Recursion Theory of Ramsey's Theorem

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Session IV

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Cohesiveness

Let $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1 \rangle \models \mathrm{RCA}_0$.

Definition

Let $R \in \mathbb{X}$. A set C is *cohesive* for the array $\{R_i\}_{i \in M}$, where $R_i = \{s | (i, s) \in R\}$, if for each i, either $C \cap R_i$ or $C \cap \bar{R}_i$ is $\mathcal{M}$-finite. C is said to be R-cohesive.

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$\mathcal{M} \models \mathrm{COH}$ if for any array R in $\mathbb{X}$, there is an R-cohesive set in $\mathbb{X}$.

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Proof. Let $\mathcal{M} \models \mathsf{RCA}_0 + \mathsf{SRT}^2_2 + \mathrm{COH}$ and $f : [M]^2 \to 2$ a 2-coloring of pairs where f is in $\mathbb{X}$.

- Let $R_i = \{s | f(i,s) = 1\}$. By COH there is an R-cohesive set C, where $R = \{(i,s) | (i,s) \in R_i\}$. Let $f_C(i,s) = 1$ if $f(i, x_s) = 1$, where x_s is the sth element of C, and 0 otherwise. Then f_C is stable and by SRT^2_2 there is an $H_{f_C} \in \mathbb{X}$. Then $(s | x_s \in H_{f_C}\}$ is a set homogeneous for f.

- Conversely, let R be an array. For each s, let the i-state of s be $\{j \leq i | s \in R_j\}$, where $i \leq s$. Order i-states lexicographically. Let $f(i,s) = 1$ if the i-state of s is greater than or equal to the i-state of i, and 0 otherwise. Then any H_f is R-cohesive.

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COH Preserving $I\Sigma_2^0$

Theorem (4.1)

(Cholak, Jockusch and Slaman [2001]) *Let* $\mathcal{M} \models \mathrm{RCA}_0 + I\Sigma_2^0$*.* *Then* $\mathcal{M}$ *is an* M*-submodel of an* $\mathcal{M}^* \models \mathrm{RCA}_0 + I\Sigma_2^0 + \mathrm{COH}$*.*

Definition

Let $T_1 \subset T_2$ be subsystems of of sedcond order arithmetic. T_2 is Π_1^1-*conservative* over T_1 if every Π_1^1 sentence that is provable over T_2 is already provable over T_1.

Corollary

$\mathrm{RCA}_0 + I\Sigma_2^0 + \mathrm{COH}$ *is* Π_1^1*-conservative over* $\mathrm{RCA}_0 + I\Sigma_2^0$*.*

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An R'-recursive Tree of Cohesive Paths

Fix $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$. We present a different proof whose idea will be used again for the $B\Sigma^0_2$ case.

Lemma (4.2)

Let $\mathcal{M} \models \mathrm{RCA}_0 + I\Sigma^0_2$ and $R \in \mathbb{X}$ an array. Then there is an $\mathcal{M}$-infinite R'-recursive tree T such that every $\mathcal{M}$-infinite path G is R-cohesive and generalized R-low (i.e. $G' \leq_T G \oplus R'$).

Proof. Let $R_i = \{s | (i, s) \in R\}$.

- $\nu \in 2^s$ is *fulfilled* if there is a σ such that (i) $\sigma(x) = 1$ for some $x \leq \mathrm{lth}(\sigma)$ and (ii) $\forall x \sigma(x) = 1$ if and only if $x \in \bigcap_{\nu(i)=1} R_i \cap \bigcap_{\nu(i)=0} \bar{R}_i$.
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- Let $C_0[\sigma_\nu] = \{\sigma \geq \sigma_\nu | \sigma \subset \bigcap_{\nu(i)=1} R_i \cap \bigcap_{\nu(i)=0} \bar{R}_i\}$ where $\nu \in 2^2$.
- Let $C_0 = \bigcup_{\sigma_\nu \in F_0} C_0[\sigma_\nu]$.
  Claim 1. C_0 is $\mathcal{M}$-infinite.
- Otherwise, let s_0 be the least upper bound and let $\sigma = \bigcap_{\nu(i)=1} R_i \cap \bigcap_{\nu(i)=0} \bar{R}_i \restriction s_0$.

