Lemma (5.1)

(Hirst [1987]) \( \text{RCA}_0 \vdash \text{RT}_1^2 \rightarrow B\Sigma_2^0 \).

Proof. Let \( h : X \rightarrow M \) be a \( \Sigma_2^0(M) \) function total on an \( M \)-finite set \( X \) in a model \( M \) of \( \text{RCA}_0 + \text{RT}_2^2 \).
Assume \( f(x) = y \leftrightarrow \exists u \forall v \varphi(u, v, x, y) \) where \( \varphi \) is \( \Delta_0^0(M) \).
Suppose for the sake of contradiction that the image is not bounded. For each \( s \), let \( x_s \) be the least \( x \in X \) such that for all \( y, u \preceq s \), there is a \( v \) such that \( \neg \varphi(u, v, x, y) \). \( B\Sigma_2^0 \) implies that \( x_s \) exists for all \( s \).
Let \( f(s, t) = 1 \) if \( x_s = x_t \), and 0 otherwise. Then since \( X \) is \( M \)-finite, by pigeonhole principle any \( H_f \) satisfies \( f([H_f]^2) = 1 \).
Let \( x \) be such that \( x = x_s \) for all \( s \in H_f \). Then \( f(x) \uparrow \), a contradiction.
Lemma (5.1)

(Hirst [1987]) RCA₀ ⊢ RT₂¹ → BΣ₂⁰.

Proof. Let \( h : X \rightarrow M \) be a \( \Sigma^0_2(M) \) function total on an \( M \)-finite set \( X \) in a model \( M \) of RCA₀ + RT₂².

Assume \( f(x) = y \Leftrightarrow \exists u \forall v \varphi(u, v, x, y) \) where \( \varphi \) is \( \Delta^0_0(M) \).

Suppose for the sake of contradiction that the image is not bounded. For each \( s \), let \( x_s \) be the least \( x \in X \) such that for all \( y, u \leq s \), there is a \( v \) such that \( \neg \varphi(u, v, x, y) \). \( BΣ₂⁰ \) implies that \( x_s \) exists for all \( s \).

Let \( f(s, t) = 1 \) if \( x_s = x_t \), and 0 otherwise. Then since \( X \) is \( M \)-finite, by pigeonhole principle any \( H_f \) satisfies \( f([H_f]^2) = 1 \).

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Lemma (5.1)

(Hirst [1987]) RCA$_0$ + RT$_1^2$ $\vdash$ BΣ$_2^0$.

Proof. Let $h : X \rightarrow M$ be a $\Sigma^0_2(M)$ function total on an $M$-finite set $X$ in a model $M$ of RCA$_0$ + RT$_2^2$.
Assume $f(x) = y \iff \exists u \forall v \varphi(u, v, x, y)$ where $\varphi$ is $\Delta^0_0(M)$.
Suppose for the sake of contradiction that the image is not bounded. For each $s$, let $x_s$ be the least $x \in X$ such that for all $y, u \leq s$, there is a $v$ such that $\neg \varphi(u, v, x, y)$. BΣ$_2^0$ implies that $x_s$ exists for all $s$.
Let $f(s, t) = 1$ if $x_s = x_t$, and 0 otherwise. Then since $X$ is $M$-finite, by pigeonhole principle any $H_f$ satisfies $f([H_f]^2) = 1$.
Let $x$ be such that $x = x_s$ for all $s \in H_f$. Then $f(x) \uparrow$, a contradiction.
Lemma (5.1)

(Hirst [1987]) $\text{RCA}_0 \vdash \text{RT}_1^2 \rightarrow B\Sigma_2^0$.

