HYPERIMMUNE-FREE DEGREES BEYOND ω

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ABSTRACT. An α-degree is hyperimmune-free if it does not contain a hyperimmune set. We study the existence problem of a hyperimmune-free α-degree for an admissible ordinal α, and show that the underlying structure of an admissible ordinal determines the existence and cardinality of the collection of hyperimmune-free α-degrees. The combinatorial principle $\diamondsuit^+$ is demonstrated to play a key role in answering this question.

1. INTRODUCTION

An ordinal is admissible if L_α, Gödel's constructible hierarchy up to level α, is a model of Σ_1 replacement. Throughout this paper we fix α to be an admissible ordinal. An α-degree is hyperimmune-free if it does not contain a hyperimmune set, i.e. there is no set A in the α-degree such that for some total α-recursive function h, $D_{h(s)} \cap A \neq \emptyset$ for all $s < \alpha$, where $D_{h(s)}$ is the $h(s)$-th α-finite set, and $D_{h(s)} \cap D_{h(i)} = \emptyset$ for $s \neq t$.

Hyperimmune-free degrees have been studied extensively for $\alpha = \omega$, beginning with the work of Miller and Martin [6], and Jockusch and Soare [4] who showed that every Π^0_1 class of reals contains a set of hyperimmune-free degree. Sets of hyperimmune-free degrees also play an important role in various parts of recursion theory, including most recently in the study of algorithmic randomness (see, for example [2]).

For $\alpha = \omega$, a typical construction of an A of hyperimmune-free degree is to ensure that if Φ^A is total, then it is dominated by a recursive function. This may be achieved using the technique of perfect set forcing (cf. [4]). Thus if T_e is a recursive perfect set, then T_{e+1} is obtained as a recursive perfect subtree of T_e such that either one of the following two possibilities holds: (1) There is a binary string $\sigma \in T_e$ such that T_{e+1} is the set of all strings in T_e extending σ, and no path on T_{e+1} is total for Φ_e, or (2) T_{e+1} is recursively generated from T_e so that Φ^X_e is total and uniformly truth-table reducible to X for each infinite path X on T_{e+1}. The domination of Φ^X_e, for each infinite path X on T_{e+1}, by a recursive function follows immediately. A will then be in the intersection of the T_e's. The use of perfect trees in the construction is similar to that of a set M of minimal degree which is also obtained as the intersection of a nested sequence of perfect trees, where at stage $e+1$, a recursive perfect tree $T_{e+1} \subseteq T_e$ is obtained so that either all paths X lying on T_{e+1} agree on Φ_e for each input x (if defined), or they pairwise disagree on some input x. There are two key features that these constructions share, arising from the use of a nested sequence of perfect trees. The first is that the recursive trees are recursively bounded at every level. In fact at level n there are 2^n many nodes. The second is that the decision to choose between a subtree T_{e+1} all of whose paths are partial on Φ_e or all of whose paths are total on Φ_e (through a truth table reduction) in the case of a hyperimmune-free degree construction, and to choose between a subtree T_{e+1} all of whose paths agree on Φ_e or all of whose paths pairwise disagree on Φ_e in the case of a minimal degree construction, is a Σ_2/Π_2 decision. For $\alpha = \omega$, the first feature is not a matter of concern since $n \mapsto 2^n$ is a recursive function, while the second feature does not pose a problem since ω is Σ_n admissible for all n. The situation changes, sometimes quite dramatically, for $\alpha > \omega$. Interesting and even challenging problems arise.

Firstly, a perfect set over L_α, for $\alpha > \omega$, where every level σ has $2^{|\sigma|}$ nodes, is necessarily non-α-finite at level $\sigma > \omega$. This means that for such levels only a bounded number of nodes may be enumerated at any stage. This phenomenon does not cause a serious problem in the minimal degree construction (cf. Shore [8]). By contrast, it will thwart any attempt at achieving truth table reducibility in the construction of an A with hyperimmune-free degree. It turns out that this difficulty is also the source that eventually restricts the cardinality of the collection of hyperimmune-free α-degrees, even when α is a regular cardinal. On the other hand, there is a wide class of admissible ordinals with an abundance of hyperimmune-free degrees (just like ω which has 2^ω many). Interestingly, Jensen's combinatorial principle $\diamond^+$ is the tool used to establish this.

The second feature, that of making a Σ_2/Π_2 decision for the next recursive prefect subtree, presents a different problem. In general, asking a Σ_1 admissible ordinal to perform tasks that are seemingly more complex than what the ordinal is endowed for, or capable of doing, is always a challenge. In the case of minimal degrees, ordering the set of requirements in an ω sequence removes the technical difficulty for countable α (MacIntyre [5]), while Σ_2 admissibility resolves it for uncountable α as well (see [8]) but otherwise the problem remains open, more than forty years after the question of the existence of a minimal α-degree (for all α) was posed. For hyperimmune-free degrees, we overcome this difficulty for countable α, by first arranging the requirements in an ω sequence, and then working with non-perfect trees with recursively selected α-finite set of nodes at every level. With additional strategies to handle more complex situations, one is able to construct a set of hyperimmune-free α-degree for Σ_2 admissible α. This approach, however, does not always extend to constructing a "large class" (in terms of cardinality) of hyperimmune-free degrees, even in the case when α has sufficient admissibility to carry out, in principle, complex constructions. Furthermore, as we shall show, for some α the Σ_2/Π_2 dichotomy leads to the collapse of any construction of a hyperimmune-free degree for many Σ_2 inadmissible ordinals, including singular cardinals of uncountable cofinality (such as $\aleph^L_{\omega_1}$).

Our motivation for this work comes from two sources: We wish to understand how two classes of sets with seemingly similar constructions differ in existential strength when viewed from the vantage point of a higher ordinal. In particular, from the methodological point of view, whether the failure of an approach implies the nonexistence of a set with the prescribed property (in this case, hyperimmune-free degree versus minimal degree). Secondly, hyperimmune-freeness is a notion of great interest in various parts of recursion theory, and studying it in the context of ordinals could shed light on the subject *ordinal algorithmic randomness* which has yet to be developed.

We assume $V = L$ throughout this paper.

The organization of this paper is as follows. In §2 we review some facts and definitions from α-recursion theory. We introduce the notion of an f-good tree, which will play a key role in the construction of sets with hyperimmune-free degree. We also recall Jensen's $\diamond^+$ principle. Section 3 is devoted to the proof that every countable admissible ordinal has a hyperimmune-free degree. Section 4 provides a careful analysis of the class of trees that will be used in §§5–7 where we show that every Σ_2 admissible ordinal has a hyperimmune-free degree. In §8 we study the cardinality of the collection of hyperimmune-free degrees for a wide class of admissible ordinals, including all successor cardinals and many regular limit cardinals. We show that if κ is a regular cardinal in which $\diamond^+_\kappa$ holds effectively, then there exist 2^κ many hyperimmune-free degrees. In contrast, this fails when such an effective $\diamond^+$ -sequence does not exist. In §9 we show that for singular cardinals κ of uncountable cofinality, all hyperimmune-free degrees are recursive in the degree of the cofinality function. In particular, there is no hyperimmune-free degree when κ is a Σ_2 inadmissible cardinal of uncountable cofinality (e.g. $\aleph_{\omega_1}$). We end the section with further results on cardinality bounds of hyperimmune-free degrees. The paper concludes with a list of open problems.

2. Preliminaries

We assume that the reader is familiar with the rudiments of α-recursion theory. Sacks [7] is a good source for background material. Throughout this paper, let $\alpha > \omega$ be an admissible ordinal. We will suppress the use of the prefix α when referring to objects defined over L_α if the context is clear with little possibility of confusion.

Definition 2.1. *$A \subset \alpha$ is* regular *if $A \upharpoonright \gamma$ is α-finite for each $\gamma < \alpha$. A is* hyperregular *if $L_\alpha[A]$ is a Σ_1 admissible structure.*

2.1. Hyperimmune Freeness.

Our interest will be in the existence of regular sets with hyperimmune property. This allows us to avoid dealing with irregular sets which in many interesting cases, such as cardinals, do not exist.

Recall the definition of hyperimmunity given in the Introduction. A degree **a** is hyperimmune if it contains a hyperimmune set. Otherwise it is called hyperimmune-free. As for $\alpha = \omega$, there are various characterizations of a hyperimmune degree. In particular, the degree of $A \subset \alpha$ is hyperimmune if there is a function $f \leq_\alpha A$ such that for all total recursive functions g, $g(x) < f(x)$ for unboundedly many $x < \alpha$. It follows, as pointed out in [4], a degree is hyperimmune-free if and only if there is a set A in that degree that fails to pass the "hyperimmune test". We define accordingly the following notion.

Definition 2.2. *A regular set A is hyperimmune-free (HIF) if for every total $f \leq_{w\alpha} A$, there is a total recursive function g such that $f(x) < g(x)$ for all x.*

Part (2) of the following lemma is an analog of the result for ω. Part (3) implies that a HIF set grows at a "slow" rate, and this fact will be exploited in the study of such sets.

Lemma 2.3. *Let A be HIF. Then*
(1) A is hyperregular;
(2) If $A \leq_\alpha \emptyset'$ is HIF then A is recursive;
(3) Let $g_A(x)$ be the $<_L$-rank of $A \upharpoonright x$. Then g_A is majorized by a recursive function.

Proof. Let A be hyperimmune.

For (1), A will be hyperregular if the image of an α-finite set under an f weakly α-recursive in A is bounded. Suppose $f \leq_{w\alpha} A$ is total on γ with a view to proving that $f''\gamma$ is bounded. Define $\Phi^A(x) = f(x)$ if $x < \gamma$ and equal to 0 otherwise. Since A is hyperimmune, $\Phi^A(x) < g(x)$ for all x, for some total recursive function g. Thus A is hyperregular.

For (2), let $f : \alpha \times \alpha \to 2$ be a recursive function such that for every $x \in \alpha$

$$A(x) = \lim_s f(x, s).$$

Let $\langle \cdot, \cdot \rangle$ be a recursive pairing function. Define

$$g(\langle s, x \rangle) = \begin{cases} \mu t > s(f(x,t) = A(x)), & s > x; \\ 0, & \text{otherwise.} \end{cases}$$

Since A is HIF. Then there exists a recursive $h : \alpha \to \alpha$ with $h(\langle s, x \rangle) > g(\langle s, x \rangle)$ for all $\langle s, x \rangle \in \alpha$. To compute $A(x)$, find $s > x$ such that

$$\forall t(s < t < h(\langle s, x \rangle) \to f(x,t) = f(x,s)).$$

Then $A(x) = f(x, s)$ and thus A is recursive.

