Recursion Theory of Ramsey’s Theorem

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COH in $B\Sigma^0_2$ Models

**Definition**

$\mathcal{M} \models \text{RCA}_0$ is a $B\Sigma^0_2$ model if it satisfies $B\Sigma^0_2$ but not $I\Sigma^0_2$.

**Theorem (6.1)**

(Chong, Slaman and Yang [2006]) $\text{RCA}_0 + \text{COH} + B\Sigma^0_2$ is $\Pi^1_1$-conservative over $\text{RCA}_0 + B\Sigma^0_2$.

By Löwenheim-Skolem, this is a consequence of

**Theorem (6.2)**

*Every countable model $\mathcal{M}$ of $\text{RCA}_0 + B\Sigma^0_2$ is an $\mathcal{M}$-submodel of one satisfying COH.*
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*Every countable model \( \mathcal{M} \) of \( \text{RCA}_0 + B\Sigma^0_2 \) is an \( M \)-submodel of one satisfying \( \text{COH} \).*
Building a $Y'$-recursive Tree of $R$-cohesive Paths

Let $\mathcal{M}$ be a $B\Sigma^0_2$ model and fix $Y \in \mathcal{M}$ so that $I$ is a $\Sigma^0_2(Y)$ cut and $g : I \rightarrow \mathcal{M}$ is $\Sigma^0_2(Y)$, cofinal and increasing.

Lemma (6.1)

Let $R \in \mathbb{X}$ be an array. There is an $\mathcal{M}$-infinite $Y'$-recursive tree $T$ such that every $\mathcal{M}$-infinite path $G$ on $T$ is $R$-cohesive and generalized low relative to $Y$, i.e. $G' \leq_T G \oplus Y'$. 
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**Proof.** We may assume that $R \leq_T Y$. Let $R_s = \{ s \mid (s, t) \in R \}$.

- $\nu \in 2^{g(0)}$ is *fulfilled* if there is a $\sigma$ such that (i) $\sigma(x) = 1$ for some $x \leq \text{lth}(\sigma)$ and (ii) $\forall x \leq \text{lth}(\sigma) \sigma(x) = 1$ if and only if $x \in \bigcap_{\nu(i) = 1} R_i \cap \bigcap_{\nu(i) = 0} \bar{R}_i$.
- If $\nu$ is fulfilled, let $\sigma_\nu$ be the least $\sigma$ that witnesses it. Let $F_0 = \{ \sigma_\nu \mid \nu \in 2^{g(0)} \}$.
- Let $C_0[\sigma_\nu] = \{ \sigma \geq \sigma_\nu \mid \sigma \subset \bigcap_{\nu(i) = 1} R_i \cap \bigcap_{\nu(i) = 0} \bar{R}_i \}$.
- Let $C_0 = \bigcup_{\sigma_\nu \in F_0} C_0[\sigma_\nu]$.
- **Claim 1.** $C_0$ is $\mathcal{M}$-infinite.
- Otherwise, let $b$ be the least upper bound and let $\sigma = \bigcap_{\nu(i) = 1} R_i \cap \bigcap_{\nu(i) = 0} \bar{R}_i \upharpoonright s_0$ for some $\nu \in F_0$. Define $\mu$ recursively in $Y'$:
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Internal Forcing

Let $\mu(0) = 1$ if $R_0$ has at least $2^{g(0)}g(0)$ elements above $b$, and $0$ otherwise. If $\mu(s)$ is defined and $S = \bigcap_{\mu(t)=1} R_t \cap \bigcap_{\mu(t)=0} \bar{R}_t$ has at least $2^{g(0) - s}g(0)$ elements above $b$, let $\mu(s + 1) = 1$ if $S \cap R_{s+1}$ has at least $2^{g(0) - (s+1)}$ elements above $b$, and $0$ otherwise.

$\mu \in 2^{g(0)}$ is $\mathcal{M}$-finite and $\bigcap_{\mu(t)=1} R_t \cap \bigcap_{\mu(t)=0} \bar{R}_t$ has at least $g(0)$ elements above $b$. Then $\sigma_\mu$ shows $b$ is not least upper bound.

At least one $C_0[\sigma_\nu]$ is $\mathcal{M}$-infinite.

