

THE STRENGTH OF PROJECTIVE MARTIN CONJECTURE

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ABSTRACT. We show that Martin's conjecture on Π_1^1 -functions uniformly \leq_T -order preserving on a cone implies Π_1^1 Turing Determinacy over $ZF + DC$. In addition, it is also proved that for $n \geq 0$, this conjecture for uniformly degree invariant Π_{2n+1}^1 -functions is equivalent over ZFC to Σ_{2n+2}^1 -Axiom of Determinacy. As a corollary, the consistency of the conjecture for Π_1^1 -uniformly degree invariant functions implies the consistency of the existence of a Woodin cardinal.

1. INTRODUCTION

A cone C of reals with base z is a set of the form $\{x \mid x \geq_T z\}$ where \leq_T denotes Turing reducibility. A function $F : 2^\omega \rightarrow 2^\omega$ is *degree invariant* on C if any two reals $x, y \geq_T z$ of the same Turing degree satisfy $F(x) \equiv_T F(y)$. The degree invariance is *uniform* on C if there is a function t such that if $x, y \geq_T z$, then $\Phi_i^x = y$ and $\Phi_j^y = x$ implies $\Phi_m^{F(x)} = F(y)$ and $\Phi_n^{F(y)} = F(x)$, where $t(i, j) = (m, n)$. F is *increasing* on C if $F(x) \geq_T x$ for all $x \geq z$. F is *order preserving* on C if $z \leq_T x \leq_T y$ implies $F(z) \leq_T F(x) \leq_T F(y)$. If this order preservation is witnessed by a function $t : \omega \rightarrow \omega$, i.e., $\Phi_e^x = y \geq_T z$ implies $\Phi_{t(e)}^{F(x)} = F(y)$, then it is *uniform* (note that a uniformly order preserving function is necessarily uniformly degree invariant). Finally, given functions F and G degree invariant on a cone, write $F \geq_M G$ if $F(x) \geq_T G(x)$ on a cone. Donald A. Martin conjectured that, under the assumption of ZF set theory plus the Axiom of Determinacy (AD) and Dependent Choice (DC),

- (1) Every degree invariant function that is not increasing on a cone is a constant on a cone, and
- (2) \leq_M prewellorders degree invariant functions which are increasing on a cone. Furthermore, if the \leq_M -rank of F is α , then F' has \leq_M -rank $\alpha + 1$, where $F'(x) = (F(x))'$, the Turing jump of $F(x)$.

Slaman and Steel [7] proved (1) for functions which are uniformly degree invariant on a cone and (2) for Borel functions which are increasing and order preserving. In [8] Steel showed (2) for uniformly degree invariant functions and conjectured that every function degree invariant on a cone is uniformly degree invariant on a cone.

While Martin [4] has shown that Borel determinacy is a theorem of $ZF + DC$ (hence conjectures (1) and (2) hold for Δ_1^1 functions that are uniformly degree invariant),

2000 *Mathematics Subject Classification.* 03D28, 03E15, 03E35, 28A20.

The research of Chong and Wang were supported in part by NUS grant WBS 146-000-054-123, and Yu was supported by NSF of China No. 10701041 as well as Research Fund for Doctoral Program of Higher Education No. 0070284043.

it is known that AD in the analytical hierarchy beyond Δ_1^1 is a large cardinal axiom. An analysis of the proof in [8] shows that conjecture (2) for uniformly degree invariant Π_{2n+1}^1 functions follows from Δ_{2n+2}^1 Determinacy. Thus a natural question for Martin's Conjectures (1) and (2) is their set-theoretic strength for uniformly degree invariant functions beyond Δ_1^1 in the analytical hierarchy. There is also a related question concerning the more restrictive uniformly order preserving functions, i.e. while (2) holds for such functions under AD according to Steel [8], the set-theoretic strength of (2) for these functions has not been considered.