For (3) notice that the regularity of A ensures that g_A is well-defined. It is then a consequence of the definition of HIF. □

The proposition below is an analogy of the equivalence between the two characterizations of HIF in classic case. Interestingly, the proof is non-uniform for admissible $\alpha > \omega$.

Proposition 2.4. *Let A be regular. Then A is* not *HIF according to Definition 2.2 if and only if there exists $B \equiv_{w\alpha} A$ such that no strong array always intersects B.*

Proof. If $L_\alpha[A]$ is inadmissible, let $\beta < \alpha$ and $f \leq_{w\alpha} A$ be such that f increasingly maps β cofinally into α. Define

$$g(x) = \begin{cases} f(x), & x < \beta; \\ 0, & \text{otherwise.} \end{cases}$$

Then $g \leq_{w\alpha} A$ witnesses A not HIF. On the other hand, define

$$B = \{\langle \gamma, f(\gamma), A \upharpoonright f(\gamma)\rangle | \gamma < \beta\}.$$

Then $B \leq_{w\alpha} A \leq_\alpha B$ is the desired set.

For the rest, we assume that $L_\alpha[A]$ is admissible.

Let $f \leq_{w\alpha} A$. Then $f \leq_\alpha A$ and may be assumed strictly increasing as $L_\alpha[A]$ is admissible. If f is not majorized by any recursive function, define

$$B = \{\langle \gamma, f(\gamma), A \upharpoonright \gamma\rangle | \gamma < \alpha\}.$$

Then $B \equiv_\alpha A$ and no strong array always intersects B.

Finally let $B \equiv_{w\alpha} A$ be such that no strong array always intersects B. Then $B \equiv_\alpha A$ as $L_\alpha[A]$ is admissible. As in the classical case, define $f = p_B$ to be the function enumerating elements of B in strictly increasing order. Then f is as desired. □

2.2. **Binary Sequences.** An *α-finite binary sequence* or *binary sequence* for short is an α-finite function $\sigma : \gamma \to 2$ for some $\gamma < \alpha$. We call the domain of σ its *length* and write $|\sigma| = dom(\sigma)$.

If σ is a binary sequence of length $\beta < \alpha$ and $\gamma \leq \beta$, then $\rho = \sigma \upharpoonright \gamma$ is such that $dom(\rho) = \gamma$ and $\rho(x) = \sigma(x)$ for $x < \gamma$. If σ and τ are binary sequences and $|\sigma| \leq |\tau|$, then we write $\sigma \preccurlyeq \tau$ if $\sigma = \tau \upharpoonright |\sigma|$, and $\sigma \prec \tau$ if in addition $\sigma \neq \tau$. If either $\sigma \preccurlyeq \tau$ or $\tau \preccurlyeq \sigma$ then they are *comparable*, otherwise they are *incomparable*.

The *concatenation* of two binary sequences σ and τ is the binary sequence $\rho = \sigma\hat{\ }\tau$ such that $|\rho| = |\sigma| + |\tau|$ and

$$\rho(x) = \begin{cases} \sigma(x), & x < |\sigma|; \\ \tau(y), & (x = |\sigma| + y) \wedge (y < |\tau|). \end{cases}$$

If $\{\sigma_i | i \in I\}$ is a family of binary sequences in which any two elements are comparable, then the *union* $\bigcup_{i \in I} \sigma_i$ is the function $f : \sup_{i \in I} |\sigma_i| \to 2$ with $f(x) = \sigma_i(x)$ whenever $x < |\sigma_i|$.

We use $\langle i^\gamma \rangle$ to denote the binary sequence $\rho : \gamma \to \{i\}$ for $i < 2$ and $\gamma < \alpha$, and $\langle i \rangle = \langle i^1 \rangle$.

2.3. **Trees.** A *tree* T is a partial function mapping binary sequences to binary sequences and has the following properties:

(1) for $\sigma, \tau \in dom(T)$, $T(\sigma) \preccurlyeq T(\tau)$ if and only if $\sigma \preccurlyeq \tau$,
(2) if $\sigma\hat{\ }\langle i\rangle \in dom(T)$ for $i < 2$, then $\sigma, \sigma\hat{\ }\langle 1-i\rangle \in dom(T)$, and
(3) if $\sigma \in dom(T)$ is of length some limit ordinal β, then $\sigma \upharpoonright \gamma \in dom(T)$ for $\gamma < \beta$ and $T(\sigma) = \bigcup_{\gamma<\beta} T(\sigma \upharpoonright \gamma)$.

The η*-th level* of a tree T is defined to be

$$T_\eta = \{(\sigma, T(\sigma)) | \sigma \in dom(T) \wedge |\sigma| = \eta\}.$$

$T_{<\eta} = \bigcup_{\zeta<\eta} T_\zeta$ and $T_{\leq\eta} = \bigcup_{\zeta\leq\eta} T_\zeta$. Define the *height* of T as $ht(T) = \inf\{\eta | T_\eta = \emptyset\}$.

The hyperimmune-free sets are to be obtained through tree constructions. However, instead of letting the tree T grow rapidly at every level, which will make it unmanageable as far as achieving our objective is concerned (i.e. if Φ^A is total then it is dominated by a total recursive function), at each level we only select strings satisfying a prescribed bound dictated by a recursive function. This will ensure that each tree being considered has only α-finitely many nodes uniformly at each level, and allows one to verify that in the end the construction succeeds. The notion of an f-good tree is introduced for this purpose.

If $f : \alpha \to \alpha$ is recursive and T is a recursive tree, then T is f*-good* if

(T1) if $\sigma \in dom(T)$, then $\sigma\hat{\ }\langle i\rangle \in dom(T)$ and $T(\sigma\hat{\ }\langle i\rangle)(|\sigma|) = i$ for $i < 2$,
(T2) if $\sigma \in dom(T)$ is of length γ and $\beta < \alpha$ is a limit ordinal greater than γ, then the binary sequence of length β obtained from appending 0's to σ is also in $dom(T)$, and
(T3) if $\sigma, \rho \in dom(T)$ are of same length and $T(\sigma)$ is of length γ, then $T(\rho)$ is also of length γ and $\{\sigma, T(\sigma), \rho, T(\rho)\} \subset L_{f(\gamma)}$.

T is *good*, if T is f-good for $f : x \mapsto x + 1$.

If $f(x) \geq x + 1$ for all $x < \alpha$, then a typical f-good tree, always denoted by T^f, is defined to be the identity function on its domain, which is defined by induction:

(1) $\emptyset \in dom(T^0)$,
(2) if $\sigma \in dom(T^0)$ then $\sigma\hat{\ }\langle 0\rangle, \sigma\hat{\ }\langle 1\rangle \in dom(T^0)$, and
(3) for ρ a binary sequence of length $\beta < \alpha$ a limit ordinal, $\rho \in dom(T^0)$ if and only if $\rho \upharpoonright \gamma \in dom(T^0)$ for each $\gamma < \beta$ and $\rho \in L_{f(\beta)}$.

If $f : x \mapsto x + 1$, then we call T^f the *canonical good tree.*

A pair (ϵ_0, ϵ_1) *defines an* f*-good tree*, if there is an f-good tree T such that

$$T(\sigma) = \tau \Leftrightarrow L_\alpha \models \varphi(\epsilon_0, \langle \sigma, \tau\rangle)$$

and

$$\sigma \notin dom(T) \Leftrightarrow L_\alpha \models \varphi(\epsilon_1, \sigma)$$

where φ is the universal Σ_1-predicate. (ϵ_0, ϵ_1) is called the Δ_1*-index* of T.

2.4. **Good Levels.** Let f be a recursive function defined by ϵ^f, i.e.

$$f(x) = y \Leftrightarrow L_\alpha \models \varphi(\epsilon^f, \langle x, y\rangle)$$

where φ is the universal Σ_1 formula. Let T be a Σ_1 α-tree, and ϵ_0 be its Σ_1-index, i.e.

$$\sigma \in dom(T) \Leftrightarrow L_\alpha \models \exists\tau\varphi(\epsilon_0, \langle\sigma, \tau\rangle)$$

and

$$T(\sigma) = \tau \Leftrightarrow L_\alpha \models \varphi(\epsilon_0, \langle\sigma, \tau\rangle).$$

In addition, assume that a possible Σ_1-index for the complement of $dom(T)$, say ϵ_1, is also given. In a priority argument, this possible index could later be found wrong and replaced by a new one.

Let $C^f(T)$ be the set of limit ordinals $\gamma < \alpha$ such that $\epsilon^f, \epsilon_0, \epsilon_1 \in L_\gamma$, and

(C1) $L_\gamma \models ((\epsilon_0, \epsilon_1)$ defines an f-good tree),
(C2) $L_\gamma \models \varphi(\epsilon_0, \langle\sigma, \tau\rangle) \Rightarrow L_\alpha \models \varphi(\epsilon_0, \langle\sigma, \tau\rangle)$, and
(C3) f is closed in L_γ, i.e. $L_\gamma \models \forall x\exists y\varphi(\epsilon^f, \langle x, y\rangle)$.

It is straightforward to verify that $C^f(T)$ is a Σ_1 closed subset of α.

Lemma 2.5. *If (ϵ_0, ϵ_1) defines an f-good tree T then $C^f(T)$ is closed unbounded in α.*

Proof. Assume that $b < \alpha$ is given.

Let $\gamma_0 = |T(\langle 0^b\rangle)|$, and let $\beta_0 \geq \gamma_0 + \omega$ be the least ordinal such that for any binary sequence $\sigma \in L_{f(\gamma_0)}$ of length at most b

$$L_{\beta_0} \models \exists\tau\varphi(\epsilon_0, \langle\sigma, \tau\rangle) \text{ or } L_{\beta_0} \models \varphi(\epsilon_1, \sigma),$$

and $L_{\beta_0} \models \forall x < b\exists y\varphi(\epsilon^f, \langle x, y\rangle)$.

Suppose that γ_n and β_n are defined. Let $\gamma_{n+1} = |T(\langle 0^{\beta_n}\rangle)|$, and let $\beta_{n+1} \geq \gamma_{n+1} + \omega$ be the least ordinal such that for any binary sequence $\sigma \in L_{f(\gamma_{n+1})}$ of length at most β_n

$$L_{\beta_{n+1}} \models \exists\tau\varphi(\epsilon_0, \langle\sigma, \tau\rangle) \text{ or } L_{\beta_{n+1}} \models \varphi(\epsilon_1, \sigma),$$

and $L_{\beta_{n+1}} \models \forall x < \beta_n\exists y\varphi(\epsilon^f, \langle x, y\rangle)$.

