Recursion Theory of Ramsey's Theorem

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Session V

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Σ^0_2 Bounding $B\Sigma^0_2$

Lemma (5.1)

(Hirst [1987]) $\mathrm{RCA}_0 \vdash \mathrm{RT}^2_1 \rightarrow B\Sigma^0_2$.

Proof. Let $h : X \rightarrow M$ be a $\Sigma^0_2(\mathcal{M})$ function total on an $\mathcal{M}$-finite set X in a model $\mathcal{M}$ of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$.
Assume $f(x) = y \leftrightarrow \exists u \forall v \varphi(u, v, x, y)$ where φ is $\Delta^0_0(\mathcal{M})$.
Suppose for the sake of contradiction that the image is not bounded. For each s, let x_s be the least $x \in X$ such that for all $y, u \leq s$, there is a v such that $\neg\varphi(u, v, x, y)$. $B\Sigma^0_2$ implies that x_s exists for all s.
Let $f(s, t) = 1$ if $x_s = x_t$, and 0 otherwise. Then since X is $\mathcal{M}$-finite, by pigeonhole principle any H_f satisfies $f([H_f]^2) = 1$. Let x be such that $x = x_s$ for all $s \in H_f$. Then $f(x) \uparrow$, a contradiction.

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Strength of RT^2_2

Questions.

- Is $\mathsf{RCA}_0 + \mathsf{RT}^2_2 + B\Sigma^0_2$ Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2$?
- Is $\mathsf{RCA}_0 + \mathsf{COH} + B\Sigma^0_2$ Π^1_1-conservative over $\mathsf{RCA}_0 + B\Sigma^0_2$?
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Working Over $B\Sigma^0_2$ Models

Fact. Every model $\mathcal{M}$ of $\mathsf{RCA}_0 + I\Sigma^0_2$ is an M-submodel of an $\mathcal{M}^*$ such that

- Any array $R \in \mathcal{M}^*$ has an R-cohesive set recursive in R''.
- Any 2-coloring of pairs $f \in \mathcal{M}^*$ has an $H_f \leq_T f''$.

Indeed for ω-models $\mathcal{M}$, low$_2$ solutions exist in $\mathcal{M}^*$ (Jockusch, Stephan, Cholak, Slaman): There is a low$_2$ r-cohesive set, and every 2-coloring of pairs has a homogeneous set low$_2$ in the code of the coloring.

These fail in models of $\mathsf{RCA}_0 + B\Sigma^0_2$ without $I\Sigma^0_2$.

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Double Jump Basis in $\mathrm{RCA}_0 + B\Sigma^0_2$ Models

Definition

$\mathcal{M} \models \mathrm{RCA}_0 + B\Sigma^0_2$ is a *double-jump* basis for φ if there is a solution to φ in $\mathcal{M}$ recursive in the double jump of the parameters in φ.

Theorem (5.1)

(Chong [2006]) *No model of* $\mathrm{RCA}_0 + \mathrm{COH} + B\Sigma^0_2$ *without* $I\Sigma^0_2$ *has a double-jump basis for* COH.

Corollary

The following are equivalent:

- $\mathcal{M} \models \mathrm{RCA}_0 + I\Sigma^0_2$.
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Regularity and Hyperregularity

From now on $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ always denotes a model of $\mathsf{RCA}_0 + B\Sigma^0_2 + \neg I\Sigma^0_2$ with $\Sigma^0_2(\mathcal{M})$ cut I. Let $g : I \to M$ be $\Sigma^0_2(\mathcal{M})$, increasing and cofinal.

Definition

$X \subset M$ is *regular* if $X \upharpoonright b$ is $\mathcal{M}$-finite for all b.

Definition

$X \subset M$ is *hyperregular* if every function $\Delta^0_1(\mathcal{M})$ in X maps a bounded set into a bounded set.

Note. Any set X in $\mathbb{X}$ is regular, hyperreular and satisfies $B\Sigma^0_1(X')$.

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Degrees Below Y''

Fix $Y \in \mathbb{X}$ so that the graph of g is r.e. in Y'.

Theorem (5.2)

If $C \leq_T Y''$, then $C' <_T Y'$, i.e. C is Y-low.

Lemma (5.2)

$(C \oplus Y)'$ is regular.