A set of reals is degree invariant if it is closed under Turing equivalence. Martin [3] showed that under AD, every degree invariant set of reals either contains or is disjoint from a cone. By Π_{2n+1}^1 -Turing Determinacy (Π_{2n+1}^1 -TD) we mean the assertion that every Π_{2n+1}^1 set of reals that is degree invariant either contains or is disjoint from a cone. We show in this paper that Conjecture (2) for uniformly order preserving Π_1^1 functions implies the existence of $0^\#$. Relativizing the argument to arbitrary reals x leads to the conclusion that $x^\#$ exists for every x , so that by Harrington [1] the truth of Conjecture (2) for Π_1^1 uniformly order preserving functions implies Π_1^1 -TD. We also show that in general, for $n \geq 0$, Conjecture (2) for Π_{2n+1}^1 -uniformly degree invariant functions implies Σ_{2n+2}^1 -TD, assuming Π_{2n+1}^1 -uniformization when $n \geq 1$. In fact, by this, Steel [8] and an unpublished work of W. H. Woodin, one concludes that Martin's conjecture (2) for Π_{2n+1}^1 -uniformly degree invariant functions is equivalent to Σ_{2n+2}^1 -AD.

We recall some facts and notations (see Sacks [6] which is used as the standard reference in this paper). For each real x , ω_1^x denotes the least ordinal α for which $L_\alpha[x]$ is admissible. Kleene constructed a $\Pi_1^1(x)$ complete set \mathcal{O}^x with a $\Pi_1^1(x)$ well founded relation $<_{\mathcal{O}^x}$ on \mathcal{O}^x . \mathcal{O}^x is the hyperjump of x . The height of the ordering $<_{\mathcal{O}^x}$ on \mathcal{O}^x is exactly ω_1^x . Furthermore, Kleene's construction of \mathcal{O}^x is uniform. In other words, the relation $\{(x, \mathcal{O}^x) \mid x \in 2^\omega\}$ is Π_1^1 . A fact that will be used implicitly is that given reals x and y , x is hyperarithmetical in y (written $x \leq_h y$) if and only if x is Δ_1^1 in y , and this is in turn equivalent to $x \in L_{\omega_1^y}[y]$. We work under ZF + DC. As we will only be concerned with Conjecture (2), it will be referred to as the \leq_M Conjecture from here on.

2. THE \leq_M CONJECTURE FOR UNIFORMLY ORDER PRESERVING Π_1^1 FUNCTIONS

Let

$$\mathcal{F} = \{x \mid \forall \alpha < \omega_1^x \forall a \subseteq \alpha (a \in L_{\omega_1^x} \implies a \in L_{\alpha+3}[x])\}.$$

\mathcal{F} is a degree invariant Σ_1^1 set introduced by H. Friedman [2]. We give a simpler proof of the following result given as Lemma 7.17 in [2].

Lemma 2.1. *\mathcal{F} is cofinal in the Turing degrees.*

Proof. For any real z , let

$$\mathcal{F}(z) = \{x \oplus z \mid \forall \alpha < \omega_1^z \forall a \subseteq \alpha (a \in L_{\omega_1^z} \implies a \in L_{\alpha+3}[x \oplus z])\}$$

be a degree invariant $\Sigma_1^1(z)$ set. Obviously $\mathcal{F}(z)$ is not empty. By the Gandy Basis Theorem relativized to z , there is an x such that $\omega_1^{x \oplus z} = \omega_1^z$ and $x \oplus z \in \mathcal{F}(z)$. Then $x \oplus z \in \mathcal{F}$. \square

The proof of the following lemma follows from Lemmas 7.20–7.22 in [2].

Lemma 2.2. *If 0^\sharp does not exist, then $\bar{\mathcal{F}} = 2^\omega - \mathcal{F}$ is cofinal in the Turing degrees.*

For x a real and $n \in \omega$, let $x^{[n]}$ be the real such that $x^{[n]}(i) = x(\langle n, i \rangle)$.

Theorem 2.3. *If the \leq_M Conjecture holds for Π_1^1 uniformly order preserving functions, then 0^\sharp exists.*

Proof. If 0^\sharp does not exist, then by Lemmas 2.1 and 2.2, both \mathcal{F} and $\bar{\mathcal{F}}$ are cofinal. For a contradiction, we will define a Π_1^1 -function G that is uniformly order preserving such that $\{x | G(x) = \mathcal{O}^{\mathcal{O}^x}\}$ and $\{x | G(x) = \mathcal{O}^{\mathcal{O}^{\mathcal{O}^x}}\}$ are both cofinal in the Turing degrees,

Let $P(x, y)$ be an arithmetic predicate such that

$$x \in \bar{\mathcal{F}} \Leftrightarrow \forall y P(x, y).$$

Claim 2.4. *If $x \leq_T y$ are such that $x \in \mathcal{F}$ and $y \in \bar{\mathcal{F}}$, then $\mathcal{O}^x \leq_h y$.*

As $y \notin \mathcal{F}$, there are $\alpha < \omega_1^y$ and $a \subseteq \alpha$ with $a \in L_{\omega_1^y} \setminus L_{\alpha+3}[y]$. So $a \notin L_{\alpha+3}[x]$ and $\omega_1^x \leq \alpha < \omega_1^y$. As $x \leq_T y$, $\mathcal{O}^x \leq_h y$.