The definition of $(\beta_n : n < \omega)$ does not break because of the Σ_1-admissibility of α and also of the α-recursiveness of f and the goodness of T.

Finally let $\beta = \sup_{n<\omega} \beta_n$. Obviously $b < \beta$ and β meets (C1-C3). As the map $n \mapsto \beta_n$ is an α-recursive function from ω into α, the bound $\beta < \alpha$. Hence $C^f(T)$ is unbounded. □

Ordinals in $C^f(T)$ are *good levels* of T. From the definition of $C^f(T)$, if $\sigma \in dom(T)$ is of length $\gamma \in C^f(T)$ then $T(\sigma)$ is also of length γ.

If $f : x \mapsto x + 1$ then we write $C(T)$ for $C^f(T)$.

2.5. **e-Total Trees.** Let $f, T, \epsilon_0, \epsilon_1$ be as in the above subsection. In addition, assume that $e \in L_\alpha$ and $\lambda < \alpha$ are also given.

We define $\hat{T} = Tot^f(T, e, \lambda)$ by induction.

Let γ_0 be the least $\gamma \in C^f(T) - \lambda$ such that $(\epsilon_0, \epsilon_1) \in L_{\gamma_0}$ and

$$\exists \rho, \tau \in L_\gamma(\varphi(\epsilon_0, \langle \rho, \tau \rangle)^{L_\gamma} \wedge \sigma \hat{\ } \langle i \rangle \subset \rho \wedge \Phi_e(\tau; x) \downarrow)$$

for each $\sigma \in dom(T_{<\gamma})$, $i < 2$ and $x \in \gamma$. Let

$$\hat{T}(\emptyset) = T(\rho)$$

where $\rho = \langle 0^{\gamma_0} \rangle$ and $\hat{T}_0 = \{(\emptyset, \hat{T}(\emptyset)\}$.

Suppose that we have defined $(\hat{T}_\nu : \nu \leq \mu < \alpha)$ as a subtree of T. Let $\gamma_{\mu+1}$ be the least $\gamma \in C^f(T)$ such that $(\hat{T}_\nu : \nu \leq \mu < \alpha) \in L_\gamma$, and

$$\exists \rho, \tau \in L_\gamma(\varphi(\epsilon_0, \langle \rho, \tau \rangle)^{L_\gamma} \wedge \hat{T}_\mu(\sigma) \hat{\ } \langle i \rangle \subset \tau \wedge \Phi_e(\tau; \mu) \downarrow)$$

for each $\sigma \in dom(\hat{T}_\mu)$, $i < 2$. For each $\sigma \in dom(\hat{T}_\mu)$ and $i < 2$, let $\rho(\sigma, i)$ be the $<_L$-least $\gamma_{\mu+1}$-path along $dom(T_{<\gamma_{\mu+1}})$ witnessing the above formula, and let

$$\hat{T}_{\mu+1}(\sigma \hat{\ } \langle i \rangle) = T(\rho(\sigma, i)).$$

Let $\hat{T}_{\mu+1} = \{(\sigma \hat{\ } \langle i \rangle, \hat{T}_{\mu+1}(\sigma \hat{\ } \langle i \rangle)) | \sigma \in dom(\hat{T}_\mu), i < 2\}$.

If $\mu < \alpha$ is a limit ordinal and $\hat{T}_{<\mu} = (\hat{T}_\nu : \nu < \mu < \alpha)$ is defined, then let $\gamma_\mu = \sup_{\nu<\mu} \gamma_\nu$. For each path σ along $dom(\hat{T}_{<\mu})$, if $\sigma, \bigcup_{\nu<\mu} \hat{T}_\nu(\sigma \upharpoonright \nu) \in L_{f(\gamma_\mu)}$ then define

$$\hat{T}_\mu(\sigma) = \bigcup_{\nu<\mu} \hat{T}_\nu(\sigma \upharpoonright \nu).$$

Otherwise $\hat{T}_\mu(\sigma)$ is undefined. Note that $\gamma_\mu \in C^f(T)$ as $C^f(T)$ is closed.

As the above procedure is Σ_1, we may recursively calculate the Σ_1-index $\hat{\epsilon}_0$ for $\hat{T}$ and also a specific Σ_1-index $\hat{\epsilon}_1$ for the complement of $dom(\hat{T})$. The pair $(\hat{\epsilon}_0, \hat{\epsilon}_1)$ searches for $\gamma \in C^f(T)$ inductively. Either it keeps searching at some level, or it finds some γ and defines $\hat{T}$ up to this level in L_γ using (ϵ_0, ϵ_1).

Let $Tot^f(T, e, \lambda) = \hat{T}$. Again we write Tot for Tot^f if $f : x \mapsto x + 1$.

Lemma 2.6. *Let f be recursive, and let (ϵ_0, ϵ_1) define an f-good tree T. Suppose that $e < \alpha$ and for all $\sigma \in dom(T)$ and $x < \alpha$ there exists $\rho \in dom(T)$ with $\sigma \prec \rho$ and $\Phi_e(T(\rho); x) \downarrow$. Then $(\hat{\epsilon}_0, \hat{\epsilon}_1)$ also defines an f-good tree $Tot^f(T, e, \lambda)$.*

Proof. From the construction it is obvious that (T1, T3) of subsection 2.3 hold for $\hat{T} = Tot^f(T, e, \lambda)$. We show (T2) by induction. In addition we simultaneously show that if $\sigma \in dom(\hat{T})$ then

$$\{(\eta, \hat{T}_\eta) | \eta < |\sigma|\} \in L_{|\hat{T}(\sigma)|}.$$

The successor steps are easy by the assumption of e.

Now assume that $\sigma \in dom(\hat{T})$, $\eta \in \alpha - (|\sigma|+1)$ is a limit ordinal, and $\rho = \sigma\hat{\ }\langle 0^{\bar{\eta}}\rangle$ is a binary sequence of length η such that for each $\zeta < \eta$, $(\rho \upharpoonright \zeta) \in dom(\hat{T})$ and

$$\{(\xi, \hat{T}_\xi))|\xi < \zeta\} \in L_{|\hat{T}(\rho\upharpoonright\zeta)|} \subset L_\gamma$$

where $\gamma = \sup_{\zeta<\eta} |\hat{T}(\rho \upharpoonright \zeta)|$. Obviously $\gamma \in C^f(T)$ and the construction of $\hat{T}_\zeta$ for $\zeta < \eta$ can be carried out in L_γ.

Hence $\bigcup_{\zeta<\eta} \hat{T}(\rho \upharpoonright \zeta), \{(\zeta, \hat{T}_\zeta) : \zeta < \eta\} \in L_{\gamma+1} \subset L_{f(\gamma)}$. It follows that $\rho \in dom(\hat{T})$ and $\hat{T}_\eta, \{(\zeta, \hat{T}_\zeta) : \zeta \le \eta\} \in L_{\gamma+\omega} \subset L_{|\hat{T}(\rho\hat{\ }\langle 0\rangle)|}$. □

This yields the following result which guarantees hyperimmune-freeness with respect to Φ_e for any path A lying on such a tree T:

Lemma 2.7. *Let α, T and e be as in Lemma 2.6. Then there is a recursive function f_e such that*

$$\forall x < \alpha(f_e(x) > \Phi_e(X; x))$$

whenever X is a path on $\hat{T} = Tot^f(T, e, \lambda)$ for some λ.

Proof. For each $x < \alpha$, let

$$f_e(x) = \sup\{\Phi_e(\hat{T}(\sigma); x) + 1 | \sigma \in dom(\hat{T}) \text{ is of length } x + 1\}.$$

Note that each $\Phi_e(\hat{T}(\sigma); x)$ is defined by the construction of $\hat{T}$. The bound is defined by the Σ_1-admissibility of α. Hence $f_e(x)$ is defined.

Obviously f_e is as desired. □

2.6. ***e*-Partial Trees.** Let $f, T, \epsilon_0, \epsilon_1, e$ and λ be as in the above subsection. Let $C = C^f(T)$.

If there exists $\sigma \in dom(T)$ and $x \in \alpha$ such that

$$\forall \rho \in dom(T)(\sigma \subset \rho \to \Phi_e(T(\rho); x) \uparrow),$$

then let (σ_0, x_0) be the $<_L$-least (σ, x) as above. Let γ_0 be the least $\gamma \in C - \lambda$ such that $(\epsilon_0, \epsilon_1), (\sigma_0, x_0) \in L_\gamma$ and

$$\forall (\sigma, x) <_L (\sigma_0, x_0) \exists \rho, \tau \in L_\gamma (L_\gamma \models \varphi(\epsilon_0, \langle \rho, \tau\rangle) \wedge \Phi_e(\tau; x) \downarrow).$$

Let $\sigma \in L_{\gamma_0} \cap dom(T_{\gamma_0})$ be the $<_L$-least extension of σ_0. Define

$$\hat{T}_0 = \{(\emptyset, T_{\gamma_0}(\sigma)\}.$$

Suppose that $\hat{T}_l$ is defined and $ran(\hat{T}_l) \subset ran(T_{\gamma_l})$ for some $\gamma_l \in C$. Let γ_{l+1} be the least $\gamma \in C - (\gamma_l + 1)$. For each $\sigma \in ran(\hat{T}_l)$ and $i < 2$, let ρ_σ be such that $\hat{T}_l(\sigma) = T(\rho_\sigma)$ and $\rho_{\sigma,i} \in dom(T_{\gamma_{l+1}})$ be obtained from appending many 0's to $\rho_\sigma\hat{\ }\langle i\rangle$. Define

$$\hat{T}_{l+1} = \{(\sigma\hat{\ }\langle i\rangle, T(\rho_{\sigma,i}))|\sigma \in dom(\hat{T}_l)\}.$$

At limit level l, let $\gamma_l = \sup_{k<l} \gamma_k \in C$. If $\sigma \in L_{f(\gamma_l)}$ is of length l, $\sigma \upharpoonright k \in dom(\hat{T}_k)$ for $k < l$ and $\tau = \bigcup_{k<l} \hat{T}_k(\sigma \upharpoonright k) \in L_{f(\gamma_l)}$, then define $\hat{T}_l(\sigma) = \rho$. Otherwise $\hat{T}_l(\sigma)$ is undefined.

Let $Par^f(T, e, \lambda) = \hat{T}$. We may also recursively calculate the corresponding $(\hat{\epsilon}_0, \hat{\epsilon}_1)$ from (ϵ_0, ϵ_1), e and λ. If (ϵ_0, ϵ_1) defines a good tree T, then $Par^f(T, e, \lambda)$ is also a good tree defined by $(\hat{\epsilon}_0, \hat{\epsilon}_1)$.