**Proof.** Let $h : X \rightarrow M$ be a $\Sigma_2^0(M)$ function total on an $\mathcal{M}$-finite set $X$ in a model $\mathcal{M}$ of $\text{RCA}_0 + \text{RT}_2^2$. Assume $f(x) = y \leftrightarrow \exists u \forall v \varphi(u, v, x, y)$ where $\varphi$ is $\Delta_0^0(M)$. Suppose for the sake of contradiction that the image is not bounded. For each $s$, let $x_s$ be the least $x \in X$ such that for all $y, u \leq s$, there is a $v$ such that $\neg \varphi(u, v, x, y)$. $B\Sigma_2^0$ implies that $x_s$ exists for all $s$.

Let $f(s, t) = 1$ if $x_s = x_t$, and 0 otherwise. Then since $X$ is $\mathcal{M}$-finite, by pigeonhole principle any $H_f$ satisfies $f([H_f]^2) = 1$. Let $x$ be such that $x = x_s$ for all $s \in H_f$. Then $f(x) \uparrow$, a contradiction.
Strength of $RT_2^2$

Questions.
- Is $RCA_0 + RT_2^2 + B\Sigma_2^0 \Pi_1^1$-conservative over $RCA_0 + B\Sigma_2^0$?
- Is $RCA_0 + COH + B\Sigma_2^0 \Pi_1^1$-conservative over $RCA_0 + B\Sigma_2^0$?
- Does $RT_2^2$ prove $I\Sigma_2^0$ over $RCA_0$?
- Does $SRT_2^2$ prove $RT_2^2$ over $RCA_0$?
Strength of $\text{RT}_2^2$

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- Is $\text{RCA}_0 + \text{RT}_2^2 + B\Sigma_2^0 \Pi_1^1$-conservative over $\text{RCA}_0 + B\Sigma_2^0$?
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- Is RCA₀ + RT₈² + BΣ₂ Π₁⁻ conservative over RCA₀ + BΣ₂⁰?
- Is RCA₀ + COH + BΣ₂ Π₁⁻ conservative over RCA₀ + BΣ₂⁰?
- Does RT₈² prove IΣ₂⁰ over RCA₀?
- Does SRT₈² prove RT₈² over RCA₀?
Working Over $B\Sigma^0_2$ Models

**Fact.** Every model $\mathcal{M}$ of RCA$_0 + I\Sigma^0_2$ is an $\mathcal{M}$-submodel of an $\mathcal{M}^*$ such that

- Any array $R \in \mathcal{M}^*$ has an $R$-cohesive set recursive in $R''$.
- Any 2-coloring of pairs $f \in \mathcal{M}^*$ has an $H_f \leq_T f''$.

Indeed for $\omega$-models $\mathcal{M}$, low$_2$ solutions exist in $\mathcal{M}^*$ (Jockusch, Stephan, Cholak, Slaman): There is a low$_2$ $r$-cohesive set, and every 2-coloring of pairs has a homogeneous set low$_2$ in the code of the coloring.

These fail in models of RCA$_0 + B\Sigma^0_2$ without $I\Sigma^0_2$. 
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Indeed for $\omega$-models $\mathcal{M}$, $\text{low}_2$ solutions exist in $\mathcal{M}^*$ (Jockusch, Stephan, Cholak, Slaman): There is a $\text{low}_2$ $r$-cohesive set, and every 2-coloring of pairs has a homogeneous set $\text{low}_2$ in the code of the coloring.

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These fail in models of $\text{RCA}_0 + B\Sigma^0_2$ without $I\Sigma^0_2$. 
Double Jump Basis in RCA$_0 + B\Sigma^0_2$ Models

**Definition**

$\mathcal{M} \models RCA_0 + B\Sigma^0_2$ is a *double-jump* basis for $\varphi$ if there is a solution to $\varphi$ in $\mathcal{M}$ recursive in the double jump of the parameters in $\varphi$.

**Theorem (5.1)**

(Chong [2006]) *No model of RCA$_0 + COH + B\Sigma^0_2$ without I$\Sigma^0_2$ has a double-jump basis for COH.*

**Corollary**

*The following are equivalent:*

- $\mathcal{M} \models RCA_0 + I\Sigma^0_2$.
- $\mathcal{M}$ is an $M$-submodel of $\mathcal{M}^* \models RCA_0 + I\Sigma^0_2$ that is a double-jump basis for COH.