By Claim 2.4, if $x \leq_T y$ are such that $x \in \mathcal{F}$ and $y \in \bar{\mathcal{F}}$, then $\mathcal{O}^{\mathcal{O}^x} \leq_T \mathcal{O}^y$.

Now define $G(x) = y$ as follows:

- (1) $y^{[0]} = \langle 0 \rangle \wedge \mathcal{O}^{\mathcal{O}^x} \wedge x \in \bar{\mathcal{F}}$ or $y^{[0]} = \langle 1 \rangle \wedge \mathcal{O}^{\mathcal{O}^{\mathcal{O}^x}} \wedge \exists v \leq_T y^{[0]} \neg P(x, v)$. Thus $y^{[0]}$ gives a Π_1^1 differentiation between $x \in \bar{\mathcal{F}}$ and $x \in \mathcal{F}$;
- (2) $y^{[1]} = \mathcal{O}^x$,
- (3) If Φ_e^x is partial then let $y^{[e+2]} = \emptyset$,
- (4) If Φ_e^x is total and equal to u , the following three cases differentiate in a Π_1^1 way between $u \in \mathcal{F}$ and $x \in \bar{\mathcal{F}}$, $u, x \in \mathcal{F}$, and $u \in \bar{\mathcal{F}}$ for all $u \leq_T x$:
 - (a) $y^{[0]}(0) = 0 \wedge \exists v \leq_T y^{[1]} \neg P(u, v) \wedge y^{[e+2]} = \langle 1 \rangle \wedge \mathcal{O}^{\Phi_i^{y^{[1]}}}$ where i is the least index so that $\mathcal{O}^{\mathcal{O}^u} = \mathcal{O}^{\Phi_i^{y^{[1]}}}$, or
 - (b) $y^{[0]}(0) = 1 \wedge \exists v \leq_T y^{[1]} \neg P(u, v) \wedge y^{[e+2]} = \langle 1 \rangle \wedge \mathcal{O}^{\mathcal{O}^{\mathcal{O}^u}}$, or
 - (c) $\forall v \leq_T y^{[1]} P(u, v) \wedge y^{[e+2]} = \langle 0 \rangle \wedge \mathcal{O}^{\mathcal{O}^u}$.

$G(x)$ is obviously Π_1^1 .

Claim 2.5. *If $x \in \mathcal{F}$ then $G(x) \equiv_T \mathcal{O}^{\mathcal{O}^x}$.*

Clearly $x \in \mathcal{F}$ implies $\mathcal{O}^{\mathcal{O}^x} \leq_T G(x)$ and $G(x)^{[0]} \oplus G(x)^{[1]} \leq_T \mathcal{O}^{\mathcal{O}^x}$.

Given $e < \omega$, $\mathcal{O}^{\mathcal{O}^x}$ can uniformly decide whether Φ_e^x is total. Suppose that Φ_e^x is total. To calculate $G(x)^{[e+2]}(n)$, one verifies clauses (4b, 4c) above. But the predicate $\forall v \leq_T y^{[1]} P(u, v)$ is $\Delta_1^1(\mathcal{O}^x)$, hence recursive in $\mathcal{O}^{\mathcal{O}^x}$. Once this predicate is decided, $\mathcal{O}^{\mathcal{O}^x}$ may use recursive functions f and g , where $u = \Phi_e^w \rightarrow \mathcal{O}^{\mathcal{O}^u} = \Phi_{f(e)}^{\mathcal{O}^{\mathcal{O}^w}}$ and $u = \Phi_e^w \rightarrow \mathcal{O}^{\mathcal{O}^u} = \Phi_{g(e)}^{\mathcal{O}^{\mathcal{O}^w}}$, to finish the calculation.

Claim 2.6. *If $x \in \bar{\mathcal{F}}$ then $G(x) \equiv_T \mathcal{O}^x$.*

This is similar to the above claim, except for the final step calculating $G(x)^{[e+2]}(n)$.