As usual we write Par for Par^f if $f : x \mapsto x + 1$.

Lemma 2.8. *Let (ϵ_0, ϵ_1) define a good tree T and $e < \alpha$. Suppose that there exists $\sigma \in dom(T)$ and $x < \alpha$ such that*

$$\forall \rho \in dom(T)(\sigma \prec \rho \to \Phi_e(T(\rho); x) \uparrow).$$

Then $\hat{T} = Par^f(T, e, \lambda)$ is a good tree defined by $(\hat{\epsilon}_0, \hat{\epsilon}_1)$ and $\Phi_e(X; x)$ diverges for every path X on $\hat{T}$.

Proof. Similar to Lemma 2.6 and 2.7. □

2.7. Combinatorial Principles.

We briefly introduce some combinatorial principles useful for studying the cardinality of the class of HIF sets. For a comprehensive treatment, we refer the reader to [1].

Let κ be a regular cardinal.

Definition 2.9. $(S_\gamma : \gamma < \kappa)$ *is* a $\diamond^+_\kappa$-sequence *if*

1. *each S_γ is a collection of at most $|\gamma|$ many subsets of γ, and*
2. *for each $X \subset \kappa$ there exists a club $C \subset \kappa$ such that $C \cap \gamma, X \cap \gamma \in S_\gamma$ whenever $\gamma \in C$.*

$\diamond^+_\kappa$ is also taken as the assertion that there exists a $\diamond^+_\kappa$-sequence. We write $\diamond^+$ for $\diamond^+_{\omega_1}$. If the function $\gamma \mapsto S_\gamma$ is κ-recursive then the $\diamond^+_\kappa$-sequence is κ-recursive.

Definition 2.10. *A* κ-Kurepa family *is a family $\mathcal{F}$ of subsets of κ such that $|\mathcal{F}| \geq \kappa^+$ but $|\{X \cap \gamma | X \in \mathcal{F}\}| \leq |\gamma|$ for each $\gamma < \kappa$. $KH(\kappa)$ is the assertion that there exists a κ-Kurepa family. KH is the abbreviation for $KH(\omega_1)$.*

In [1, Theorem VII.3.2] it was proved that $\diamond^+_\kappa$ implies $KH(\kappa)$. If a $\diamond^+_\kappa$-sequence $(S_\gamma : \gamma < \kappa)$ is given, then let $f(\gamma)$ be the least Σ_1-admissible α with $S_\gamma \in L_\alpha$. The proof of [1, Theorem VII.3.2] also works for

$$\mathcal{F} = \{X \subset \kappa | \forall \gamma (X \cap \gamma \in L_{f(\gamma)})\}.$$

As f is κ-recursive in the map $\gamma \mapsto S_\gamma$, we may say that each $\diamond^+_\kappa$-sequence computes a κ-Kurepa family.

3. HIF Degrees for Countable Admissible Ordinals

In this section we fix α to be a countable admissible ordinal and let $(e_n : n < \omega)$ be an enumeration of α. We define a nested sequence of good trees $(T^n : n < \omega)$ and pick A as the path on every T^n. A will be hyperimmune-free.

We begin with T^0 being the canonical good tree.

Suppose that T^n is defined to be a good tree by $(\epsilon_0^n, \epsilon_1^n)$.

Case 1. For any $\beta < \alpha$ and $\sigma \in dom(T^n)$ there exist τ_0 and τ_1 such that

$$\forall i < 2(\sigma\hat{}\langle i\rangle \subset \tau_i \wedge \Phi_{e_n}(\tau_i;\beta)\downarrow).$$

Let $T^{n+1} = Tot(T^n, e_n, 0)$.

Case 2. Case 1 fails.

Let $T^{n+1} = Par(T^n, e_n, 0)$

This ends the construction of $(T^n : n < \omega)$.

Let $A = \bigcup_{n<\omega} T^n(\emptyset)$. Then by Lemma 2.6, 2.7 and 2.8, A is of length α and HIF. We show that A is automatically not recursive.

Lemma 3.1. *$A \neq \Phi_e(\emptyset)$ for each $e < \alpha$.*

Proof. If $\Phi_e(\emptyset)$ is partial then it is trivial.

Assume that $\Phi_e(\emptyset)$ is total. Define $\bar{e}$ to be such that

$$\Phi_{\bar{e}}(\tau; x)\downarrow \Leftrightarrow L_{|\tau|} \models \exists y(\Phi_e(\emptyset; y)\downarrow = 1 - \tau(y)).$$

For any $n < \omega$ and $\sigma \in dom(T^n)$, $T^n(\sigma)$ has infinitely many incomparable extensions on T^n by (T2) in subsection 2.3. Hence $\alpha, \bar{e}$ and T^n always satisfy the assumption of Lemma 2.6. The assertion follows immediately as A is on some $Tot(T^n, \bar{e}, \lambda)$. □

Theorem 3.2. *If α is a countable Σ_1-admissible ordinal then there exists a non-recursive $A \subset \alpha$ which is HIF.*

Proof. Immediate from the above argument. □

4. Hyperimmune-Free Degree for Σ_2-admissible Ordinals

In the next three sections we show that there exist non-recursive HIF sets for Σ_2-admissible α.

Theorem 4.1. *If α is Σ_2-admissible then there exists a non-recursive HIF α-degree.*

As in the construction of a minimal α-degree in [8], we divide the proof into three cases:

(1) $\rho_\alpha^2 < \alpha$ where ρ_α^2 is the Σ_2 projectum of α,
(2) $\rho_\alpha^2 = \alpha$ and there exists a greatest α-cardinal below α, and
(3) neither of the above two cases holds.

We shall need some general properties of Σ_2-admissible ordinals, and refer the reader to [8, §3] for these. We remark that while the overall approach appears to run in parallel with that given in [8], there is significant difference in the execution of the construction of a HIF set, since the trees considered here are not perfect trees and different problems are presented.

As in the previous section we always build nested good trees by approximations. At successor stages, the definitions of Tot and Par operators (together with Lemmata 2.6 and 2.8) show that goodness property is preserved. But when α is uncountable, there is a need to handle the limit stages during construction. The subsection below shows how intersection of trees may be defined properly and Lemma 4.2 then guarantees that goodness is also preserved at limit stages.

The issue then becomes one of satisfying the hypothesis of Lemma 4.2. For this, we have to make the trees defined at limit stages capable of decoding the construction at each of its earlier stages (which in essence is an approximation procedure) from the root of the tree. As the approximation procedure is carried out in general using a priority argument, it becomes necessary ro code appropriate priority assignments into these roots. This results in some major differences in the coding strategy for the three cases of α in our construction.

Until further notice, α will denote a Σ_2-admissible.

4.1. **Recursive Intersection of Trees.** Let $(T^d : d < e)$ be a sequence of trees. Assume that in addition,

(1) $ran(T^d) \subset ran(T^c)$ for $c < d < e$, and
(2) $T^d(\sigma) = T^c(\rho) \wedge \sigma\hat{\ }\langle i\rangle \in dom(T^d) \rightarrow \rho\hat{\ }\langle i\rangle \in dom(T^c) \wedge T^c(\rho\hat{\ }\langle i\rangle) \preccurlyeq T^d(\sigma\hat{\ }\langle i\rangle)$ for all $i < 2$ and $c < d < e$.

Then $(T^d : d < e)$ is *a nested sequence of trees.*

Assume that f is recursive, and that $(T^d : d < e)$ is a uniform Σ_1 nested sequence of trees such that $ran(T^d_l) \subset ran(T^c_\gamma)$ for some $\gamma \in C^f(T^c)$ whenever $c < d < e$.

If $\bigcup_{d<e} T^d(\emptyset) \in L_{f(\gamma_0)}$ where $\gamma_0 = |\bigcup_{d<e} T^d(\emptyset)|$, then let $\hat{T}_0(\emptyset) = \bigcup_{d<e} T^d(\emptyset)$.

Suppose that $\hat{T}_l$ is defined and $ran(\hat{T}_l) \subset \bigcap_{d<e} ran(T^d)$. For each $\sigma \in dom(\hat{T}_l)$ and $d < e$, let ρ_d be such that $\hat{T}_l(\sigma) = T^d(\rho_d)$; for $i < 2$ let $\tau_i = \bigcup_{d<e} T^d(\rho_d\hat{\ }\langle i\rangle)$. If $\tau_0, \tau_1 \in L_{f(\gamma)}$ where $\gamma = |\tau_0| = |\tau_1|$, then let $\hat{T}_{l+1}(\sigma\hat{\ }\langle i\rangle) = \tau_i$ for $i < 2$.

At limit level l, suppose that $\hat{T}_k$ is defined for $k < l$. If $|\sigma| = l$, $\sigma \upharpoonright k \in dom(\hat{T}_k)$ for $k < l$ and $\sigma, \bigcup_{k<l} \hat{T}_k(\sigma \upharpoonright k) \in L_{f(\gamma)}$ where $\gamma = |\bigcup_{k<l} \hat{T}_k(\sigma \upharpoonright k)|$, then define $\hat{T}_l(\sigma) = \bigcup_{k<l} \hat{T}_k(\sigma \upharpoonright k)$.

Let $RI^f(T^d : d < e) = \hat{T}$. Write RI for RI^f if $f : x \mapsto x + 1$.

Lemma 4.2. *Let β, γ and $((\epsilon^d_0, \epsilon^d_1) : d < e)$ be such that*

(1) $((\epsilon^d_0, \epsilon^d_1) : d < e) \in L_{\gamma+1} \subset L_\beta \prec_1 L_\alpha$,
(2) $L_\beta \models ((\epsilon^d_0, \epsilon^d_1)$ *defines an f-good tree) for each $d < e$,*

(3) $\gamma = |\bigcup_{d<e} T^d(\emptyset)|$ *where* T^d *is the tree defined by* $(\epsilon_0^d, \epsilon_1^d)$ *for* $d < e$. *Then* $(\epsilon_0^e, \epsilon_1^e) \in L_\beta \models ((\epsilon_0^e, \epsilon_1^e)$ *defines an* f*-good tree), where* $(\epsilon_0^e, \epsilon_1^e)$ *is the index given by* $RI^f(T^d : d < e)$.

Proof. Let $T = RI^f(T^d : d < e)$. By induction, we show simultaneously that in L_β,

(i) $T(\emptyset)$ is defined,
(ii) if $\sigma \in dom(T)$ and $\eta > |\sigma|$ then $\sigma\hat{\ }\langle i0^{\bar\eta}\rangle \in dom(T)$ with $i < 2$ and $\eta = |\sigma| + 1 + \bar\eta$, and
(iii) if $\sigma \in dom(T)$ and $|T(\sigma)| = \gamma_\sigma$, then $\gamma_\sigma \in \bigcap_{d<e} C^f(T^d)$ and $(T(\sigma \upharpoonright \zeta) : \zeta \le \eta) \in L_{\gamma_\sigma+1}$ where $\eta = |\sigma|$.