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The following are equivalent:

- $\mathcal{M} \models \text{RCA}_0 + I\Sigma^0_2$.
- $\mathcal{M}$ is an $M$-submodel of $\mathcal{M}^* \models \text{RCA}_0 + I\Sigma^0_2$ that is a double-jump basis for COH.
Double Jump Basis in RCA\(_0 + B\Sigma^0_2\) Models

**Definition**

\(\mathcal{M} \models \text{RCA}_0 + B\Sigma^0_2\) is a **double-jump** basis for \(\varphi\) if there is a solution to \(\varphi\) in \(\mathcal{M}\) recursive in the double jump of the parameters in \(\varphi\).

**Theorem (5.1)**

(Chong [2006]) **No model of** RCA\(_0 + \text{COH} + B\Sigma^0_2\) **without I\(\Sigma^0_2\)** has a double-jump basis for COH.

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*The following are equivalent:*

- \(\mathcal{M} \models \text{RCA}_0 + I\Sigma^0_2\).
- \(\mathcal{M}\) is an \(M\)-submodel of \(\mathcal{M}^* \models \text{RCA}_0 + I\Sigma^0_2\) that is a double-jump basis for COH.
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The following are equivalent:

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From now on $\mathcal{M} = \langle M, \emptyset, +, \times, 0, 1 \rangle$ always denotes a model of $\text{RCA}_0 + B\Sigma^0_2 + \neg I\Sigma^0_2$ with $\Sigma^0_2(\mathcal{M})$ cut $I$. Let $g : I \rightarrow M$ be $\Sigma^0_2(\mathcal{M})$, increasing and cofinal.

**Definition**

$X \subset M$ is regular if $X \upharpoonright b$ is $\mathcal{M}$-finite for all $b$.

**Definition**

$X \subset M$ is hyperregular if every function $\Delta^0_1(\mathcal{M})$ in $X$ maps a bounded set into a bounded set.

*Note.* Any set $X$ in $\emptyset$ is regular, hyperregular and satisfies $B\Sigma^0_1(X')$. 
From now on $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1 \rangle$ always denotes a model of $\text{RCA}_0 + \text{B}\Sigma_2^0 + \neg \text{I}\Sigma_2^0$ with $\Sigma^0_2(\mathcal{M})$ cut $I$. Let $g : I \to M$ be $\Sigma^0_2(\mathcal{M})$, increasing and cofinal.

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Regularity and Hyperregularity

From now on $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1 \rangle$ always denotes a model of $\text{RCA}_0 + B\Sigma^0_2 + \neg I\Sigma^0_2$ with $\Sigma^0_2(\mathcal{M})$ cut $I$. Let $g : I \to M$ be $\Sigma^0_2(\mathcal{M})$, increasing and cofinal.

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$X \subset M$ is hyperregular if every function $\Delta^0_1(\mathcal{M})$ in $X$ maps a bounded set into a bounded set.

*Note.* Any set $X$ in $\mathbb{X}$ is regular, hyperreular and satisfies $B\Sigma^0_1(X')$. 
Fix \( Y \in X \) so that the graph of \( g \) is r.e. in \( Y' \).

**Theorem (5.2)**

If \( C \leq_T Y'' \), then \( C' <_T Y' \), i.e. \( C \) is \( Y \)-low.

**Lemma (5.2)**

\((C \oplus Y)' \) is regular.

**Proof.** \( C \oplus Y \in X \) hence regular and hyperregular. Thus for each \( i \in I \) there is a \( j \in I \) such that 
\[(C \oplus Y)' \upharpoonright g(i) = (C \oplus Y)'_{g(j)} \upharpoonright g(i). \]
Hence \((C \oplus Y)' \) is regular.