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For $\sigma \in C_0$, let $C_0[\sigma] = \{\tau \in C_0 | \sigma \leq \tau\}$. Let σ^* be the least τ in $C_0[\sigma]$ such that $0 \in \tau'$, if this exists. Let it be σ otherwise. Let $C_0^* = \bigcup_{\sigma^* \in C_0} C_0[\sigma^*]$. Let $T_0 = \{\sigma^* | \sigma^* \in C_0^*$. This is an $\mathcal{M}$-finite set. Let s_0 be the maximum of T_0.

If C_j^* and T_j s_j are defined and $s \geq s_j$, let

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Define T_{j+1} from C_{j+1}^* similarly. Let $T = \bigcup_j T_j$. Then T is R'-recursive. Every unbounded path in T is R-cohesive, and generalized R-low.

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Preserving $I\Sigma^0_2$ in the Path

Lemma (4.3)

If $\mathcal{M} \models \mathrm{RCA}_0 + I\Sigma^0_2$ is countable and $R \in \mathbb{X}$, then any $\mathcal{M}$-infinite R'-recursive tree T has an $\mathcal{M}$-infinite path G such that $\mathcal{M}[G] \models I\Sigma^0_2$.

Proof. We will select a path through T that satisfies $I\Sigma^0_2$.

- Let $\exists x \varphi_n(x, y, G \oplus \emptyset')$ be the nth Σ^0_1 with free variable y.
- Suppose that $T_{n-1} \subset T$ is $\mathcal{M}$-infinite and $\sigma_{n-1} \in T_{n-1}$ are defined so that (i) T_{n-1} is R'-recursive, (ii) all strings in T_{n-1} extend σ_{n-1} and (iii) for all $m < n$, either there is a least $c \leq \mathrm{lth}(\sigma_{n-1})$ such that $\exists x \varphi_m(x, c, \sigma_{n-1} \oplus R')$, or no $\sigma \geq \sigma_{n-1}$ in T_{n-1} satisfies $\exists x \varphi_m(x, c, \sigma \oplus R')$ for any c.

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Preserving $I\Sigma_2^0$ in the Path

- For each c, let

$$U_c = \{\sigma \in T | \forall x \leq \mathrm{lth}(\sigma) \forall c' \leq c \neg\varphi_n(x, c', \sigma \oplus R')\}.$$

- “$F = \{c | U_c$ is $\mathcal{M}$-finite$\}$” is $\Sigma_1^0(R')$. By $I\Sigma_2^0$ in $\mathcal{M}$, if $F \neq \emptyset$ it has a least element c_n.
- Choose the least σ such that $\sigma \in U_{c_n - 1}$, $\exists x \leq \mathrm{lth}(\sigma)\varphi_n(x, c_n, \sigma \oplus R')$ and $U_{c_n-1}[\sigma] = \{\tau \in U_{c_n-1} | \tau \geq \sigma\}$ is $\mathcal{M}$-infinite. Call this σ_n and let $T_n = T[\sigma_n]$.
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As before the above can be iterated to obtain $\mathcal{M}^* \models \mathrm{RCA}_0 + \mathrm{COH} + I\Sigma^0_2$.

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Π^1_1-conservation of $\mathrm{RCA}_0 + \mathrm{COH} + I\Sigma^0_2$

Proof of Corollary to Theorem 4.1.

Let $\forall X \varphi$ be Π^1_1 so that φ is arithmetical and assume $\mathrm{RCA}_0 + \mathrm{COH} + I\Sigma^0_2 \vdash \forall X \varphi$.
Suppose $\mathcal{M}_0 \models \neg\forall X \varphi$ for some $\mathcal{M}_0 \models \mathrm{RCA}_0 + I\Sigma_2$. By Löwenheim-Skolem Theorem we may assume that $\mathcal{M}_0$ is countable. So $\mathcal{M}_0 \models \exists X \neg\varphi$.
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SRT^2_2 and $I\Sigma^0_2$

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(Cholak, Jockusch and Slaman [2001]) *Let* $\mathcal{M} \models \mathrm{RCA}_0 + I\Sigma^0_2$.
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We give a different proof of this theorem.