By (2), $\gamma \in \bigcap_{d<e} C^f(T^d)$. Together with (1), $\bigcup_{d<e} T^d(\emptyset) \in L_{\gamma+1}$. Hence $\emptyset \in dom(T)$. (iii) is obvious for $\sigma = \emptyset$.

If $T(\sigma)$ is defined then $T(\sigma) \in L_{\gamma_\sigma+1}$ where $\gamma_\sigma = |T(\sigma)| \in \bigcap_{d<e} C^f(T^d)$. By (1, 2), $\{(d, \sigma_d) | d < e \wedge T^d(\sigma_d) = T(\sigma)\} \in L_{\gamma_\sigma+2}$. Let

$$\gamma_\sigma^+ = |\bigcup_{d<e} T^d(\sigma_d\hat{\ }\langle 0\rangle)| \in \bigcap_{d<e} C^f(T^d).$$

Hence $\bigcup_{d<e} T^d(\sigma_d\hat{\ }\langle 0\rangle) \in L_{\gamma_\sigma^+ + 1}$ and $\sigma\hat{\ }\langle 0\rangle \in dom(T)$. Similarly $\sigma\hat{\ }\langle 1\rangle \in dom(T)$. (iii) is again obvious.

Finally assume that $\sigma \in dom(T)$, $\rho = \sigma\hat{\ }\langle 0^{\bar\eta}\rangle$, $\eta = |\sigma| + i + \bar\eta$ is a limit ordinal, and $\rho \upharpoonright \zeta \in dom(T)$ is defined for each $\zeta < \eta$. Let $\gamma_\zeta = |T(\rho \upharpoonright \zeta)|$ and $\gamma_\eta = \sup_{\zeta<\eta} \gamma_\zeta$. Then $\eta \le \gamma_\eta \in \bigcap_{d<e} C^f(T^d)$. Now a binary sequence $\pi \prec \bigcup_{\zeta<\eta} T(\rho \upharpoonright \zeta)$, if and only if there exist $\zeta < \eta$ and $(\tau_\xi : \xi \le \zeta) \in L_{\gamma_\eta}$ such that

- $\tau_0 = T(\sigma)$,
- $\tau_{\nu+1} = \bigcup_{d<e} T^d(\sigma_d^\nu\hat{\ }\langle 0\rangle)$ where $T^d(\sigma_d^\nu) = \tau_\nu$, if $\nu \in (\zeta + 1) - \xi$,
- $\tau_\mu = \bigcup_{\nu<\mu} \tau_\nu$ if $\mu \in (\zeta + 1) - \xi$ is a limit, and
- $\pi \preccurlyeq \tau_\zeta$.

This shows that $\bigcup_{\zeta<\eta} T(\rho \upharpoonright \zeta) \in L_{\gamma_\eta+1} \subset L_{f(\gamma_\eta)}$ and $T(\rho)$ is defined. A similar definition yields (iii). □

4.2. **Enumerations of Σ_1-stable Ordinals.** As α is Σ_2-admissible, the collection of Σ_1-stable ordinals $\bar\alpha$ below α (i.e. $L_{\bar\alpha} \prec_1 L_\alpha$) is unbounded, and we may define α_β to be the β-th Σ_1-stable ordinal for each $\beta < \alpha$.

5. HIF Degree for Σ_2-admissible Ordinals: Case I

Theorem 5.1. *If* α *is a* Σ_2*-admissible ordinal with* $\rho_\alpha^2 < \alpha$ *then there is a non-recursive, where* ρ_α^2 *is the* Σ_2*-projectum of* α.

5.1. **Priority Assignment.** Let $\varphi(u, v, w, p)$ be a Π_1 formula such that $\exists w \varphi(u, v, w)$ defines a partial function $f(u) = v$ mapping ρ_α^2 onto α, where p is the only parameter.

Let $k(\beta, x) = v$ if $L_{\alpha_\beta} \models \exists w \varphi(x, v, w)$, and $k(\beta, x) = -1$ otherwise.

5.2. **The Construction.** For all $\beta < \alpha$, let $T^{0,\beta}$ be the canonical good tree defined in §2.3.

At stage $\beta < \alpha$, suppose that we have defined the following objects

(1) $(T^{e,\beta'} : e < l(\beta'))$ is a nested sequence of trees for $\beta' < \beta$,
(2) $\epsilon_0^{e,\beta'}$ as Σ_1-index of $T^{e,\beta'}$ for each $e < l(\beta')$ and $\beta' < \beta$,
(3) $\epsilon_1^{e,\beta'}$ as a possible Σ_1-index of the complement of $dom(T^{e,\beta'})$ for each $e < l(\beta')$ and $\beta' < \beta$, and
(4) $(I(e, \beta') : e + 1 < l(\beta'), \beta' < \beta)$ such that $I(e, \beta') \in 2$,

$$I(e, \beta') = 0 \to \exists\lambda(T^{e+1,\beta'} = Tot(T^{e,\beta'}, e, \lambda))$$

and

$$I(e, \beta') = 1 \to \exists\lambda(T^{e+1,\beta'} = Par(T^{e,\beta'}, e, \lambda)).$$

For each $e < \beta$, if $\lim_{\beta'<\beta}(\epsilon_0^{e,\beta'}, \epsilon_1^{e,\beta'})$ exists, i.e. $(\epsilon_0^{e,\beta'}, \epsilon_1^{e,\beta'})$ is fixed from some point on, then let the limit be $(\hat{\epsilon}_0^{e,\beta}, \hat{\epsilon}_1^{e,\beta})$. Let

$$l_0(\beta) = \sup\{e + 1 | (\hat{\epsilon}_0^{e,\beta}, \hat{\epsilon}_1^{e,\beta}) \text{ is defined}\}.$$

Note that for $e + 1 < l_0(\beta)$, by (4) above $\lim_{\beta'<\beta} I(e, \beta')$ exists.

Furthermore, let

$$l_1(\beta) = \sup\{e + 1 | e < l_0(\beta), k(\beta, e) = \lim_{\beta'<\beta} k(\beta', e)\}.$$

Let $e(\beta)$ be the least $e \leq l_1(\beta)$ such that

(E1) $e = l_1(\beta)$, or
(E2) $e < l_1(\beta)$ and $L_{\alpha_\beta} \models ((\hat{\epsilon}_0^{e,\beta}, \hat{\epsilon}_1^{e,\beta})$ does not define a good tree).

For $e < e(\beta)$,

(1) let $(\epsilon_0^{e,\beta}, \epsilon_1^{e,\beta}) = (\hat{\epsilon}_0^{e,\beta}, \hat{\epsilon}_1^{e,\beta})$,
(2) let $I(e, \beta) = \lim_{\beta'<\beta} I(e, \beta')$ if $e + 1 < e(\beta)$.

For $e = e(\beta)$, there are several cases.

Case 1. $e = l_1(\beta)$ is a limit ordinal.

Let $T^{e,\beta} = RI(T^{e',\beta} : e' < e)$ and $(\epsilon_0^{e,\beta}, \epsilon_1^{e,\beta})$ be the corresponding indices obtained from taking recursive intersection.

Case 2. $e = l_1(\beta) = e^- + 1$ for some e^-.

If $k(\beta, e^-) > -1$, then let $T^{e,\beta} = Tot(T^{e^-,\beta}, k(\beta, e^-), \alpha_\beta)$ and $I(e^-, \beta) = 0$.

If $k(\beta, e^-) = -1$, then let $T^{e,\beta} = T^{e^-,\beta}$ and $I(e^-, \beta) = 0$.

Case 3. $e < l_1(\beta)$. Then $e = e^- + 1$ for some e^- and $\lim_{\beta'<\beta} I(e^-, \beta') = 0$. Let $T^{e,\beta} = Par(T^{e^-,\beta}, k(\beta, e^-), \alpha_\beta)$ and $I(e^-, \beta) = 1$.

Finally let $l(\beta) = e(\beta)$. This ends the construction at stage β.

5.3. **The Verification.** We simultaneously show by induction on $e < \rho_\alpha^2$ that

(V1) $I_e = \{\beta < \alpha | l(\beta) \leq e\}$ is α-finite;
(V2) $(\epsilon_0^e, \epsilon_1^e) = \lim_{\beta<\alpha}(\epsilon_0^{e,\beta}, \epsilon_1^{e,\beta})$ defines a good α-tree, denoted by T^e;

(V3) if $e < l(\beta)$ then $((\epsilon_0^{d,\beta}, \epsilon_1^{d,\beta}) : d \le e) \in L_{\alpha_{\beta+1}}$ and $T^{e,\beta}(\emptyset)$ is either undefined or in $L_{\alpha_{\beta+1}}$.

(V1-3) hold trivially for $e = 0$.

Assume that $e = e^- + 1$ and (V1-3) hold for $d < e$.

Lemma 5.2. *If $\beta_0, \beta_1 \in I_{e^-}$ are such that $\beta_0 < \beta_1$ and $I_{e^-} \cap (\beta_1 - \beta_0) = 0$, then $|I_e \cap (\beta_1 - \beta_0)| \le 2$.*

Proof. Note that $k(\beta, e^-)$ could change at most once between β_0 and β_1, since we evaluate k in $L_{\alpha_\beta} \prec_1 L_\alpha$. Now assume that $\beta \in \beta_1 - \beta_0$ and k does not change on e^- at stage β.

If $k(\beta, e^-) = -1$, then $(\epsilon_0^{e,\beta}, \epsilon_1^{e,\beta}) = (\epsilon_0^{e^-,\beta}, \epsilon_1^{e^-,\beta})$ by the construction. So L_{α_β} observes that $T^{e,\beta}$ is not a good tree, if and only if it observes that $T^{e^-,\beta}$ is not a good tree.

If $k(\beta, e^-) > -1$ and (E2) holds with $e(\beta) = e$, it suffices to verify that $\lim_{\beta' < \beta} I(e^-, \beta') = 0$ as claimed by Case 3 of the construction.