**Recall:** (Chong and Mourad [1991]) In a model of \( \text{RCA}_0 + B\Sigma^0_2 \) without \( I\Sigma^0_2 \), every subset of \( I \) that is \( \Delta^0_2(M) \) on \( I \) is coded on \( I \).
Fix $Y \in \mathbb{X}$ so that the graph of $g$ is r.e. in $Y'$.

**Theorem (5.2)**

If $C \leq_T Y''$, then $C' <_T Y'$, i.e. $C$ is $Y$-low.

**Lemma (5.2)**

$(C \oplus Y)'$ is regular.

**Proof.** $C \oplus Y \in \mathbb{X}$ hence regular and hyperregular. Thus for each $i \in I$ there is a $j \in I$ such that

$$(C \oplus Y)' \restriction g(i) = (C \oplus Y)'_{g(j)} \restriction g(i).$$

Hence $(C \oplus Y)'$ is regular.

**Recall:** (Chong and Mourad [1991]) In a model of $\text{RCA}_0 + B\Sigma^0_2$ without $/\Sigma^0_2$, every subset of $I$ that is $\Delta^0_2(\mathcal{M})$ on $I$ is coded on $I$. 
Fix $Y \in X$ so that the graph of $g$ is r.e. in $Y'$.

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*Recall:* (Chong and Mourad [1991]) In a model of $\text{RCA}_0 + B\Sigma^0_2$ without $I\Sigma^0_2$, every subset of $I$ that is $\Delta^0_2(M)$ on $I$ is coded on $I$. 
Fix $Y \in X$ so that the graph of $g$ is r.e. in $Y'$.

**Theorem (5.2)**

If $C \leq_T Y''$, then $C' <_T Y'$, i.e. $C$ is $Y$-low.

**Lemma (5.2)**

$(C \oplus Y)'$ is regular.

**Proof.** $C \oplus Y \in X$ hence regular and hyperregular. Thus for each $i \in l$ there is a $j \in l$ such that

$$(C \oplus Y)' \upharpoonright g(i) = (C \oplus Y)'_{g(j)} \upharpoonright g(i).$$

Hence $(C \oplus Y)'$ is regular.

**Recall:** (Chong and Mourad [1991]) In a model of RCA$_0 + B\Sigma^0_2$ without $I\Sigma^0_2$, every subset of $l$ that is $\Delta^0_2(M)$ on $l$ is coded on $l$. 

Fact. $B\Sigma_1^0(Y')$ implies: If $D \subset Y''$ is $\mathcal{M}$-finite, then there is an $i_D \in \omega$ such that $D \subset Y''_{g(i_D)}$.

Lemma (5.3)

If $X \leq_T Y''$ is regular, then $X \leq_T I \oplus Y'$.

Proof. Let $\Phi^{Y''} = X$. Since $X$ is regular, for each $i \in I$, there is a neighborhood condition $\langle P, N \rangle$ of $Y''$ ($P \subset Y''$ and $N \subset \bar{Y}''$) so that $\Phi^{\langle P, N \rangle} \upharpoonright g(i) = X \upharpoonright g(i)$. There is a least $(j, j') \in I$, denoted $(j_i, j'_i)$, such that $P \subset Y''_{g(j)}$ and any $N^- \subset \bar{Y}''_{g(j)} \cap \bar{Y}''_{g(j')}$ satisfies $\Phi^{\langle P, N^- \rangle} \upharpoonright g(i) = X \upharpoonright g(i)$. The set

$$Z = \{(i, j, j')|\forall N^- \subset \bar{Y}''_{g(j)}[N^- \cap Y''_{g(j')} \neq \emptyset]\}$$

is $\Delta^0_2(\mathcal{M})$ on $I \times I$ hence coded on $I \times I$. Then $X \upharpoonright g(i) = \Phi^{\langle Y''_{g(j)_i}, \bar{Y}''_{g(j'_i)} \rangle}_{g(j'_i)} \upharpoonright g(i)$, and $i \mapsto (j_i, j'_i)$ is recursive in $I$ via the code. proving the lemma.
Degrees Below $Y''$

**Fact.** $B\Sigma^0_1(Y')$ implies: If $D \subset Y''$ is $\mathcal{M}$-finite, then there is an $i_D \in \omega$ such that $D \subset Y''_{g(i_D)}$.