Now \mathcal{O}^x is able to decide whether (4a) or (4c) holds, as in the above claim. If (4c) holds, the calculation is the same. If (4a) holds, then $u \in \mathcal{F}$. By Claim 2.4, $\mathcal{O}^u \leq_h x$ and thus $\mathcal{O}^u \leq_T \mathcal{O}^x = G(x)^{[1]}$. So i exists. Moreover, the search for i is a procedure uniformly $\Pi_1^1(\mathcal{O}^x)$. Hence \mathcal{O}^x uniformly computes $G(x)^{[e+2]}(n)$.

It follows from the above two claims that G is degree invariant. Moreover G preserves \leq_T by Claim 2.4.

To show that G is uniformly order preserving, let h be a recursive function such that $\forall x, y, e (x = \Phi_e^y \rightarrow \mathcal{O}^x = \Phi_{h(e)}^y)$. In addition, let s be recursive with

$$\forall x, y, e, i (x = \Phi_e^y \rightarrow \Phi_i^x = \Phi_{s(e,i)}^y).$$

Suppose that $x = \Phi_e^y$. Then

- (1) $G(x)^{[0]} = G(y)^{[e+2]}$,
- (2) $G(x)^{[1]} = \Phi_{h(e)}^{G(y)^{[1]}}$, and
- (3) $G(x)^{[i+2]} = G(y)^{[s(e,i)+2]}$.

Hence G is as desired. \square

The above proof easily relativizes to any real x to guarantee the existence of $x^\#$. Since Harrington [1] has shown that the existence of sharps implies Π_1^1 -TD, we have

Corollary 2.7. *If the \leq_M Conjecture holds for Π_1^1 functions which are uniformly order preserving, then Π_1^1 -TD is true.*

3. THE \leq_M CONJECTURE FOR Π_{2n+1}^1 FUNCTIONS AND Σ_{2n+2}^1 -TD

Lemma 3.1. *Π_{2n+1}^1 -uniformization and Δ_{2n+2}^1 -TD imply Σ_{2n+2}^1 -TD for $n \in \omega$.*

Proof. Let $A \in \Sigma_{2n+2}^1$ be degree invariant and \leq_T -cofinal. Define

$$R(x, y) \Leftrightarrow x \leq_T y \wedge y \in A.$$

So $R(x, y) \in \Sigma_{2n+2}^1$. Note that Π_{2n+1}^1 -uniformization implies Σ_{2n+2}^1 -uniformization. Let $F \in \Sigma_{2n+2}^1$ uniformize R . F is actually a Δ_{2n+2}^1 function.

Define

$$B = \{u \mid \exists x \leq_T u, y \equiv_T u (F(x) = y)\}.$$

Then B is Δ_{2n+2}^1 , degree invariant and \leq_T -cofinal. Moreover $B \subseteq A$. By Δ_{2n+2}^1 -TD, B contains a cone of Turing degrees. Hence so does A . \square

Corollary 3.2. *Δ_2^1 -TD implies Σ_2^1 -TD.*

Proof. As Π_1^1 -uniformization is a theorem of ZFC, the corollary follows immediately from Theorem 3.1. \square

We prove the next result for the lightface version. The proof for the boldface version follows with obvious changes.

Theorem 3.3. *Assume Π_{2n+1}^1 -uniformization. If the \leq_M Conjecture holds for uniformly degree invariant Π_{2n+1}^1 functions, then Σ_{2n+2}^1 -TD holds.*

Proof. Let $A \in \Delta_{2n+2}^1$, and $P, Q \in \Pi_{2n+1}^1$ are such that

$$x \in A \Leftrightarrow \exists y P(x, y) \Leftrightarrow \forall y \neg Q(x, y).$$

Let $R(x, y) \Leftrightarrow P(x, y) \vee Q(x, y)$. By Π_{2n+1}^1 -uniformization, let $F \in \Pi_{2n+1}^1$ uniformize R . Define $J_0(x) = z$ if and only if $z^{[0]} = F(x)$ and

$$\forall e ((\Phi_e^x \text{ is total} \rightarrow z^{[e+1]} = F(\Phi_e^x)) \wedge (\Phi_e^x \text{ is partial} \rightarrow z^{[e+1]} = \emptyset)).$$

Obviously $J_0 \in \Pi_{2n+1}^1$ is total. Moreover J_0 is uniformly order preserving. To see this, let f be a recursive function such that

$$\forall e, x_0, x_1 (x_0 = \Phi_e^{x_1} \rightarrow \forall i (\Phi_{f(e,i)}^{x_1} \simeq \Phi_i^{x_0})).$$

Suppose $x_0 = \Phi_e^{x_1}$. Then $(J_0(x_0))^{[0]} = (J_0(x_1))^{[e]}$ and $(J_0(x_0))^{[i+1]} = (J_0(x_1))^{[f(e,i)+1]}$. Thus $J_0(x_0)$ may be effectively computed from $J_0(x_1)$.