Otherwise, $(\hat{\epsilon}_0^{e,\beta}, \hat{\epsilon}_1^{e,\beta}) \in L_{\alpha_\beta}$ defines some $Par(T^{e^-,\beta}, k(\beta, e^-), \lambda)$. By the definition of Par and $L_{\alpha_\beta} \prec_1 L_\alpha$, if L_{α_β} observes that $(\hat{\epsilon}_0^{e,\beta}, \hat{\epsilon}_1^{e,\beta})$ does not define a good tree then it also observes that $(\hat{\epsilon}_0^{e^-,\beta}, \hat{\epsilon}_1^{e^-,\beta})$ does not define a good tree. □

(V1) follows immediately from the above lemma. Thus

$$(\epsilon_0^e, \epsilon_1^e) = \lim_{\beta < \alpha} (\epsilon_0^{e,\beta}, \epsilon_1^{e,\beta})$$

exists. (V2) then follows from the definition of Tot and Par. Note that in the construction, when $(\epsilon_0^{e,\beta}, \epsilon_1^{e,\beta})$ is defined, only parameters in $L_{\alpha_\beta+1}$ are involved by inductive hypothesis. Hence (V3) holds.

Now we turn to limit $e < \rho_\alpha^2$. First assume that β_0 and β_1 are such that $\beta_0 < \beta_1 \in I_{<e} = \bigcup_{d<e} I_d$, $\beta_0 = \sup(I_{<e} \cap \beta_0)$ and $I_{<e} \cap (\beta_1 - \beta_0) = 0$.

Lemma 5.3. *If $\beta_0 \le \beta < \beta_1$ and $e(\beta) \ge e$ then $e(\beta) > e$.*

Proof. For each $d < e$, let $\beta_d < \beta$ be the least ordinal with

$$\forall \beta' (\beta_d \le \beta' < \beta \rightarrow (\epsilon_0^{d,\beta'}, \epsilon_1^{d,\beta'}) = (\epsilon_0^{d,\beta}, \epsilon_1^{d,\beta})).$$

Moreover let $\bar{\alpha}_d = \alpha_{\beta_d}$ and $\hat{\alpha}_d = \alpha_{\beta_d+1}$.

By the inductive hypothesis,

$$\forall d < e(((\epsilon_0^{c,\beta}, \epsilon_1^{c,\beta}) : c \le d) \in L_{\hat{\alpha}_d} \subset L_{\alpha_\beta}).$$

It follows that $T^{d,\beta}(\emptyset) \in L_{\hat{\alpha}_d}$ is of length between $\bar{\alpha}_d$ and $\bar{\alpha}_{d+1}$ for each $d < e$, as $\hat{\alpha}_d$ is Σ_1-stable and $\bar{\alpha}_d \le \bar{\alpha}_{d+1}$. Hence $\tau = \bigcup_{d<e} T^{d,\beta}(\emptyset)$ is of length $\bar{\alpha} = \sup_{d<e} \bar{\alpha}_d = \alpha_{\beta_0} \in \bigcap_{d<e} C(T^{d,\beta})$.

By Case 2 of the construction,

$$k(\beta, d) = y \Leftrightarrow L_{\bar{\alpha}} \models \exists w \varphi(d, y, w)$$

for $d < e$. By the construction and the definitions of Tot and Par, $T^{d+1,\beta} = Par(T^{d,\beta}, k(\beta, d), \lambda)$ for some λ, if

- $(\epsilon_0^{d,\beta}, \epsilon_1^{d,\beta}) \in L_\lambda$ and $k(\lambda, c) = k(\bar{\alpha}, c)$ for $c \leq d$,
- there exist $\sigma \in dom(T^{d,\beta}) \cap L_\lambda$ and $x < \lambda$ such that $\Phi_{k(\beta,d)}(T^{d,\beta}(\sigma'); x) \uparrow$ for any $\sigma' \in dom(T^{d,\beta}) \cap L_\lambda$ extending σ, and
- λ is the least ordinal with the above properties and $L_\lambda \prec_1 L_{\bar{\alpha}}$.

Otherwise for the least λ with $L_\lambda \prec_1 L_{\bar{\alpha}}$, $k(\lambda, c) = k(\bar{\alpha}, c)$ for $c \leq d$ and $(\epsilon_0^{d,\beta}, \epsilon_1^{d,\beta}) \in L_\lambda$, we have $T^{d+1,\beta} = Tot(T^{d,\beta}, k(\lambda, d), \lambda)$.

So the construction of $(T^{d,\beta} : d < e)$ could be carried out in $L_{\bar{\alpha}}$. As a consequence $((\epsilon_0^{d,\beta}, \epsilon_1^{d,\beta}) : d < e) \in L_{\bar{\alpha}+1}$. By Lemma 4.2, $L_{\alpha_\beta} \models$ ($T^{e,\beta}$ is a good tree) and (E2) does not happen at stage β to e. □

It follows from the above lemma that I_e and is α-finite. (V2) then follows immediately. In the proof of the above lemma, also observe that (V1) holds. (V3) follows from the last paragraph of the above proof.

Finally, A is HIF by Lemmata 2.7 and 2.8. It is not recursive by an argument similar to that of Lemma 3.1.

6. HIF Degrees for Σ_2-admissible Ordinals: Case II

Theorem 6.1. *If α is Σ_2-admissible, $\rho^2_\alpha = \alpha$ and there exists a greatest α-cardinal $\kappa < \alpha$, then there is a non-recursive HIF degree.*

6.1. **Priority Assignment.** We define an approximation of priority ordering recursive in $\emptyset'$, as in Shore [8, §5 Case 3] with a slight modification. Let h be the canonical Σ_1-Skolem function of $(L_\alpha, \emptyset')$, and

(1) $L_{\delta_0} = L_\kappa$,
(2) $L_{\delta_{\xi+1}} = h''(L_{\delta_\xi} \cup \{L_{\delta_\xi}\})$, and
(3) $L_{\delta_\zeta} = \bigcup_{\xi<\zeta} L_{\delta_\xi}$ when $\zeta < \alpha$ is a limit ordinal.

By the Σ_2-admissibility of α, the above sequence is unbounded in α. Let f_ξ be the $<_L$-least surjection mapping κ to $\delta_{\xi+1}$.

For each $\eta < \alpha$, h^η is the computation of h in $(L_{\alpha_\eta}, \emptyset' \upharpoonright \alpha_\eta)$. Using h^η we may define δ^η_ξ and f^η_ξ. Note that in [8, §5 Case 3], h^η is evaluated in $(L_\eta, \emptyset' \upharpoonright \eta)$.

So we define an approximation of priority ordering as

$$k(\eta, \kappa \times \xi + \zeta) = \begin{cases} f^\eta_\xi(\zeta), & \text{if } \eta \text{ is a successor;} \\ f^\eta_\xi(\zeta), & \text{if } \eta \text{ is a limit and } f^\eta_\xi(\zeta) = \lim_{\rho<\eta} f^\rho_\xi(\zeta); \\ f^\eta_\xi(\zeta) + 1, & \text{otherwise.} \end{cases}$$

6.2. **The Construction.** At the beginning, let $T^{0,0}$ be the canonical good tree.

At stage $\beta < \alpha$, suppose that we have defined the following objects:

(1) $(T^{e,\beta'} : e < l(\beta'))$ is a nested sequence of trees for $\beta' < \beta$,
(2) $\epsilon_0^{e,\beta'}$ as Σ_1-index of $T^{e,\beta'}$ for each $e < l(\beta')$ and $\beta' < \beta$,

(3) $\epsilon_1^{e,\beta'}$ as a speculated Σ_1-index of the complement of $dom(T^{e,\beta'})$ for each $e < l(\beta')$ and $\beta' < \beta$, and
(4) $(I(e,\beta') : e+1 < l(\beta'), \beta' < \beta)$ such that $I(e,\beta') \in 2$,

$$I(e,\beta') = 0 \rightarrow \exists\lambda(T^{e+1,\beta'} = Tot(T^{e,\beta'}, k(\beta', e), \lambda))$$

and

$$I(e,\beta') = 1 \rightarrow \exists\lambda(T^{e+1,\beta'} = Par(T^{e,\beta'}, k(\beta', e), \lambda)).$$

Firstly, we take a subsequence of trees which have stabilized. Let $l_0(\beta)$ be the supremum of $e+1$ such that both $\hat{\epsilon}_0^{e,\beta} = \lim_{\beta'<\beta} \epsilon_0^{e,\beta'}$ and $\hat{\epsilon}_1^{e,\beta} = \lim_{\beta'<\beta} \epsilon_1^{e,\beta'}$ exist.

Secondly, we require that the priority ordering k has also stabilized on some initial segment and take a further subsequence accordingly. Let

$$\delta(\beta) = \sup\{\delta_\xi^\beta | \delta_\xi^\beta = \lim_{\beta'<\beta} \delta_\xi^{\beta'}\} = \delta_{\xi(\beta)}^\beta,$$

and let

$$l_1(\beta) = \min\{l_0(\beta), \kappa \times \xi(\beta) + 1\}.$$

For each $e < l_1(\beta)$ let $\hat{T}^{e,\beta}$ be the tree defined by the Σ_1-formula of index $\hat{\epsilon}_0^{e,\beta}$.

Let $e(\beta)$ be the least $e \leq l_1(\beta)$ such that

(E1) $e = l_1(\beta)$, or
(E2) $e < l_1(\beta)$ and $L_{\alpha_\beta} \models ((\hat{\epsilon}_0^{e,\beta}, \hat{\epsilon}_1^{e,\beta})$ does not define a good tree).

For $e < e(\beta)$, let

(1) $T^{e,\beta} = \hat{T}^{e,\beta}$ and $\epsilon_i^{e,\beta} = \hat{\epsilon}_i^{e,\beta}$ for $i < 2$,
(2) $I(e,\beta) = \lim_{\beta'<\beta} I(e,\beta')$ if $e+1 < e(\beta)$.

We build or rebuild a tree for $e(\beta)$ by cases.

Case 1. $e(\beta) = l_1(\beta)$ is a limit ordinal.

Let $T^{e(\beta),\beta}$ be the recursive intersection of $(\hat{T}^{e,\beta} : e < e(\beta))$.

Case 2. $e(\beta) = l_1(\beta) = d + 1 < \kappa \times \xi(\beta) + 1$.

Let λ be the least ordinal with $((\epsilon_0^{c,\beta}, \epsilon_1^{c,\beta}) : c \leq d) \in L_\lambda$.

Let $T^{e(\beta),\beta} = Tot(T^{d,\beta}, k(\beta, d), \lambda)$, $\epsilon_i^{e(\beta),\beta}$ be the corresponding index given by Tot for $i < 2$, and $I(d,\beta) = 0$.

Case 3. $e(\beta) = l_1(\beta) = \kappa \times \xi(\beta) + 1$.

