

Lowness Properties for Randomness in Higher Recursion Theory

(Joint work with André Nies and Liang Yu)

C T Chong

National University of Singapore

chongct@math.nus.edu.sg

Π_1^1 and Δ_1^1 Randomness

- $x \in 2^\omega$ is “r.e.” over $L_{\omega_1^{\text{CK}}}$ if and only if x is Π_1^1 .
- (Spector and Gandy) $A \subset 2^\omega$ is Π_1^1 if and only if there is a Δ_0 formula φ such that $x \in A \Leftrightarrow L_{\omega_1^x}[x] \models \exists z \varphi(z, x)$.
- x is Δ_1^1 (Π_1^1) random if x is not in any Δ_1^1 (Π_1^1) null subset of 2^ω .
- x is Δ_1^1 (Π_1^1) random for Martin-Löf test if $x \notin \bigcap_n V_n$ where $\{V_n\}$ is a Δ_1^1 (Π_1^1) collection of open sets and $\mu(V_n) \leq 2^{-n}$.
- x is Δ_1^1 random $\Leftrightarrow x$ is Δ_1^1 random for ML.

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- (Hjorth and Nies) The Π_1^1 ML-random reals are cofinal in the hyperdegrees.
- If $\omega_1^x = \omega_1^{\text{CK}}$, then x is Π_1^1 random $\Leftrightarrow x$ is Π_1^1 random for ML $\Leftrightarrow x$ is Δ_1^1 random for ML. In general, only \Rightarrow holds.
- Definition y is $\Delta_1^1(x)$ random if it is not in any $\Delta_1^1(x)$ null set. x is low for Δ_1^1 randomness if any y that is Δ_1^1 random is $\Delta_1^1(x)$ random.
- Define other notions of lowness similarly.
- (Hjorth and Nies) If x is low for Π_1^1 ML randomness, then x is hyperarithmetical.

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Lowness

- Definition x is Δ_1^1 (Π_1^1) traceable if there is a nondecreasing, unbounded hyperarithmetical function h such that for all $f \leq_h x$ there is a uniformly Δ_1^1 (Π_1^1) sequence $\{T_e\}$ with (i) $|T_e| \leq h(e)$ for each e , and (ii) $f(e) \in T_e$.
- There exist 2^{\aleph_0} many Δ_1^1 traceable sets.
- The following are equivalent:
 - (i) x is low for Δ_1^1 randomness;
 - (ii) x is Δ_1^1/Π_1^1 traceable.
 - (iii) Every Π_1^1 ML random real is $\Delta_1^1(x)$ random.

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Corollary. Low for Π_1^1 ML randomness is not equivalent to low for Δ_1^1 randomness.

- (Harrington, Slaman, Nies) x is low for Π_1^1 randomness if and only if x is low for Δ_1^1 randomness and for all Π_1^1 random y , $x \oplus y \not\leq_h \mathcal{O}$.
- Is there a non-hyperarithmetical x that is low for Π_1^1 randomness?
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