Recursion Theory of Ramsey's Theorem

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Session III

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The Seetapun-Slaman Theorem

Theorem (3.1)

(Seetapun and Slaman [1995]) *Given* $\{C_i\}_{i<\omega}$ *such that* $C_i >_T \emptyset$*, and* $f : [\mathbb{N}]^2 \to 2$ *where* f *is recursive, there is an* H_f *such that* $C_i \not\leq_T H_f$ *for all* i*.*

Corollary

There is an ω*-model of* $\mathrm{RCA}_0 + \mathrm{WKL}_0 + \mathrm{RT}^2_2$ *which is not a model of* RT^n_2 *for* $n \geq 3$*. Hence* $\mathrm{RCA}_0 + \mathrm{RT}^2_2 + \mathrm{WKL}_0 \not\to \mathrm{RT}^n_2$ *for* $n \geq 3$*.*

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Mathias Forcing Avoiding $\emptyset'$

Let $Z \subset \omega$ and $f : [\mathbb{N}]^2 \to 2$ be Z-recursive.

Definition

If $f(x, y) = 0$, then $\{x, y\}$ is colored *red*. Otherwise it is colored *blue*.

Definition

Given two (finite) binary strings σ and τ, a set X is *suitable* for $\langle \sigma, \tau \rangle$ if $\{x, z\}$ is red for all $x \in \sigma$ and $z \in X$, and $\{y, z\}$ is blue for all $y \in \tau$ and $z \in X$.

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Definition

A forcing condition $p = \langle \sigma, \tau, X \rangle$ is a triple of the form

- σ, τ are finite strings;
- σ is red: $\{x, z\}$ is red for any $x, z \in \sigma$;
- τ is blue: $\{y, z\}$ is blue for any $y, z \in \tau$;
- X is infinite, $X \oplus Z \not\geq_T \emptyset'$, and is suitable for σ, τ;
- Max $\sigma \cup \tau <$ Min X.

Definition

$q = \langle \sigma_q, \tau_q, X_q \rangle$ is *stronger* than $p = \langle \sigma_p, \tau_p, X_p \rangle$ ($p \geq q$) if

- $\sigma_p \leq \sigma_q$ and $\tau_p \leq \tau_q$;
- $X_p \supset X_q$;
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- If $\sigma \cup \sigma_0$ and $\sigma \cup \sigma_1$ are red, and $\sigma \leq \sigma_0$ and $\sigma \leq \sigma_1$ are incompatible, then σ_0 and σ_1 Φ-*split* for red over σ if there is an $x \leq s = \mathrm{Max}\ \sigma_0 \cup \sigma_1$ such that $\Phi_s^{\sigma_0 \oplus Z}(x) \downarrow \neq \Phi_s^{\sigma_1 \oplus Z}(x) \downarrow$;
- Define Φ-split for blue over τ similarly.

Let G_R and G_B be generic: For all φ in the language of second order arithmetic with set constants G_R and G_B and constant Z, there is a condition $p = \langle \sigma, \tau, X \rangle$ such that $\sigma < G_R$ and $\tau < G_B$ ("$<$" means initial segment) and $p \Vdash \varphi$ or $\neg\varphi$.

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Lemma (3.1)

Given $\langle \sigma, \tau, X \rangle$, if no incompatible $\sigma_0, \sigma_1 \geq \sigma$ with $\sigma_0 \setminus \sigma, \sigma_1 \setminus \sigma \subset X$ Φ-split for red over σ and if $\Phi^{G_R \oplus Z}$ is total, then it is recursive in $X \oplus Z$. Same conclusion for τ.

A finite set D is red if $\{x, z\}$ is red for all $x, z \in D$.

Definition

Let $p = \langle \sigma, \tau, X \rangle$ be a condition. X is *blue-hyperimmune* if for any infinite $X \oplus Z$-recursive array of red finite sets $\langle D_i | i \in \omega \rangle$, there is a D_i and an infinite $Y \subset X$ such that $Y \leq_T X$, Max $D_i <$ Min Y and $\{x, z\}$ is red for all $x \in D_i$ and $z \in Y$.

Note. In this case $\langle \sigma, \tau, X \rangle \geq \langle \sigma_{D_i}, \tau, Y \rangle$, where $\sigma_{D_i}(x) = \sigma(x)$ for $x \leq \text{lth}(\sigma)$, and equal to $D_i(x)$ otherwise.

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Avoiding $\emptyset'$

Lemma (3.2)

If the set of conditions $\langle \sigma, \tau, X \rangle$ with X blue-hyperimmune is dense, then any G_R is an H_f satisfying $G_R \oplus Z \not\geq_T \emptyset'$.

