Recursion Theory of Ramsey's Theorem

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COH in $B\Sigma^0_2$ Models

Definition

$\mathcal{M} \models \mathrm{RCA}_0$ is a $B\Sigma^0_2$ model if it satisfies $B\Sigma^0_2$ but not $I\Sigma^0_2$.

Theorem (6.1)

(Chong, Slaman and Yang [2006]) $\mathrm{RCA}_0 + \mathrm{COH} + B\Sigma^0_2$ *is* Π^1_1*-conservative over* $\mathrm{RCA}_0 + B\Sigma^0_2$.

By Löwenheim-Skolem, this is a consequence of

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Building a Y'-recursive Tree of R-cohesive Paths

Let $\mathcal{M}$ be a $B\Sigma^0_2$ model and fix $Y \in \mathcal{M}$ so that I is a $\Sigma^0_2(Y)$ cut and $g : I \to M$ is $\Sigma^0_2(Y)$, cofinal and increasing.

Lemma (6.1)

Let $R \in \mathbb{X}$ be an array. There is an $\mathcal{M}$-infinite Y'-recursive tree T such that every $\mathcal{M}$-infinite path G on T is R-cohesive and generalized low relative to Y, i.e. $G' \leq_T G \oplus Y'$.

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Proof. We may assume that $R \leq_T Y$. Let $R_s = \{s | (s,t) \in R\}$.

- $\nu \in 2^{g(0)}$ is *fulfilled* if there is a σ such that (i) $\sigma(x) = 1$ for some $x \leq \mathrm{lth}(\sigma)$ and (ii) $\forall x \leq \mathrm{lth}(\sigma) \sigma(x) = 1$ if and only if $x \in \bigcap_{\nu(i)=1} R_i \cap \bigcap_{\nu(i)=0} \bar{R}_i$.
- If ν is fulfilled, let σ_ν be the least σ that witnesses it. Let $F_0 = \{\sigma_\nu | \nu \in 2^{g(0)}\}$.
- Let $C_0[\sigma_\nu] = \{\sigma \geq \sigma_\nu | \sigma \subset \bigcap_{\nu(i)=1} R_i \cap \bigcap_{\nu(i)=0} \bar{R}_i\}$.
- Let $C_0 = \bigcup_{\sigma_\nu \in F_0} C_0[\sigma_\nu]$.
- **Claim 1.** C_0 is $\mathcal{M}$-infinite.
- Otherwise, let b be the least upper bound and let $\sigma = \bigcap_{\nu(i)=1} R_i \cap \bigcap_{\nu(i)=0} \bar{R}_i \upharpoonright s_0$ for some $\nu \in F_0$. Define μ recursively in Y':

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Internal Forcing

- Let $\mu(0) = 1$ if R_0 has at least $2^{g(0)} g(0)$ elements above b, and 0 otherwise. If $\mu(s)$ is defined and $S = \bigcap_{\mu(t)=1} R_r \cap \bigcap_{\mu(t)=0} \bar{R}_t$ has at least $2^{g(0)-s} g(0)$ elements above b, let $\mu(s+1) = 1$ if $S \cap R_{s+1}$ has at least $2^{g(0)-(s+1)}$ elements above b, and 0 otherwise.
- $\mu \in 2^{g(0)}$ is $\mathcal{M}$-finite and $\bigcap_{\mu(t)=1} R_t \cap \bigcap_{\mu(t)=0} \bar{R}_t$ has at least $g(0)$ elements above b. Then σ_μ shows b is not least upper bound.
- At least one $C_0[\sigma_\nu]$ is $\mathcal{M}$-infinite.
- **Claim 2.** *Forcing the jump of generic paths.* Each $C_0[\sigma_\nu]$, if $\mathcal{M}$-infinite, contains a $\sigma_\nu^* \geq \sigma_\nu$ such that for all $e \leq g(0)$, either (i): $\Phi_e^{\sigma_\nu^*}(e)\downarrow$, or (ii): no $\sigma \geq \sigma_\nu^*$ in $C_0[\sigma_\nu]$ satisfies $\Phi_e^{\sigma}(e)\downarrow$.

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- Sequentially construct σ_e, $e \leq g(0)$, to force either (i) or (ii) in $C_0[\sigma_\nu]$. If $C_0[\sigma_\nu]$ is $\mathcal{M}$-infinite, and the sequence, which is Y'-recursive, is not total on $g(0)$, then it is defined on a $\Sigma^0_2(\mathcal{M})$ cut J.
- The set $\{e|\sigma_e$ satisfies (i)$\}$ is $\Delta^0_2(Y)$ on J and therefore coded on J.
- If $\bigcup_{e\in J}\sigma_e$ is unbounded, then the code provides a Y-recursive way of computing σ_e, $e \in J$, contradicting $I\Sigma^0_1(Y)$.
- Thus if $C_0[\sigma_\nu]$ is $\mathcal{M}$-infinite, σ^*_ν is defined.
- Hence Y' decides recursively whether $C_0[\sigma_\nu]$ is $\mathcal{M}$-finite or σ^*_ν exists. $B\Sigma^0_1(Y')$ allows Y to choose an s_0 such that $\mathrm{lth}(\sigma^*_\nu) \leq s_0$ whenever it is defined.

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- Let T_0 be the set of all σ_ν^*'s (this is an $\mathcal{M}$-finite set).
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- Inductively get s_j, T_j and C_j. Let $J^* = \{j | T_j \text{ is defined}\}$ (a $\Sigma_1^0(Y')$ cut). Let $T = \bigcup_{j \in J^*} T_j$.
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External Forcing

Lemma (6.2)

Let $\mathcal{M}$ be countable $B\Sigma^0_2$ model and T an $\mathcal{M}$-infinite Y'-recursive tree in which every $\mathcal{M}$-infinite path G satisfies $G' \leq_T G \oplus Y'$. Then T has an $\mathcal{M}$-infinite path such that $\mathcal{M}[G] \models B\Sigma^0_2$.

Proof. Let $\langle \exists x \varphi_n \rangle_{n \in \omega}$ be a countable list of Σ^0_1 formulas with parameters, where φ_n is Δ^0_0. Let $\langle D_n \rangle_{n \in \omega}$ be a list of $\mathcal{M}$-finite sets. Define a nested sequence of trees U_n as follows:

- For $c \in D_0$, consider

$$X_{c,0} = \{\sigma \in T | \forall x \leq \text{lth}(\sigma) \neg \varphi_0(x, c, \sigma \oplus Y')\}.$$

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- Otherwise, by $B\Sigma_2^0$ in $\mathcal{M}$, there is an s_0 where for all $c \in D_0$ and all $\tau \in T$ of length at least s_0, $\exists x \varphi_0(x, c, \tau \oplus Y')$ holds. Let σ_0 be a string of length at least s_0 so that $T[\sigma_0] = \{\tau \in T | \tau \geq \sigma_0\}$ is $\mathcal{M}$-infinite. Let $U_0 = T[\sigma_0]$.
- Define $\sigma_{n+1} \geq \sigma_n$ and U_{n+1} from U_n similarly, replacing T by U_n and φ_0 by φ_{n+1} in the respective definitions.
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An iteration of Lemmas 6.1 and 6.2 yields Theorem 6.2 and therefore Theorem 6.1.

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- If $X_{c,0}$ is $\mathcal{M}$-infinite for some $c \in D_0$, let U_0 be one such $X_{c,0}$ and set $\sigma_0 = \emptyset$. Then $B\Sigma^0_1(G')$ fails for $\exists x \varphi_0$ on D_0, for any G that is a path in U_0.
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Session VI:

L’estremità