Claim 2. Forcing the jump of generic paths. Each $C_0[\sigma_\nu]$, if $\mathcal{M}$-infinite, contains a $\sigma^*_\nu \geq \sigma_\nu$ such that for all $e \leq g(0)$, either (i): $\Phi_{\mu^*_\nu}(e) \downarrow$, or (ii): no $\sigma \geq \sigma^*_\nu$ in $C_0[\sigma_\nu]$ satisfies $\Phi_{\mu}(e) \downarrow$. 

Let $\mu(0) = 1$ if $R_0$ has at least $2^{g(0)} g(0)$ elements above $b$, and 0 otherwise. If $\mu(s)$ is defined and $S = \bigcap_{\mu(t) = 1} R_t \cap \bigcap_{\mu(t) = 0} \bar{R}_t$ has at least $2^{g(0)} - s g(0)$ elements above $b$, let $\mu(s + 1) = 1$ if $S \cap R_{s+1}$ has at least $2^{g(0)} - (s+1)$ elements above $b$, and 0 otherwise.

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- Let $\mu(0) = 1$ if $R_0$ has at least $2^{g(0)}g(0)$ elements above $b$, and 0 otherwise. If $\mu(s)$ is defined and $S = \bigcap_{\mu(t)=1} R_t \cap \bigcap_{\mu(t)=0} \bar{R}_t$ has at least $2^{g(0)}-s g(0)$ elements above $b$, let $\mu(s+1) = 1$ if $S \cap R_{s+1}$ has at least $2^{g(0)}-(s+1)$ elements above $b$, and 0 otherwise.

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Internal Forcing

- Sequentially construct $\sigma_e, e \leq g(0)$, to force either (i) or (ii) in $C_0[\sigma_\nu]$. If $C_0[\sigma_\nu]$ is $\mathcal{M}$-infinite, and the sequence, which is $Y'$-recursive, is not total on $g(0)$, then it is defined on a $\Sigma^0_2(\mathcal{M})$ cut $J$.

- The set $\{e | \sigma_e \text{ satisfies (i)}\}$ is $\Delta^0_2(Y)$ on $J$ and therefore coded on $J$.

- If $\bigcup_{e \in J} \sigma_e$ is unbounded, then the code provides a $Y$-recursive way of computing $\sigma_e, e \in J$, contradicting $I\Sigma^0_1(Y)$.

- Thus if $C_0[\sigma_\nu]$ is $\mathcal{M}$-infinite, $\sigma^*_\nu$ is defined.

- Hence $Y'$ decides recursively whether $C_0[\sigma_\nu]$ is $\mathcal{M}$-finite or $\sigma^*_\nu$ exists. $B\Sigma^0_1(Y')$ allows $Y$ to choose an $s_0$ such that $lth(\sigma^*_\nu) \leq s_0$ whenever it is defined.
Sequentially construct $\sigma_e, e \leq g(0)$, to force either (i) or (ii) in $C_0[\sigma_\nu]$. If $C_0[\sigma_\nu]$ is $\mathcal{M}$-infinite, and the sequence, which is $Y'$-recursive, is not total on $g(0)$, then it is defined on a $\Sigma^0_2(\mathcal{M})$ cut $J$.

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Inductively get $s_j$, $T_j$ and $C_j$. Let $J^* = \{j | T_j$ is defined\} (a $\Sigma^0_1(Y')$ cut). Let $T = \bigcup_{j \in J^*} T_j$.

$T$ is $Y'$-recursive, $\mathcal{M}$-infinite and every $\mathcal{M}$-infinite path on $T$ is $R$-cohesive.

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Let $T_0$ be the set of all $\sigma^*_\nu$’s (this is an $\mathcal{M}$-finite set).

Let $\sigma^*_\nu$ be defined. For $\mu \in 2^{g(1)}$ such that $\mu > \nu$, let $C_1[\sigma_\mu]$ be defined similar to $C_0[\sigma^*_\nu]$, but replacing $\nu$ by $\mu$. $C_1$ is defined in the obvious way. We then get $\sigma^*_\mu$ for appropriate $\mu \in 2^{g(1)}$, an upper bound $s_1$ for all such $\sigma^*_\mu$’s, and $T_1$ each of whose elements extends some string in $T_0$ and forces $\Phi^G_e(e)$ for every $e \leq \text{Max}\{s_0, g(1)\}$ and every generic $G$.