Proof. Suppose $\Phi^{G_R \oplus Z} = \emptyset'$. Then there is a $p = \langle \sigma, \tau, X \rangle$ such that $p \Vdash \Phi^{G_R \oplus Z} = \emptyset'$, $\sigma < G_R$, and X is blue-hyperimmune. Enumerate $X \oplus Z$-recursively pairwise disjoint red finite subsets $\langle D_{i,0}, D_{i,1} \rangle$ of X such that $\langle \sigma_{D_{i,0}}, \sigma_{D_{i,1}} \rangle$ Φ-split for red over σ.

- Case 1. There are only finitely many such i's. Then there is an s such that if $Y = X \cap \{x | x > s\}$, no pair of strings that Φ-split for red over σ exists by choosing them from finite subsets of Y. Now $q = \langle \sigma, \tau, Y \rangle \leq p$ and so $q \Vdash \Phi^{G_R \oplus Z} = \emptyset'$. However, Lemma 3.1 says $q \Vdash \Phi^{G_R \oplus Z} \leq_T Y \oplus Z$. Since $Y \oplus Z \not\geq_T \emptyset'$, we have a contradiction.

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Avoiding $\emptyset'$

- Case 2. There are infinitely many i's. Then since X is blue-hyperimmune, there exist i and infinite $Y \subset X$ with $Y \leq_T X$ such that $\{x, z\}$ is red for all $x \in D_{i,0} \cup D_{i,1}$ and $z \in Y$, where Max $D_{i,0} \cup D_{i,1} <$ Min Y. Then $\langle \sigma_{D_{i,0}}, \tau, Y \rangle$ and $\langle \sigma_{D_{i,1}}, \tau, Y \rangle \leq p$ and $\langle \sigma_{D_{i,0}}, \sigma_{D_{i,1}} \rangle$ Φ-split for red over σ, contradicting the assumption that $p \Vdash \Phi^{G_R \oplus Z} = \emptyset'$.

Definition

Let X be infinite. A *Seetapun tree* for X is an infinite X-recursively bounded X-recursive tree T such that for each n, there is a string $\nu \in T$ of length n satisfying: for all $i \leq n$ and all infinite $Y \subset X$ with $Y \leq_T X$, $\{z \in Y | \{\nu(i), z\} \text{is blue}\}$ is infinite.

Seetapun Tree

Lemma (3.3)

Assume that $\{\langle\sigma,\tau,X\rangle | X \text{ is blue-hyperimmune}\}$ *is not dense. Then there is a* G_B *that is an* H_f *and* $G_B \oplus Z \not\geq_T \emptyset'$.

Proof Fix $p = \langle\sigma_p, \tau_p, X_p\rangle$ so that if $q = \langle\sigma_q, \tau_q, X_q\rangle \leq p$, then X_q is not blue-hyperimmune. Let G_B be such that $\tau_q < G_B$. Suppose $\Phi^{G_B \oplus Z} = \emptyset'$ and let $q \Vdash \Phi^{G_B \oplus Z} = \emptyset'$ with $q \leq p$.

Let $\langle D_i \rangle$ be an infinite X_q-recursive pairwise disjoint array of finite subsets of X_q that witnesses the non-blue-hypeerimmunity of X_q. Define T to be the collection of all strings ν such that if $n = \text{lth}(\nu)$, then $\nu(i) \in D_i$ for $i \leq n$. We may assume that Max $D_i <$ Min D_{i+1}.

T is a Seetapun tree.

Seetapun Tree

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Seetapun Tree

Let

$U = \{\nu \in T | \forall E_0, E_1 \subset \nu [\langle \tau_{E_0}, \tau_{E_1} \rangle$ do not Φ-split for blue over $\tau_q\}$,

where $\tau_{E_i}(x) = \tau(x)$ for $x \leq \text{lth}(\tau)$, and equal to $E_i(x)$ otherwise. [$E \subset \nu$ means $\forall x(x \in E \leftrightarrow \nu(i) = x)$ for some $i \leq \text{lth}(\nu)$.]

- Case 1. U is finite. Fix s_0 such that for all ν of length greater than s_0, there exist $E_0, E_1 \subset \nu$ with $\langle \tau_{E_0}, \tau_{E_1} \rangle$ Φ-split for blue over τ. Since X_q is not blue-hyperimmune, there is $\langle x_0, Y_0 \rangle$ such that
  - $x_0 \in D_0$;
  - $Y_0 \subset X_q$ is infinite and $Y_0 \leq_T X_0$;
  - $\{x_0, z\}$ is blue for all $z \in Y_0$.