**Lemma (5.3)**

*If $X \leq_T Y''$ is regular, then $X \leq_T I \oplus Y'$.***

**Proof.** Let $\Phi^{Y''} = X$. Since $X$ is regular, for each $i \in I$, there is a neighborhood condition $\langle P, N \rangle$ of $Y''$ ($P \subset Y''$ and $N \subset \bar{Y}''$) so that $\Phi^{\langle P, N \rangle} \upharpoonright g(i) = X \upharpoonright g(i)$. There is a least $(j, j') \in I$, denoted $(j_i, j'_i)$, such that $P \subset Y''_{g(j)}$ and any $N^- \subset \bar{Y}''_{g(j)} \cap \bar{Y}''_{g(j')}$ satisfies $\Phi_{g(j')}^{\langle P, N^- \rangle} \upharpoonright g(i) = X \upharpoonright g(i)$. The set $Z = \{(i, j, j')|\forall N^- \subset \bar{Y}''_{g(j)}[N^- \cap Y''_{g(j')} \neq \emptyset]\}$ is $\Delta^0_2(\mathcal{M})$ on $I \times I$ hence coded on $I \times I$. Then $X \upharpoonright g(i) = \Phi_{g(j'_i)}^{\langle Y''_{g(j)}, \bar{Y}''_{g(j)} \rangle} \upharpoonright g(i)$, and $i \mapsto (j_i, j'_i)$ is recursive in $I$ via the code, proving the lemma.
Fact. $B\Sigma^0_1(Y')$ implies: If $D \subset Y''$ is $\mathcal{M}$-finite, then there is an $i_D \in \omega$ such that $D \subset Y''_{g(i_D)}$.

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If $X \leq_T Y''$ is regular, then $X \leq_T I \oplus Y'$.

Proof. Let $\Phi^{Y''} = X$. Since $X$ is regular, for each $i \in I$, there is a neighborhood condition $\langle P, N \rangle$ of $Y''$ ($P \subset Y''$ and $N \subset Y''$) so that $\Phi^{\langle P, N \rangle} \upharpoonright g(i) = X \upharpoonright g(i)$. There is a least $(j, j') \in I$, denoted $(j_i, j'_i)$, such that $P \subset Y''_{g(j)}$ and any $N^- \subset \bar{Y}''_{g(j)} \cap \bar{Y}''_{g(j')}$ satisfies $\Phi^{\langle P, N^- \rangle} \upharpoonright g(i) = X \upharpoonright g(i)$. The set $Z = \{(i, j, j') | \forall N^- \subset \bar{Y}''_{g(j)}[N^- \cap Y''_{g(j')} \neq \emptyset]\}$ is $\Delta^0_2(\mathcal{M})$ on $I \times I$ hence coded on $I \times I$. Then $X \upharpoonright g(i) = \Phi^{\langle Y''_{g(j_i)}, \bar{Y}''_{g(j'_i)} \rangle}_{g(j'_i)} \upharpoonright g(i)$, and $i \mapsto (j_i, j'_i)$ is recursive in $I$ via the code. proving the lemma.
Degrees Below $Y''$

Setting $X = C \oplus Y$ and applying Lemmas 5.2 and 5.3 yields

**Lemma (5.4)**

$$(C \oplus Y)' \leq_T I \oplus Y'.$$

*Note.* A neighborhood condition of $I$ is a pair $(c, d) \in I \times \overline{I}$.

**Lemma (5.5)**

$$(C \oplus Y)' \leq_T Y'.$$

*Proof.* Let $\Phi^{I \oplus Y'} = (C \oplus Y)'$.