Inductively get $s_j$, $T_j$ and $C_j$. Let $J^* = \{j | T_j$ is defined} (a $\Sigma^0_1(Y’)$ cut). Let $T = \bigcup_{j \in J^*} T_j$.

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Every $\mathcal{M}$-infinite path $G$ in $T$ is generalized $Y$-low by construction.
Lemma (6.2)

Let \( \mathcal{M} \) be countable \( B\Sigma^0_2 \) model and \( T \) an \( \mathcal{M} \)-infinite \( Y' \)-recursive tree in which every \( \mathcal{M} \)-infinite path \( G \) satisfies \( G' \leq_T G \oplus Y' \). Then \( T \) has an \( \mathcal{M} \)-infinite path such that \( \mathcal{M}[G] \models B\Sigma^0_2 \).

Proof. Let \( \langle \exists x \varphi_n \rangle_{n \in \omega} \) be a countable list of \( \Sigma^0_1 \) formulas with parameters, where \( \varphi_n \) is \( \Delta^0_0 \). Let \( \langle D_n \rangle_{n \in \omega} \) be a list of \( \mathcal{M} \)-finite sets. Define a nested sequence of trees \( U_n \) as follows:

- For \( c \in D_0 \), consider
  \[
  X_{c,0} = \{ \sigma \in T | \forall x \leq \text{lth}(\sigma) \neg \varphi_0(x, c, \sigma \oplus Y') \}.
  \]
Lemma (6.2)

Let $\mathcal{M}$ be countable $B\Sigma^0_2$ model and $T$ an $\mathcal{M}$-infinite $Y'$-recursive tree in which every $\mathcal{M}$-infinite path $G$ satisfies $G' \leq^T G \oplus Y'$. Then $T$ has an $\mathcal{M}$-infinite path such that $\mathcal{M}[G] \models B\Sigma^0_2$.

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$$X_{c,0} = \{ \sigma \in T | \forall x \leq \text{lth}(\sigma) \neg \varphi_0(x, c, \sigma \oplus Y') \}.$$
If $X_{c,0}$ is $\mathcal{M}$-infinite for some $c \in D_0$, let $U_0$ be one such $X_{c,0}$ and set $\sigma_0 = \emptyset$. Then $B\Sigma_1^0(G')$ fails for $\exists x \varphi_0$ on $D_0$, for any $G$ that is a path in $U_0$.

Otherwise, by $B\Sigma_2^0$ in $\mathcal{M}$, there is an $s_0$ where for all $c \in D_0$ and all $\tau \in T$ of length at least $s_0$, $\exists x \varphi_0(x, c, \tau \oplus Y')$ holds. Let $\sigma_0$ be a string of length at least $s_0$ so that $T[\sigma_0] = \{\tau \in T|\tau \geq \sigma_0\}$ is $\mathcal{M}$-infinite. Let $U_0 = T[\sigma_0]$.

Define $\sigma_{n+1} \geq \sigma_n$ and $U_{n+1}$ from $U_n$ similarly, replacing $T$ by $U_n$ and $\varphi_0$ by $\varphi_{n+1}$ in the respective definitions.

Let $G = \bigcup_n \sigma_n$. Then $\mathcal{M}[G] \models B\Sigma_1^0[G \oplus Y']$, hence $B\Sigma_1^0[G']$ since $G' \leq_T G \oplus Y'$. Thus $\mathcal{M}[G] \models B\Sigma_2^0$.

An iteration of Lemmas 6.1 and 6.2 yields Theorem 6.2 and therefore Theorem 6.1.
If $X_{c,0}$ is $\mathcal{M}$-infinite for some $c \in D_0$, let $U_0$ be one such $X_{c,0}$ and set $\sigma_0 = \emptyset$. Then $\text{B}\Sigma^0_1(G')$ fails for $\exists x \varphi_0$ on $D_0$, for any $G$ that is a path in $U_0$.

Otherwise, by $\text{B}\Sigma^0_2$ in $\mathcal{M}$, there is an $s_0$ where for all $c \in D_0$ and all $\tau \in T$ of length at least $s_0$, $\exists x \varphi_0(x, c, \tau \oplus Y')$ holds. Let $\sigma_0$ be a string of length at least $s_0$ so that $T[\sigma_0] = \{\tau \in T | \tau \geq \sigma_0\}$ is $\mathcal{M}$-infinite. Let $U_0 = T[\sigma_0]$.