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$$U = \{\nu \in T | \forall E_0, E_1 \subset \nu[\langle \tau_{E_0}, \tau_{E_1} \rangle \text{ do not } \Phi\text{-split for blue over } \tau_q\},$$

where $\tau_{E_i}(x) = \tau(x)$ for $x \leq \text{lth}(\tau)$, and equal to $E_i(x)$ otherwise. [$E \subset \nu$ means $\forall x(x \in E \leftrightarrow \nu(i) = x)$ for some $i \leq \text{lth}(\nu)$.]

- Case 1. U is finite. Fix s_0 such that for all ν of length greater than s_0, there exist $E_0, E_1 \subset \nu$ with $\langle \tau_{E_0}, \tau_{E_1} \rangle$ Φ-split for blue over τ. Since X_q is not blue-hyperimmune, there is $\langle x_0, Y_0 \rangle$ such that
  - $x_0 \in D_0$;
  - $Y_0 \subset X_q$ is infinite and $Y_0 \leq_T X_0$;
  - $\{x_0, z\}$ is blue for all $z \in Y_0$.

Seetapun Tree

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Avoiding $\emptyset'$

If $\langle x_i, Y_i \rangle$ is defined so that $x_i \in D_i$, $Y_i \subset X_q$ is infinite, $Y_i \leq_T X_q$ and $\{x_i, z\}$ is blue for all $z \in Y_i$, then there exists $\langle x_{i+1}, Y_{i+1} \rangle$ such that

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Let $\nu^*(i) = x_i$ for $i \leq s_0 + 1$. Then there exist $E_0, E_1 \subset \nu^*$ so that $\langle \tau_{E_0}, \tau_{E_1} \rangle$ Φ-split for blue over τ_q. Then $\langle \sigma, \tau_{E_0}, Y_{s_0+1} \rangle$ and $\langle \sigma, \tau_{E_1}, Y_{s_0+1} \rangle$ are two incompatible extensions of q that split Φ, contradiction.

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- Case 2. U is infinite. By the Low Basis Theorem relativized to $X_q \oplus Z$, U has a infinite path W that is low in $X_q \oplus Z$ not above $\emptyset'$ (since $X \oplus Z \not\geq_T \emptyset'$). Then $\langle \sigma_q, \tau_q, W \rangle \leq q$ and for any $E_0, E_1 \subset W$, $\langle \tau_{E_0}, \tau_{E_1} \rangle$ does not Φ-split for blue over τ_q. Lemma 3.1 then implies that $\Phi^{G_B \oplus Z}$ is recursive in $X_q \oplus Z$, a contradiction.

The above may be iterated to produce an ω-model of $RCA_0 + WKL_0 + RT^2_2$ that avoids $\emptyset'$, as follows:

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An ω-model of $\mathrm{RCA}_0 + \mathrm{WKL}_0 + \mathrm{RT}^2_2$

Lemma (3.4)

Let $\mathcal{M} = \langle \omega, \mathbb{X}, +, \times, 0, 1 \rangle$ be a model of RCA_0 such that $\mathbb{X}$ is finite. Assume that $X = \oplus_{X_i \in \mathbb{X}} X_i$ does not compute $\emptyset'$ and $X \in \mathbb{X}$. Furthermore,

(i) *T is a X-recursively bounded, X-recursive tree and*

(ii) *$f : [\mathbb{N}]^2 \to 2$ is an X-recursive 2-coloring of pairs.*

Then T has an infinite path W so that $W \oplus X \not\geq_T \emptyset'$, and there is an H_f such that $H_f \oplus W \oplus X$ does not compute $\emptyset'$.

An ω-model of $\mathsf{RCA}_0 + \mathsf{WKL}_0 + \mathsf{RT}^2_2$

The proof proceeds by first applying the relativized Low Basis Theorem to obtain W, and let $\mathbb{X}^* = \mathbb{X} \cup \{W\}$. Then apply Theorem 3.1 with Z as the joint X of all $X_i \in \mathbb{X}$.

Proof of corollary to Theorem 3.1. Begin with an ω-model $\mathcal{M}_0$ where $\mathbb{X} = \emptyset$. Recursively define $\mathcal{M}_n$ so that an infinite path for a tree $T \in \mathbb{X}_n$ that does not compute $\emptyset'$ is first added to $\mathbb{X}_{n+1}$, followed by an H_f for a 2-coloring of pairs that satisfies the conclusion of Lemma 3.4. This can be arranged in such a way that $\mathcal{M} = \bigcup_n \mathcal{M}_n$ is the desired model.

An ω-model of $\mathsf{RCA}_0 + \mathsf{WKL}_0 + \mathsf{RT}^2_2$

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Session III:

L’estremità