Corollary

$\mathrm{RCA}_0 + \mathrm{SRT}^2_2 + \mathrm{COH} + I\Sigma^0_2$ *is* Π^1_1*-conservative over* $\mathrm{RCA}_0 + I\Sigma^9_2$.
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Stable Coloring and f-hyperimmunity

Lemma (4.4)

Let $f : [M]^2 \rightarrow 2$ be a stable 2-coloring of pairs coded in $\mathbb{X}$. Then there is an $\mathcal{M}$-infinite f'-recursive tree T in which every $\mathcal{M}$-infinite path G is homogeneous for f and $G' \leq_T G \oplus f'$.

Definition

Y is *f-hyperimmune* if there is no f-recursive $\mathcal{M}$-infinite array of pairwise disjoint $\mathcal{M}$-finite sets $\langle D_i \rangle_{i \in M}$ such that $D_i \cap Y \neq \emptyset$ for all i.

Definition

$A \subset M$ is *not hyperimmune on f-low sets* if for all $\mathcal{M}$-infinite, f-low set X, $A \cap X$ is not f-hyperimmune.

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Stable 2-Coloring and f-hyperimmunity

Let $A = \{i | \lim_s f(i,s) = 1\}$, the set of *eventually red* elements. Assume that for all $\mathcal{M}$-infinite f-low set X, both $A \cap X$ and $\bar{A} \cap X$ are $\mathcal{M}$-infinite. The proof splits into two cases.

Case 1. $A \cap X$ is f-hyperimmune for some $\mathcal{M}$-infinite f-low set X. Let

$$H_0 = \{E \in X | \Phi^E_{0,\mathrm{Max}(E)}(0) \downarrow\}.$$

We may assume that H_0 is an array of pairwise disjoint $\mathcal{M}$-finite sets E. Then "H_0 is $\mathcal{M}$-finite" is r.e. in f'. In this case f' may decide if there is an $E \in H_0 \cap \bar{A}$. If H_0 is $\mathcal{M}$-infinite, by hyperimmunity of $A \cap X$, there is an $E \in H_0$ such that $E \subset \bar{A}$. In either way, f' decides effectively which situation holds. Let σ_0 be σ_E, where $E \subset \bar{A}$ is the least, if it exists in either situation, and $\emptyset$ otherwise.

If σ_s is defined, define σ_{s+1} analogously, replacing 0 by $s+1$, and requiring $\sigma_s < E$, i.e. $E(x) = \sigma_s(x)$ for $x \leq \mathrm{lth}(\sigma_s)$.

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Let $T = \{\sigma_s | s \in M\}$. Then T satisfies the requirement. T has a unique $\mathcal{M}$-infinite path $G \subset \bar{A}$.

Case 2. $A \cap X$ is not f-hyperimmune for any $\mathcal{M}$-infinite, f-low X. Then for any f-low X, there is an f-recursive array that witnesses the failure of f-hyperimmunity of $A \cap X$.

- For each $\mathcal{M}$-infinite f-low X with index e (denoted X_e), the least index t for such an array $H(t)$ exists by $I\Sigma^0_2$ and is denoted $h(e)$.
- Given $H(h(e))$ and $\sigma \subset A$, there is a Seetapun tree S_e so that $\tau \in S_e \to \tau \subset \bigcup_s D_s$, where $D_s \in H(h(e))$, and for all s there is a $\tau \in S_e \cap A$ of length s. Let