Let g be a recursive function such that $x_0 = \Phi_e^{x_1} \rightarrow J_0(x_0) = \Phi_{g(e)}^{J_0(x_1)}$.

Define $J(x) = x \oplus z_0 \oplus z_1$ if and only if $z_0 = J_0(x)$ and

$$(P(x, z_0^{[0]}) \wedge z_1 = \emptyset) \vee (Q(x, z_0^{[0]}) \wedge z_1 = \langle 1 \rangle \hat{\ } (x \oplus z_0)').$$

Note that $J \in \Pi_{2n+1}^1$. We claim that J is uniformly degree invariant. To see this, let h be a recursive function such that $x_0 = \Phi_e(x_1) \rightarrow (x_0 \oplus J_0(x_0))' = \Phi_{h(e)}^{(x_1 \oplus J_0(x_1))'}$. For each e , let $t(e)$ be the index of the procedure Ψ defined by:

- (1) $(\Psi^z)^{[0]} = \Phi_e^{z^{[0]}}$ and $(\Psi^z)^{[1]} = \Phi_{g(e)}^{z^{[1]}}$, and
- (2) if $z^{[2]}(0) = 0$ then $(\Psi^z)^{[2]} = \emptyset$. Otherwise $(\Psi^z)^{[2]} = \langle 1 \rangle \hat{\ } \Phi_{h(e)}^w$, where w is such that $\langle 1 \rangle \hat{\ } w = z^{[2]}$.

If the \leq_M Conjecture holds for uniformly degree invariant Π_{2n+1}^1 functions, then eventually J is either $x \mapsto x \oplus J_0(x)$ or $x \mapsto x \oplus J_0(x) \oplus (x \oplus J_0(x))'$. Hence A either contains or avoids a cone of Turing degrees.

Thus we have Δ_{2n+2}^1 -TD. Σ_{2n+2}^1 -TD follows from Lemma 3.1. \square

Corollary 3.4. *Let $n \geq 0$. The \leq_M Conjecture for Π_{2n+1}^1 uniformly degree invariant functions is equivalent to Σ_{2n+2}^1 -AD.*

Proof. An analysis of Theorem 1 in Steel [8] shows that Σ_{2n+2}^1 -AD (in fact Δ_{2n+2}^1 -AD) implies the \leq_M Conjecture for Π_{2n+1}^1 uniformly degree invariant functions. We show the converse by induction on n : First note that if $n = 0$, then Π_1^1 -uniformization is the Kondo-Addison Theorem, so that by Theorem 3.3, Σ_2^1 -TD holds. Now assume by induction that Σ_{2n}^1 -TD is true. Woodin (unpublished) has shown that over ZFC, for $k \geq 1$, Σ_{2k}^1 -TD is equivalent to Σ_{2k}^1 -AD, and Moschovakis [5] has shown (Chapter 6) that Π_{2k+1}^1 -uniformization is a consequence of Σ_{2k}^1 -AD. Thus Π_{2n+1}^1 -uniformization holds and so Theorem 3.3 yields Σ_{2n+2}^1 -TD, hence Σ_{2n+2}^1 -AD. \square

The following corollary gives the consistency strength of the \leq_M Conjecture.

Corollary 3.5. *If it is consistent that the \leq_M Conjecture holds for Π_1^1 -uniformly degree invariant functions, then it is consistent that there is a Woodin cardinal.*

Proof. The hypothesis and Theorem 3.3 imply that Π_2^1 -TD is consistent. Woodin has shown that Π_2^1 -TD is equiconsistent with the existence of a Woodin cardinal. \square

Remark. We do not know if the conclusion of Corollary 2.7 may be strengthened to Δ_2^1 -TD (hence Δ_2^1 -AD). If this is true, then by Steel [8] it will give a characterization of the \leq_M Conjecture for Π_1^1 uniformly order preserving functions. In general, one would like to understand better the role of order preserving functions in the study of the \leq_M Conjecture. For example, it is not clear if Corollary 3.4 applies to functions which are order preserving.

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