Proof. $C \oplus Y \in \mathbb{X}$ hence regular and hyperregular. Thus for each $i \in I$ there is a $j \in I$ such that $(C \oplus Y)' \upharpoonright g(i) = (C \oplus Y)'_{g(j)} \upharpoonright g(i)$. Hence $(C \oplus Y)'$ is regular.

Recall: (Chong and Mourad [1991]) In a model of $\mathrm{RCA}_0 + B\Sigma^0_2$ without $I\Sigma^0_2$, every subset of I that is $\Delta^0_2(\mathcal{M})$ on I is coded on I.

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Fact. $B\Sigma^0_1(Y')$ implies: If $D \subset Y''$ is $\mathcal{M}$-finite, then there is an $i_D \in \omega$ such that $D \subset Y''_{g(i_D)}$.

Lemma (5.3)

If $X \leq_T Y''$ is regular, then $X \leq_T I \oplus Y'$.

Proof. Let $\Phi^{Y''} = X$. Since X is regular, for each $i \in I$, there is a neighborhood condition $\langle P, N \rangle$ of Y'' ($P \subset Y''$ and $N \subset \bar{Y}''$) so that $\Phi^{\langle P,N \rangle} \upharpoonright g(i) = X \upharpoonright g(i)$. There is a least $(j, j') \in I$, denoted (j_i, j'_i), such that $P \subset Y''_{g(j)}$ and any $N^- \subset \bar{Y}''_{g(j)} \cap \bar{Y}''_{g(j')}$ satisfies $\Phi^{\langle P,N^- \rangle}_{g(j')} \upharpoonright g(i) = X \upharpoonright g(i)$. The set $Z = \{(i, j, j') | \forall N^- \subset \bar{Y}''_{g(j)} [N^- \cap Y''_{g(j')} \neq \emptyset]\}$ is $\Delta^0_2(\mathcal{M})$ on $I \times I$ hence coded on $I \times I$. Then $X \upharpoonright g(i) = \Phi^{\langle Y''_{g(j_i)}, \bar{Y}''_{g(j_i)} \rangle}_{g(j'_i)} \upharpoonright g(i)$, and $i \mapsto (j_i, j'_i$ is recursive in I via the code. proving the lemma.

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Setting $X = C \oplus Y$ and applying Lemmas 5.2 and 5.3 yields

Lemma (5.4)

$(C \oplus Y)' \leq_T I \oplus Y'$.

Note. A neighborhood condition of I is a pair $(c, d) \in I \times \overline{I}$.

Lemma (5.5)

$(C \oplus Y)' \leq_T Y'$.

Proof. Let $\Phi^{I \oplus Y'} = (C \oplus Y)'$.
Claim. For each $i \in I$, there is a $(j, j', j'') \in I \times I \times I$ such that (i) (Correctness) any (c, d), where $c \leq j < j' \leq d$, and $Y' \upharpoonright g(j'')$ used as oracle, computes $(C \oplus Y)' \upharpoonright g(i)$ correctly (if defined), and (ii) (Existence) there is a $c \leq j$ and $d > j'$ such that $\Phi^{\langle c,d \rangle \oplus Y' \upharpoonright g(j'')} \upharpoonright g(i) = (C \oplus Y)' \upharpoonright g(i)$.

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Note. A neighborhood condition of I is a pair $(c, d) \in I \times \bar{I}$.

Lemma (5.5)

$(C \oplus Y)' \leq_T Y'$.

Proof. Let $\Phi^{I \oplus Y'} = (C \oplus Y)'$.

Claim. For each $i \in I$, there is a $(j, j', j'') \in I \times I \times I$ such that (i) (Correctness) any (c, d), where $c \leq j < j' \leq d$, and $Y' \upharpoonright g(j'')$ used as oracle, computes $(C \oplus Y)' \upharpoonright g(i)$ correctly (if defined), and (ii) (Existence) there is a $c \leq j$ and $d > j'$ such that $\Phi^{\langle c,d \rangle \oplus Y' \upharpoonright g(j'')} \upharpoonright g(i) = (C \oplus Y)' \upharpoonright g(i)$.