**Claim.** For each $i \in I$, there is a $(j, j', j'') \in I \times I \times I$ such that (i) (Correctness) any $(c, d)$, where $c \leq j < j' \leq d$, and $Y' \rhd g(j'')$ used as oracle, computes $(C \oplus Y)' \rhd g(i)$ correctly (if defined), and (ii) (Existence) there is a $c \leq j$ and $d > j'$ such that $\Phi^{\langle c, d \rangle \oplus Y' \mid g(j'')} \rhd g(i) = (C \oplus Y)' \rhd g(i)$. 
Setting $X = C \oplus Y$ and applying Lemmas 5.2 and 5.3 yields

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Setting $X = C \oplus Y$ and applying Lemmas 5.2 and 5.3 yields

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$$(C \oplus Y)' \leq_T I \oplus Y'.$$

*Note.* A neighborhood condition of $I$ is a pair $(c, d) \in I \times \bar{I}$.

**Lemma (5.5)**

$$(C \oplus Y)' \leq_T Y'.$$

**Proof.** Let $\Phi^{I \oplus Y'} = (C \oplus Y)'$.

**Claim.** For each $i \in I$, there is a $(j, j', j'') \in I \times I \times I$ such that (i) (Correctness) any $(c, d)$, where $c \leq j < j' \leq d$, and $Y' \upharpoonright g(j'')$ used as oracle, computes $(C \oplus Y)' \upharpoonright g(i)$ correctly (if defined), and (ii) (Existence) there is a $c \leq j$ and $d > j'$ such that $\phi^{(c,d) \oplus Y' \upharpoonright g(j'')} \upharpoonright g(i) = (C \oplus Y)' \upharpoonright g(i)$.
Setting $X = C \oplus Y$ and applying Lemmas 5.2 and 5.3 yields

**Lemma (5.4)**

$$(C \oplus Y)' \leq_T I \oplus Y'.$$

*Note.* A neighborhood condition of $I$ is a pair $(c, d) \in I \times \bar{I}$.

**Lemma (5.5)**

$$(C \oplus Y)' \leq_T Y'.$$

**Proof.** Let $\Phi^{I \oplus Y'} = (C \oplus Y)'$.  

**Claim.** For each $i \in I$, there is a $(j, j', j'') \in I \times I \times I$ such that (i) (Correctness) any $(c, d)$, where $c \leq j < j' \leq d$, and $Y' \upharpoonright g(j'')$ used as oracle, computes $(C \oplus Y)' \upharpoonright g(i)$ correctly (if defined), and (ii) (Existence) there is a $c \leq j$ and $d > j'$ such that $\Phi^{\langle c, d \rangle \oplus Y' \upharpoonright g(j'')} \upharpoonright g(i) = (C \oplus Y)' \upharpoonright g(i)$. 
Otherwise,

- There is an \( i \in I \) such that for all \( (j, j', j'') \in I \times I \times I \), (i) or (ii) is false.

- (ii) is true since there exist \( c = j_0 < j'_0 < d \in \bar{I} \) and \( j''_0 \) such that \( \langle c, d \rangle \oplus Y' \upharpoonright g(j''_0) \) computes \( (C \oplus Y)' \upharpoonright g(i) \) correctly. Hence (i) is false, i.e.

- For all \( j' > j_0 \) in \( I \) there is a \( (c, d) \) with \( c \leq j_0 < j' \leq d \) and \( \Phi \langle c, d \rangle \oplus Y' \upharpoonright g(j''_0) \upharpoonright g(i) \neq (C \oplus Y)' \upharpoonright g(i) \).

- Thus \( j' \in \bar{I} \) if and only if there is no \( c \leq j_0 < j' \leq d \) such that \( \Phi \langle c, d \rangle \oplus Y' \upharpoonright g(j''_0) \upharpoonright g(i) \neq (C \oplus Y)' \upharpoonright g(i) \). So \( \bar{I} \) is \( \Delta^0_1(Y') \), hence \( M \)-finite, contradiction.
Otherwise,

- There is an \( i \in I \) such that for all \( (j, j', j'') \in I \times I \times I \), (i) or (ii) is false.

