infinite, \( Y_i \leq_T X_q \) and \( \{x_i, z\} \) is blue for all \( z \in Y_i \), then there exists \( \langle x_{i+1}, Y_{i+1} \rangle \) such that

- \( x_{i+1} \in D_{i+1} \);
- \( Y_{i+1} \subset Y_i \) is infinite and \( Y_{i+1} \leq_T Y_i \);
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Let \( \nu^*(i) = x_i \) for \( i \leq s_0 + 1 \). Then there exist \( E_0, E_1 \subset \nu^* \) so that \( \langle \tau_{E_0}, \tau_{E_1} \rangle \) \( \Phi \)-split for blue over \( \tau_q \). Then \( \langle \sigma, \tau_{E_0}, Y_{s_0+1} \rangle \) and \( \langle \sigma, \tau_{E_1}, Y_{s_0+1} \rangle \) are two incompatible extensions of \( q \) that split \( \Phi \), contradiction.
Avoiding $\emptyset'$

Case 2. $U$ is infinite. By the Low Basis Theorem relativized to $X_q \oplus Z$, $U$ has a infinite path $W$ that is low in $X_q \oplus Z$ not above $\emptyset'$ (since $X \oplus Z \not\geq_T \emptyset'$). Then $\langle \sigma_q, \tau_q, W \rangle \leq q$ and for any $E_0, E_1 \subset W$, $\langle \tau_{E_0}, \tau_{E_1} \rangle$ does not $\Phi$-split for blue over $\tau_q$. Lemma 3.1 then implies that $\Phi^{G_B \oplus Z}$ is recursive in $X_q \oplus Z$, a contradiction.

The above may be iterated to produce an $\omega$-model of $\text{RCA}_0 + \text{WKL}_0 + \text{RT}_2^2$ that avoids $\emptyset'$, as follows:
Avoiding $\emptyset'$

Case 2. $U$ is infinite. By the Low Basis Theorem relativized to $X_q \oplus Z$, $U$ has a infinite path $W$ that is low in $X_q \oplus Z$ not above $\emptyset'$ (since $X \oplus Z \not\leq_T \emptyset'$). Then $\langle \sigma_q, \tau_q, W \rangle \leq q$ and for any $E_0, E_1 \subset W$, $\langle \tau_{E_0}, \tau_{E_1} \rangle$ does not $\Phi$-split for blue over $\tau_q$. Lemma 3.1 then implies that $\Phi^{G_B \oplus Z}$ is recursive in $X_q \oplus Z$, a contradiction.

The above may be iterated to produce an $\omega$-model of $\text{RCA}_0 + \text{WKL}_0 + \text{RT}^2_2$ that avoids $\emptyset'$, as follows:
Lemma (3.4)

Let $\mathcal{M} = \langle \omega, X, +, \times, 0, 1 \rangle$ be a model of RCA$_0$ such that $X$ is finite. Assume that $X = \bigoplus_{i \in X} X_i$ does not compute $\emptyset'$ and $X \in X$. Furthermore,

(i) $T$ is a $X$-recursively bounded, $X$-recursive tree and
(ii) $f : [\mathbb{N}]^2 \to 2$ is an $X$-recursive 2-coloring of pairs.

Then $T$ has an infinite path $W$ so that $W \oplus X \not\leq_T \emptyset'$, and there is an $H_f$ such that $H_f \oplus W \oplus X$ does not compute $\emptyset'$. 

An $\omega$-model of RCA$_0 + WKL_0 + RT^2_2$
An $\omega$-model of $\text{RCA}_0 + \text{WKL}_0 + \text{RT}_2^2$

The proof proceeds by first applying the relativized Low Basis Theorem to obtain $W$, and let $X^* = X \cup \{W\}$. Then apply Theorem 3.1 with $Z$ as the joint $X$ of all $X_i \in X$.

*Proof of corollary to Theorem 3.1.* Begin with an $\omega$-model $M_0$ where $X = \emptyset$. Recursively define $M_n$ so that an infinite path for a tree $T \in X_n$ that does not compute $\emptyset'$ is first added to $X_{n+1}$, followed by an $H_f$ for a 2-coloring of pairs that satisfies the conclusion of Lemma 3.4. This can be arranged in such a way that $M = \bigcup_n M_n$ is the desired model.
The proof proceeds by first applying the relativized Low Basis Theorem to obtain $W$, and let $X^* = X \cup \{W\}$. Then apply Theorem 3.1 with $Z$ as the joint $X$ of all $X_i \in X$.

**Proof of corollary to Theorem 3.1.** Begin with an $\omega$-model $M_0$ where $X = \emptyset$. Recursively define $M_n$ so that an infinite path for a tree $T \in X_n$ that does not compute $\emptyset'$ is first added to $X_{n+1}$, followed by an $H_f$ for a 2-coloring of pairs that satisfies the conclusion of Lemma 3.4. This can be arranged in such a way that $M = \bigcup_n M_n$ is the desired model.
Session III:

L’estremità