Degrees Below Y''

Setting $X = C \oplus Y$ and applying Lemmas 5.2 and 5.3 yields

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Degrees Below Y''

Otherwise,

- There is an $i \in I$ such that for all $(j, j', j'') \in I \times I \times I$, (i) or (ii) is false.
- (ii) is true since there exist $c = j_0 < j_0' < d \in \bar{I}$ and j_0'' such that $\langle c, d \rangle \oplus Y' \upharpoonright g(j_0'')$ computes $(C \oplus Y)' \upharpoonright g(i)$ correctly. Hence (i) is false, i.e.
- For all $j' > j_0$ in I there is a (c, d) with $c \leq j_0 < j' \leq d$ and $\Phi^{\langle c,d \rangle \oplus Y' \upharpoonright g(j_0'')} \upharpoonright g(i) \neq (C \oplus Y)' \upharpoonright g(i)$.
- Thus $j' \in \bar{I}$ if and only if there is no $c \leq j_0 < j' \leq d$ such that $\Phi^{\langle c,d \rangle \oplus Y' \upharpoonright g(j_0'')} \upharpoonright g(i) \neq (C \oplus Y)' \upharpoonright g(i)$. So $\bar{I}$ is $\Delta^0_1(Y')$, hence $\mathcal{M}$-finite, contradiction.

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Sets Recursive in Y''

The set

$$Z = \{\langle i, (j, j', j'')\rangle | \langle i, (j, j', j'')\rangle \text{ satisfies (i) and (ii)}\}$$

is $\Delta^0_2(C \oplus Y)$ on $I \times I \times I \times I$, hence coded by an $\mathcal{M}$-finite set $\hat{Z}$. Now Y' may use the code to compute $(C \oplus Y)'$ from Y' via Φ.

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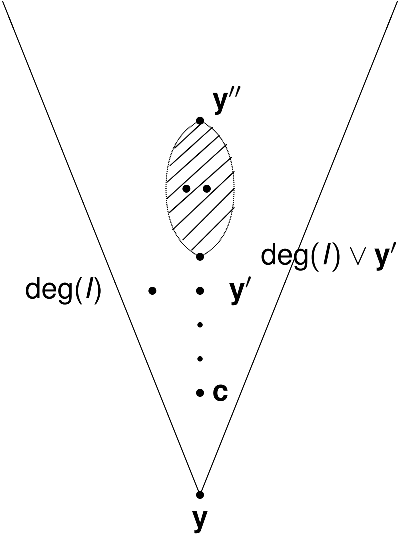

$I = \Sigma_1(Y')$ cut

Double Jump Basis in $\mathrm{RCA}_0 + B\Sigma^0_2$ Models

Lemma 5.5 implies Theorem 5.2. Together with the following lemma, it shows that there is an array $R \in \mathbb{X}$ with no R-cohesive set below R'' in the model, proving Theorem 5.1.

Lemma (5.6)

There is an array $R \in \mathbb{X}$ such that no R-cohesive set is low relative to Y.

Proof. Let $\Phi^{Y'}$ be a $\{0,1\}$-valued partial function with no extension to a Y'-recursive total function.
Let $h(s,t) = m$ if $\Phi^{Y'_t}_t(s) \downarrow = m$, and equal to 0 otherwise.
h is Y-recursive. Let $R_s = \{t | h(s,t) = 1\}$. If C is R-cohesive and $C' \leq_T Y'$, then $\hat{h}(s) = \lim_{t \in C} h(s,t)$ exists and total as well as $\{0,1\}$-valued and Y'-recursive, contradicting assumption.

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Double Jump Basis in $\mathrm{RCA}_0 + B\Sigma^0_2$ Models

Same conclusion for RT^2_2 holds:

Theorem (5.3)

If $\mathcal{M}$ is a model of $\mathrm{RCA}_0 + B\Sigma^0_2$ with Σ_2 cut I, then $\mathcal{M}$ is not a double jump basis for RT^2_2.

Proof. Again fix Y as before. By Jockusch [1972] there is a Y-recursive 2-coloring f of pairs with no $H_f \leq_T Y'$. Then in the model $\mathcal{M}$, there is no $H_f \leq_T Y''$.

Corollary

The following are equivalent:

- $\mathcal{M} \models \mathrm{RCA}_0 + I\Sigma^0_2$.
- *$\mathcal{M}$ is an M-submodel of $\mathcal{M}^* \models \mathrm{RCA}_0 + I\Sigma^0_2$ and $\mathcal{M}^*$ is a double-jump basis for RT^2_2.*

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Session V:

L’estremità