- (ii) is true since there exist \( c = j_0 < j'_0 < d \in \bar{I} \) and \( j''_0 \) such that \( \langle c, d \rangle \oplus Y' \upharpoonright g(j''_0) \) computes \( (C \oplus Y)' \upharpoonright g(i) \) correctly. Hence (i) is false, i.e.

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Thus \( j' \in \bar{I} \) if and only if there is no \( c \leq j_0 < j' \leq d \) such that \( \Phi^{\langle c, d \rangle \oplus Y' \upharpoonright g(j''_0)} \upharpoonright g(i) \neq (C \oplus Y)' \upharpoonright g(i) \). So \( \bar{I} \) is \( \Delta^0_1(Y') \), hence \( \mathcal{M} \)-finite, contradiction.
Otherwise,

- There is an \( i \in I \) such that for all \( (j, j', j'') \in I \times I \times I \), (i) or (ii) is false.

- (ii) is true since there exist \( c = j_0 < j'_0 < d \in \overline{I} \) and \( j''_0 \) such that \( \langle c, d \rangle \oplus Y' \upharpoonright g(j''_0) \) computes \( (C \oplus Y)' \upharpoonright g(i) \) correctly. Hence (i) is false, i.e.

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Sets Recursive in $Y''$

The set

$$Z = \{ \langle i, (j, j', j'') \rangle \mid \langle i, (j, j', j'') \rangle \text{ satisfies (i) and (ii)} \}$$

is $\Delta^0_2(C \oplus Y)$ on $I \times I \times I \times I$, hence coded by an $\mathcal{M}$-finite set $\hat{Z}$. Now $Y'$ may use the code to compute $(C \oplus Y)'$ from $Y'$ via $\Phi$.

This proves Lemma 5.5
The set

\[ Z = \{ \langle i, (j, j', j'') \rangle \mid \langle i, (j, j', j'') \rangle \text{ satisfies (i) and (ii)} \} \]

is $\Delta^0_2(C \oplus Y)$ on $I \times I \times I \times I$, hence coded by an $\mathcal{M}$-finite set $\hat{Z}$. Now $Y'$ may use the code to compute $(C \oplus Y)'$ from $Y'$ via $\Phi$.

This proves Lemma 5.5
Degrees Below $Y''$

\[ I = \Sigma_1(Y') \text{ cut} \]
Lemma 5.5 implies Theorem 5.2. Together with the following lemma, it shows that there is an array $R \in X$ with no $R$-cohesive set below $R''$ in the model, proving Theorem 5.1.

**Lemma (5.6)**

*There is an array $R \in X$ such that no $R$-cohesive set is low relative to $Y$.***

**Proof.** Let $\Phi_Y$ be a $\{0, 1\}$-valued partial function with no extension to a $Y'$-recursive total function. Let $h(s, t) = m$ if $\Phi^Y_t(s) \downarrow = m$, and equal to 0 otherwise. $h$ is $Y$-recursive. Let $R_s = \{ t | h(s, t) = 1 \}$. If $C$ is $R$-cohesive and $C' \leq_T Y'$, then $\hat{h}(s) = \lim_{t \in C} h(s, t)$ exists and total as well as $\{0, 1\}$-valued and $Y'$-recursive, contradicting assumption.
Lemma 5.5 implies Theorem 5.2. Together with the following lemma, it shows that there is an array $R \in \mathbb{X}$ with no $R$-cohesive set below $R''$ in the model, proving Theorem 5.1.

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Lemma 5.5 implies Theorem 5.2. Together with the following lemma, it shows that there is an array $R \in \mathbb{X}$ with no $R$-cohesive set below $R''$ in the model, proving Theorem 5.1.