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- U_e is f-recursive.
- (i) If U_e is $\mathcal{M}$-finite, then by non-hyperimmunity there is a $\tau \geq \sigma$ in S_e with $\tau \subset A$ and $\Phi^{\tau}_{e,\mathrm{lth}(\tau)}(e)\downarrow$. Then $X = X_e \cap \{s | s > \mathrm{lth}(\sigma_E)\}$ is f-low.
- (ii) If U_e is $\mathcal{M}$-infinite then by the relativized Low Basis Theorem it has an infinite f-low path X for which no $\mathcal{M}$-finite subset E satisfies $\Phi^{E}_{e}(e)\downarrow$.
- Given $H(h(e))$, an index $r(h(e))$ for X that satisfies (i) or (ii) is f'-recursively found.
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An f'-recursive Tree of Homogeneous Sets

Define an f'-recursive tree T as follows:

- Every string in any T_s is a subset of A;
- T_0 has a unique element σ^0 obtained from X_0 such that either (i) or (ii) holds. If (ii) holds, $\sigma^0 = \emptyset$;
- X_{σ^0} is $X_{r(h(0))}$ derived from either (i) or (ii);
- If $\sigma \in T_s$, then σ has at most $s+1$-many proper extensions, defined recursively in f', in T_{s+1};
- If $\sigma \in T_{s+1} \setminus T_s$, then there is a sequence $\sigma^0 < \sigma^1 < \cdots < \sigma^u < \cdots < \sigma^s < \sigma$ where $\sigma^u \in T_u \setminus T_{u-1}$ and
- There is a sequence $X_0, X_{\sigma^0}, \ldots, X_{\sigma^u}, \ldots, X_{\sigma^s}, X_\sigma$, and a $t \leq s+1$, $u \leq s$ such that $\sigma \in X_{\sigma^u}$, and X_σ is obtained from X_{σ^u} via (i) or (ii) using the tth pairwise disjoint array $H(t)$, provided that

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- X_{σ^0} is $X_{r(h(0))}$ derived from either (i) or (ii);
- If $\sigma \in T_s$, then σ has at most $s+1$-many proper extensions, defined recursively in f', in T_{s+1};
- If $\sigma \in T_{s+1} \setminus T_s$, then there is a sequence $\sigma^0 < \sigma^1 < \cdots < \sigma^u < \cdots < \sigma^s < \sigma$ where $\sigma^u \in T_u \setminus T_{u-1}$ and
- There is a sequence $X_0, X_{\sigma^0}, \ldots, X_{\sigma^u}, \ldots, X_{\sigma^s}, X_\sigma$, and a $t \le s+1$, $u \le s$ such that $\sigma \in X_{\sigma^u}$, and X_σ is obtained from X_{σ^u} via (i) or (ii) using the tth pairwise disjoint array $H(t)$, provided that

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- If no such u exists, then $\sigma = \sigma^s \frown x$ where x is the least element of A greater than σ^s.

Let $T = \bigcup_s T_s$. Then $\sigma \in T$ if and only if $\sigma \in T_s$, where $s = \mathrm{lth}(\sigma)$. Thus T is f'-recursive. T is $\mathcal{M}$-infinite since if $X_\sigma = X_e$ is f-low and $\mathcal{M}$-infinite, then by stage $h(e)$, $H(h(e))$ is an $\mathcal{M}$-infinite witness array for X_e and (i) or (ii) applies. Thus there will be a $\sigma \in T_{h(e)} \setminus T_{h(e)-1}$.

Applying Lemma 4.3, we can select an $\mathcal{M}$-infinite path G such that $\mathcal{M}[G] \models I\Sigma^0_2$. By iterating Lemma 4.4, we obtain Theorem 4.2.

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Applying Lemma 4.3, we can select an $\mathcal{M}$-infinite path G such that $\mathcal{M}[G] \models I\Sigma_2^0$. By iterating Lemma 4.4, we obtain Theorem 4.2.

RT^2_2 and WKL_0

- Since RT^2_2 is equivalent to $\mathsf{SRT}^2_2 + \mathsf{COH}$ over RCA_0, combining Lemmas 4.2 and 4.4 yields: $\mathsf{RCA}_0 + \mathsf{RT}^2_2 + I\Sigma^0_2$ is Π^1_1-conservative over $\mathsf{RCA}_0 + I\Sigma^0_2$.
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Session IV:

L’estremità