Define $\sigma_{n+1} \geq \sigma_n$ and $U_{n+1}$ from $U_n$ similarly, replacing $T$ by $U_n$ and $\varphi_0$ by $\varphi_{n+1}$ in the respective definitions.

Let $G = \bigcup_n \sigma_n$. Then $\mathcal{M}[G] \models \text{B}\Sigma^0_1[G \oplus Y']$, hence $\text{B}\Sigma^0_1[G']$ since $G' \leq_T G \oplus Y'$. Thus $\mathcal{M}[G] \models \text{B}\Sigma^0_2$.

An iteration of Lemmas 6.1 and 6.2 yields Theorem 6.2 and therefore Theorem 6.1.
If $X_{c,0}$ is $\mathcal{M}$-infinite for some $c \in D_0$, let $U_0$ be one such $X_{c,0}$ and set $\sigma_0 = \emptyset$. Then $B\Sigma^0_1(G')$ fails for $\exists x \varphi_0$ on $D_0$, for any $G$ that is a path in $U_0$.

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Let $G = \bigcup_n \sigma_n$. Then $\mathcal{M}[G] \models B\Sigma^0_1[G \oplus Y']$, hence $B\Sigma^0_1[G']$ since $G' \leq_T G \oplus Y'$. Thus $\mathcal{M}[G] \models B\Sigma^0_2$.

An iteration of Lemmas 6.1 and 6.2 yields Theorem 6.2 and therefore Theorem 6.1.
If $X_{c,0}$ is $\mathcal{M}$-infinite for some $c \in D_0$, let $U_0$ be one such $X_{c,0}$ and set $\sigma_0 = \emptyset$. Then $\Sigma^0_1(G')$ fails for $\exists x \varphi_0$ on $D_0$, for any $G$ that is a path in $U_0$.

Otherwise, by $\Sigma^0_2$ in $\mathcal{M}$, there is an $s_0$ where for all $c \in D_0$ and all $\tau \in T$ of length at least $s_0$, $\exists x \varphi_0(x, c, \tau \oplus Y')$ holds. Let $\sigma_0$ be a string of length at least $s_0$ so that $T[\sigma_0] = \{\tau \in T | \tau \geq \sigma_0\}$ is $\mathcal{M}$-infinite. Let $U_0 = T[\sigma_0]$.

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Let $G = \bigcup_n \sigma_n$. Then $\mathcal{M}[G] \models \Sigma^0_1[G \oplus Y']$, hence $\Sigma^0_1[G']$ since $G' \leq_T G \oplus Y'$. Thus $\mathcal{M}[G] \models \Sigma^0_2$.

An iteration of Lemmas 6.1 and 6.2 yields Theorem 6.2 and therefore Theorem 6.1.
External Forcing

- If $X_{c,0}$ is $\mathcal{M}$-infinite for some $c \in D_0$, let $U_0$ be one such $X_{c,0}$ and set $\sigma_0 = \emptyset$. Then $B\Sigma_1^0(G')$ fails for $\exists x \varphi_0$ on $D_0$, for any $G$ that is a path in $U_0$.

- Otherwise, by $B\Sigma_2^0$ in $\mathcal{M}$, there is an $s_0$ where for all $c \in D_0$ and all $\tau \in T$ of length at least $s_0$, $\exists x \varphi_0(x, c, \tau \oplus Y')$ holds. Let $\sigma_0$ be a string of length at least $s_0$ so that $T[\sigma_0] = \{\tau \in T | \tau \geq \sigma_0\}$ is $\mathcal{M}$-infinite. Let $U_0 = T[\sigma_0]$.

- Define $\sigma_{n+1} \geq \sigma_n$ and $U_{n+1}$ from $U_n$ similarly, replacing $T$ by $U_n$ and $\varphi_0$ by $\varphi_{n+1}$ in the respective definitions.

- Let $G = \bigcup_n \sigma_n$. Then $\mathcal{M}[G] \models B\Sigma_1^0[G \oplus Y']$, hence $B\Sigma_1^0[G']$ since $G' \leq_T G \oplus Y'$. Thus $\mathcal{M}[G] \models B\Sigma_2^0$.

An iteration of Lemmas 6.1 and 6.2 yields Theorem 6.2 and therefore Theorem 6.1.
Session VI:

L’estremità