Let $\lambda(\beta)$ be the least $\lambda > \delta_{\xi(\beta)+1}^\beta$ with $\{(x, k(\beta, x)) | x \in \delta_{\xi(\beta)+1}^\beta\} \in L_\lambda$, and let

$$T^{e(\beta),\beta} = Tot(\hat{T}^{e(\beta)-1,\beta}, k(\beta, e(\beta)), \lambda(\beta)).$$

Moreover, let $\epsilon_i^{e(\beta),\beta}$ be the corresponding index in the definition of Tot for $i < 2$, and let $I(\delta(\beta), \beta) = 0$.

Thus we have coded the priority ordering of $\delta_{\xi(\beta)+1}^\beta - \delta(\beta)$ into $T^{e(\beta),\beta}(\emptyset)$ so that the latter has the information on this ordering.

Case 4. (E2) holds. Then $e(\beta) = d+1$ for some d, and $\lim_{\beta'<\beta} I(d, \beta') = 0$. Let ξ and η be such that $e(\beta) = \kappa \times \xi + \eta + 1$, and let λ be the least ordinal with

$$\delta^{\beta}_{\xi+1}, \{(x, k(\beta, x)) | x \in \delta^{\beta}_{\xi+1}\}, ((\epsilon^{c,\beta}_0, \epsilon^{c,\beta}_1) : c \leq d) \in L_\lambda.$$

Let $T^{d+1,\beta} = Par(\hat{T}^{d,\beta}, k(\beta, d), \lambda)$, $\epsilon^{e(\beta),\beta}_i$ be the corresponding index in the definition of Par for $i < 2$, and let $I(d, \beta) = 1$.

Finally let $l(\beta) = e(\beta) + 1$.

6.3. **The Verification.** We assume that κ is Σ_2-regular and show simultaneously by induction that for $e = \kappa \times \xi + \eta$ where $\eta < \kappa$,

(V1) $I_{<e} = \{\beta < \alpha | l(\beta) < e\}, I_e = \{\beta < \alpha | l(\beta) \leq e\} \in L_{\delta_{\xi+2}}$,
(V2) $I^-_e = I_e - I_{<\kappa\times\xi}$ has $\delta_{\xi+2}$-cardinality $< \kappa$,
(V3) if $\epsilon^{e,\beta}_i$ $(i < 2)$ is defined then $\epsilon^{e,\beta}_i \in \delta_{\xi+2}$,
(V4) if $\sup I_e \leq \beta < \alpha$, then $(\epsilon^{e,\beta}_0, \epsilon^{e,\beta}_1)$ defines a good tree $T^{e,\beta}$.

(V1-V4) hold trivially for $e = 0$.

Let $e = \kappa \times \xi + \eta > 0$. Assume that (V1-V4) hold below e. (V3) is obvious because the construction of $T^{e,\beta}$ can be restricted to $L_{\delta_{\xi+2}}$ as $L_{\delta_{\xi+2}} \prec_2 L_\alpha$.

Lemma 6.2. *$I_{<e} \in L_{\delta_{\xi+2}}$ and $I^-_{<e} = I_{<e} - I_{<\kappa\times\xi} \in L_{\delta_{\xi+2}}$ is of $\delta_{\xi+2}$-cardinality $< \kappa$.*

Proof. Note that $e \leq \delta_{\xi+1}$. The map $d \mapsto I_d$ for $d < \kappa \times \xi$ is definable over $L_{\delta_{\xi+1}}$. Thus $I_{<\kappa\times\xi} \in L_{\delta_{\xi+2}}$.

The sequence $(I^-_d : \kappa \times \xi \leq d < e)$ is uniformly Σ_2 of length $\eta < \kappa$. By the inductive hypothesis and [8, Lemma 3.2],

$$I^-_{<e} = \bigcup_{\kappa\times\xi\leq d<e} I^-_d \in L_{\delta_{\xi+2}}$$

is of $\delta_{\xi+2}$-cardinality $< \kappa$.

It follows that $I_{<e} = I_{<\kappa\times\xi} \cup I^-_{<e} \in L_{\delta_{\xi+2}}$. $\square$

First we show that (E2) does not hold for $e = \kappa \times \xi + \eta$ a limit.

Lemma 6.3. *If $e < l_1(\beta)$ and $e \leq e(\beta)$ then $e < e(\beta)$.*

Proof. Let $\rho_0 = \bigcup_{d<e} T^{d,\beta}(\emptyset)$ be of length γ_0. Then $\gamma_0 \in \bigcap_{d<e} C(T^{d,\beta})$, and

$$\forall d < e(((\epsilon^{c,\beta}_0, \epsilon^{c,\beta}_1) : c \leq d < e) \in L_{\gamma_0})$$

by Case 2 and 4 in the construction.

By Case 3 in the construction, $\delta^{\gamma_0}_\zeta = \delta^{\beta}_\zeta$ for $\zeta \leq \hat{\xi}$, where $\hat{\xi} = \xi$ if $\eta = 0$ or $\hat{\xi} = \xi + 1$ if $\eta > 0$. Moreover $k(\gamma_0, \kappa \times \zeta + \nu) = k(\beta, \kappa \times \zeta + \nu)$ for $\kappa \times \zeta + \nu < \kappa \times \hat{\xi}$, thus also for $\kappa \times \zeta + \nu < e$.

By the definition of Tot and Par, $T^{c+1,\beta} = Par(T^{c,\beta}, k(\gamma_0, c), \lambda)$ for some $c < e$ and λ if and only if there exist $\sigma \in dom(T^{c,\beta}) \cap L_{\gamma_0}$ and $x \in \gamma_0$ such that $\Phi_{k(\gamma_0,c)}(T^{c,\beta}(\tau); x) \uparrow$ for any $\tau \in dom(T^{c,\beta}) \cap L_{\gamma_0}$ extending σ.

The predicate above is actually about $(\epsilon_0^{c,\beta}, \epsilon_1^{c,\beta})$. Let ψ denote this formula.

If $c = \kappa \times \zeta + \nu$, then the λ above would be the least ordinal with

$$\delta^{\gamma_0}_{\zeta+1}, \{(x, k(\gamma_0, x)) | x \in \delta^{\gamma_0}_{\zeta+1}\}, ((\epsilon_0^{c,\beta}, \epsilon_1^{c,\beta}) : c \leq d) \in L_\lambda.$$

Such λ exists below γ_0 again by Case 3 in the construction.

So the construction of $(T^{d,\beta} : d < e)$ could be carried out in L_{γ_0} and $((\epsilon_0^d, \epsilon_1^d) : d < e) \in L_{\gamma_0+1}$. By Lemma 4.2, $e(\beta) > e$. □

Thus (V2) holds. (V1, V3-V4) follow immediately.

Now assume that $\eta = \theta + 1 < \kappa$.

Lemma 6.4. *(V1-V4) hold for $e = \kappa \times \xi + \eta$.*

Proof. Let $d = \kappa \times \xi + \theta$.

It follows from the construction and the definition of Par, between two stages in $I_{<e} - \delta_{\xi+1}$, there can be at most one $\beta \in I_e$.

For $\beta \in \delta_{\xi+2} - \sup I_{<e}$, $T^{d,\beta}$ is a good α-tree and $T^{e,\beta}$ depends on whether (E2) in the construction holds. But (E2) is a Σ_2 statement with parameters in $\delta_{\xi+2}$, by the inductive hypothesis on (V4) for d. As $L_{\delta_{\xi+2}} \prec_2 L_\alpha$, either (E2) holds at some $\beta \in \delta_{\xi+2}$ or it never holds.

So there exists at most one element in $I_e - \sup(\delta_{\xi+1} \cup I_{<e})$. This proves (V1-V2) for e, and also (V4) as the above argument shows that the decision of (E2) happens below $\delta_{\xi+2}$. □

If κ is singular in L_α, then we may fix a cofinal sequence of regular cardinals below κ, and reformulate (V2) as

(V2') I_e^- has $\delta_{\xi+2}$ cardinality η^+ where $e = \kappa \times \xi + \eta$ and η^+ is the least α-regular cardinal greater than η in the fixed sequence.

The earlier argument may then be adapted to handle this situation.

Finally A is non-recursive and HIF as in §5.

7. HIF Degrees for Σ_2-admissible Ordinals: Case III

Theorem 7.1. *If α is a Σ_2-admissible ordinal with $\rho^2_\alpha = \alpha$ and there is no greatest α-cardinal, then there is a non-recursive HIF degree.*

This is the simplest case for Σ_2-admissible α.

To build a HIF set A, we construct a nested sequence of trees as before and A will be a path through each of the trees.

At stage 0, let $T^{0,0}$ be the canonical good tree.

At stage $\beta < \alpha$, suppose that we have defined the following objects

(1) $(T^{e,\beta'} : e < l(\beta'))$ for $\beta' < \beta$,

(2) $\epsilon_0^{e,\beta'}$ as Σ_1-index of $T^{e,\beta'}$ and $\epsilon_1^{e,\beta'}$ as a possible Σ_1-index of the complement of $dom(T^{e,\beta'})$ for $e < l(\beta')$ and $\beta' < \beta$,

(3) $I(e, \beta') < 2$ for $e + 1 < l(\beta')$ and $\beta' < \beta$ with

$$I(e, \beta') = 0 \to \exists\lambda(T^{e+1,\beta'} = Tot(T^{e,\beta}, e, \lambda))$$

and

$$I(e, \beta') = 1 \to \exists\lambda(T^{e+1,\beta'} = Par(T^{e,\beta}, e, \lambda)).$$

Let

$$l_0(\beta) = \sup\{e + 1 | (\hat{\epsilon}_0^{e,\beta}, \hat{\epsilon}_1^{e,\beta}) = \lim_{\beta'<\beta}(\epsilon_0^{e,\beta}, \epsilon_1^{e,\beta}) \text{ exists}\}.$$

Let $e(\beta)$ be the least $e \le l_0(\beta)$ with

(E1) $e = l_0(\beta)$, or

(E2) $e < l_0(\beta)$ and $L_{\alpha_\beta} \models ((\hat{\epsilon}_0^{e,\beta}, \hat{\epsilon}_1^{e,\beta})$ does not define a good tree).

For $e < e(\beta)$, let

(1) $(\epsilon_0^{e,\beta}, \epsilon_1^{e,\beta}) = (\hat{\epsilon}_0^{e,\beta}, \hat{\epsilon}_1^{e,\beta})$, and

(2) $I(e, \beta) = \lim_{\beta'<\beta} I(e, \beta')$ if $e + 1 < e(\beta)$.

Case 1. $e(\beta) = l_0(\beta)$ is a limit ordinal.

Let $T^{e(\beta),\beta} = RI(T^{d,\beta} : d < e(\beta))$ and $(\epsilon_0^{e(\beta),\beta}, \epsilon_1^{e(\beta),\beta})$ be the index obtained from the RI operation.