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There is an array $R \in \mathbb{X}$ such that no $R$-cohesive set is low relative to $Y$.

**Proof.** Let $\Phi^{Y'}$ be a $\{0, 1\}$-valued partial function with no extension to a $Y'$-recursive total function. Let $h(s, t) = m$ if $\Phi_t^{Y'}(s) \downarrow = m$, and equal to 0 otherwise. $h$ is $Y$-recursive. Let $R_s = \{t|h(s, t) = 1\}$. If $C$ is $R$-cohesive and $C' \leq_T Y'$, then $\hat{h}(s) = \lim_{t \in C} h(s, t)$ exists and total as well as $\{0, 1\}$-valued and $Y'$-recursive, contradicting assumption.
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Double Jump Basis in $\text{RCA}_0 + B\Sigma^0_2$ Models

Same conclusion for $\text{RT}^2_2$ holds:

**Theorem (5.3)**

*If $\mathcal{M}$ is a model of $\text{RCA}_0 + B\Sigma^0_2$ with $\Sigma_2$ cut $I$, then $\mathcal{M}$ is not a double jump basis for $\text{RT}^2_2$.***

*Proof.* Again fix $Y$ as before. By Jockusch [1972] there is a $Y$-recursive 2-coloring $f$ of pairs with no $H_f \leq_T Y'$. Then in the model $\mathcal{M}$, there is no $H_f \leq_T Y''$.

**Corollary**

The following are equivalent:

- $\mathcal{M} \models \text{RCA}_0 + I\Sigma^0_2$.
- $\mathcal{M}$ is an $M$-submodel of $\mathcal{M}^* \models \text{RCA}_0 + I\Sigma^0_2$ and $\mathcal{M}^*$ is a double-jump basis for $\text{RT}^2_2$. 
Double Jump Basis in $\text{RCA}_0 + B\Sigma^0_2$ Models

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**Theorem (5.3)**

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Double Jump Basis in $\text{RCA}_0 + B\Sigma^0_2$ Models

Same conclusion for $\text{RT}_2^2$ holds:

**Theorem (5.3)**

*If $\mathcal{M}$ is a model of $\text{RCA}_0 + B\Sigma^0_2$ with $\Sigma_2$ cut $I$, then $\mathcal{M}$ is not a double jump basis for $\text{RT}_2^2$.***

**Proof.** Again fix $Y$ as before. By Jockusch [1972] there is a $Y$-recursive 2-coloring $f$ of pairs with no $H_f \leq_T Y'$. Then in the model $\mathcal{M}$, there is no $H_f \leq_T Y''$.

**Corollary**

*The following are equivalent:*

- $\mathcal{M} \models \text{RCA}_0 + I\Sigma^0_2$.
- $\mathcal{M}$ is an $M$-submodel of $\mathcal{M}^*$ \models $\text{RCA}_0 + I\Sigma^0_2$ and $\mathcal{M}^*$ is a double-jump basis for $\text{RT}_2^2$.***
Same conclusion for $\text{RT}_2^2$ holds:

**Theorem (5.3)**

If $\mathcal{M}$ is a model of $\text{RCA}_0 + B\Sigma^0_2$ with $\Sigma_2$ cut $I$, then $\mathcal{M}$ is not a double jump basis for $\text{RT}_2^2$.

**Proof.** Again fix $Y$ as before. By Jockusch [1972] there is a $Y$-recursive 2-coloring $f$ of pairs with no $H_f \leq_T Y'$. Then in the model $\mathcal{M}$, there is no $H_f \leq_T Y''$.

**Corollary**

The following are equivalent:

- $\mathcal{M} \models \text{RCA}_0 + I\Sigma^0_2$.
- $\mathcal{M}$ is an $\mathcal{M}$-submodel of $\mathcal{M}^* \models \text{RCA}_0 + I\Sigma^0_2$ and $\mathcal{M}^*$ is a double-jump basis for $\text{RT}_2^2$. 
Session V:

L’estremità