Case 2. $e(\beta) = l_0(\beta) = d + 1$ for some d.

Let λ be the least ordinal with $e(\beta) < \lambda$ and $((\epsilon_0^{c,\beta}, \epsilon_1^{c,\beta}) : c \le d) \in L_\lambda$. Let $T^{e(\beta),\beta} = Tot(T^{d,\beta}, d, \lambda)$, and $(\epsilon_0^{e(\beta),\beta}, \epsilon_1^{e(\beta),\beta})$ be the index obtained from the Tot operation.

Case 3. (E2) happens to $e(\beta)$. Then $e(\beta) = d + 1$ for some d.

Let λ be the least ordinal with $e(\beta) < \lambda$ and $((\epsilon_0^{c,\beta}, \epsilon_1^{c,\beta}) : c \le d) \in L_\lambda$. Let $T^{e(\beta),\beta} = Par(T^{d,\beta}, d, \lambda)$, and $(\epsilon_0^{e(\beta),\beta}, \epsilon_1^{e(\beta),\beta})$ be the index obtained from the Par operation.

Let $l(\beta) = e(\beta) + 1$.

This ends the construction of nested trees. The verification is much simpler than the previous two cases and is left to the reader.

8. Regular Cardinals with Many Hyperimmune-Free Degrees

The construction in §§5–7 provides a hyperimmune-free degree below $\mathbf{0}''$ for Σ_2 admissible ordinals α. Our focus in this section is to study the cardinality of the set of hyperimmune-free degrees. Because of the special properties associated with total trees and partial trees in our hyperimmune-free considerations, the "width" of the tree at each level has to be α-finite. This would seem to naturally limit the number of possible unbounded paths through each of trees being considered during the construction, and therefore the number of possible unbounded paths eventually emerging from intersecting sequences of trees. In particular, it is conceivable that for any α, the cardinality of the set of hyperimmune-free degrees is at most $|\alpha|$. This turns out not to be so. Indeed, in the case of ω_1 for example, the situation is reminiscent of the existence of an ω_1-Kurepa tree, a tree with countable nodes at every level and yet with ω_2 unbounded paths. Such an object was first obtained by Jensen using the combinatorial principle $\diamondsuit^+_{\omega_1}$. In fact the solution to obtaining ω_2 many hyperimmune-free degrees lies in an application of $\diamondsuit^+_{\omega_1}$, and the method may be generalized to all regular κ where $\diamondsuit^+_\kappa$ holds.

First of all we introduce a new operator on trees. If T is a tree and $\sigma \in dom(T)$, then let $\hat{T} = Ext(T, \sigma)$ be the tree such that $\hat{T}(\rho) = T(\sigma\hat{\ }\rho)$ if $\sigma\hat{\ }\rho \in dom(T)$.

Theorem 8.1. *There are ω_2 many HIF for ω_1.*

Proof. Let $f : \omega_1 \to \omega_1$ be an ω_1-recursive function with $f(x) \geq x + 1$. Assume that $P \in 2^{\omega_1}$ is such that $P \restriction x \in L_{f(x)}$ for all $x < \omega_1$. We associate a hyperimmune-free $A(P) \subset \omega_1$ to P.

At the beginning let $T^0 = T^f$, and let $\lambda < \omega_1$ denote the least ordinal with L_λ containing the parameter defining f via a Σ_1 formula.

If $e < \omega_1$ and T^e is defined to be an f-good tree, then we define T^{e+1} by cases.

Case 1. $Tot^f(T^e, e, \lambda)$ is an f-good tree.

Let $\tilde{T}^{e+1} = Tot^f(T^e, e, \lambda)$ and $T^{e+1} = Ext(\tilde{T}^{e+1}, \langle P(e)\rangle)$.

Case 2. Case 1 fails.

Let $\tilde{T}^{e+1} = Par^f(T^e, e, \lambda)$ and $T^{e+1} = Ext(\tilde{T}^{e+1}, \langle P(e)\rangle)$.

If $e < \omega_1$ is a limit ordinal and $(T^d : d < e)$ is defined to be a family of f-good trees, then let $T^e = RI^f(T^d : d < e)$.

This ends the construction of a nested family of f-good trees $(T^e : e < \omega_1)$.

As before, the construction does not break and $A(P) = \bigcup_{e<\omega_1} T^e(\emptyset)$ is HIF and not ω_1-recursive.

According to the above construction, if $P, Q \in 2^{\omega_1}$ are such that $P \restriction x, Q \restriction x \in L_{f(x)}$ for all $x < \omega_1$ and $P \neq Q$, then $A(P) \neq A(Q)$.

By Jensen's proof of $L \models KH$, there exists an ω_1-recursive f such that $\{P \subset \omega_1 | \forall\gamma < \omega_1(P \cap \gamma \in L_{f(\gamma)})\}$ is a Kurepa family. The theorem follows immediately. $\square$

The above argument has an immediate generalization:

Theorem 8.2. *If κ is a regular cardinal with a κ-recursive $\diamondsuit^+_\kappa$ sequence, then there exist κ^+ many non-κ-recursive HIF $A \subset \kappa$.*

Proof. Let $f : \kappa \to \kappa$ be a κ-recursive function such that $S_x \in L_{f(x)}$ where $(S_x : x < \kappa)$ is a recursive $\diamondsuit^+_\kappa$-sequence, and let $\bar{f}(x)$ be the least Σ_1-admissible ordinal greater than $x < \kappa$.

The proof of [1, Theorem VII.3.2] actually shows that

$$\mathcal{F} = \{P \subset 2^\kappa | \forall x < \kappa(P \cap x \in L_{\bar{f}(x)})\}$$

is a κ-Kurepa family. In particular $|\mathcal{F}| = \kappa^+$.

The theorem then follows as in Theorem 8.1. □

Corollary 8.3. *If κ is a successor cardinal, then there exist κ^+ many HIF sets.*

Definition 8.4. *Let κ be a regular cardinal. If θ is $\Pi^1_1(L_\kappa)$ with parameters in L_κ such that L_κ satisfies φ but L_x does not satisfy θ for any $x < \kappa$, then we call κ* strongly Π^1_1-describable *via θ.*

Proposition 8.5. *If κ is strongly Π^1_1-describable via some θ, then there exists a recursive $\diamondsuit^+_\kappa$-sequence. Hence there are κ^+ many HIF $X \subset \kappa$.*

Proof. Let $\theta = \forall X \psi$ where ψ is a first order formula.

For each $x < \kappa$, let $f(x)$ be the least admissible $\beta > x$ such that there exists $X \in L_\beta$, $X \subset x$ and $L_\beta \models ((L_x, \in, X) \models \neg\psi)$.

Then $(L_{f(x)} \cap \mathcal{P}(x) : x < \kappa)$ is a $\diamondsuit^+_\kappa$-sequence and f is κ-recursive, as in [1, Theorem III.3.7].

The remaining part follows from Theorem 8.2. □

Corollary 8.6. *If κ is the least inaccessible cardinal, then there are κ^+ many HIF.*

Proof. Note that being the least inaccessible cardinal is a Π^1_1 property. □

9. Ordinals with Few Hyperimmune Free Degrees

Let κ be a regular cardinal and $f : \kappa \to \kappa$. A κ-Kurepa family $\mathcal{F}$ is *f-good* if and only if

$$\forall \gamma < \kappa (\mathcal{F} \restriction \gamma = \{X \cap \gamma | X \in \mathcal{F}\} \in L_{f(\gamma)}).$$

Theorem 9.1. *If there is no f-good κ-Kurepa family for any κ-recursive f, then there are at most κ many HIF $X \subset \kappa$. In particular for κ without κ-recursive $\diamondsuit^+_\kappa$-sequence there are at most κ many HIF $X \subset \kappa$.*

Proof. For $X \subset \kappa$, let $g_X(\gamma)$ be the $<_L$-rank of $X \restriction \gamma$. Then $g_X \leq_\kappa X$.

If A is hyperimmune-free then g_A is majorized by some κ-recursive $f : \kappa \to \kappa$ with $f(x) \geq x + 1$ (see Lemma 2.3). Thus A is on some T^f. As there is no f-good κ-Kurepa family, $|[T^f]| \leq \kappa$ for each κ-recursive f.

Hence there are at most κ many HIF $A \subset \kappa$.

The remaining part follows from the remark in subsection 2.7. □

The corollary below shows that not all regular cardinals κ have κ^+ many HIF degrees. Recall that $\kappa > \omega$ is *ineffable* if for any partition of two-element subsets of κ into two colors, there is a homogeneous stationary subset. It is known that ineffable cardinals do not have κ-Kurepa trees ([1]).

Corollary 9.2. *Let $\kappa > \omega$ be an ineffable cardinal. Then there are at most κ many HIF $A \subset \kappa$.*

Proof. By Theorem 9.1 together with either Theorem VII.3.1 or Exercise VII.3A of [1]. $\square$

The Σ_2-*cofinality* of an ordinal α, denoted by $\sigma 2cf(\alpha)$, is the least $\gamma < \alpha$ such that there is a $\Sigma_2(L_\alpha)$ cofinal total function $f : \gamma \to \alpha$.

Theorem 9.3. *If κ is a cardinal with $\omega < \sigma 2cf(\kappa) < \kappa$, then every HIF $A \subset \kappa$ is κ-recursive. In particular, $\aleph_{\omega_1}$ has no non-$\aleph_{\omega_1}$-recursive HIF subset.*

Proof. Under the assumption, $\lambda = \sigma 2cf(\kappa)$ is an uncountable regular cardinal and the cofinal function $f \leq_\kappa \emptyset'$. We may assume that $f(\gamma)$ is a cardinal for each $\gamma < \lambda$ and f is normal, i.e. f is strictly increasing and $f(\gamma') = \sup_{\gamma<\gamma'} f(\gamma)$ whenever $\gamma' < \lambda$ is a limit ordinal.

For $X \subset \kappa$, let g_X be as in the proof of Theorem 9.1. Define $h_X : \lambda \to \kappa$ as $h_X(\gamma) = g_X \circ f(\gamma)$. h_X is essentially the *cutoff* function defined by Sy D. Friedman [3, §1].

Assume A is HIF, then g_A is majorized by a κ-recursive function b. Define $\bar{b}(\gamma) = b \circ f(\gamma)$, then $\bar{b} : \lambda \to \kappa$ is κ-recursive in $\emptyset'$ and majorizes h_A.

Using an argument similar to the proof of [3, Lemma 2], we can conclude that $A \leq_\kappa \emptyset'$. This is done as follows: For each $\gamma < \lambda$, let $c_\gamma : \bar{b}(\gamma) \to |\bar{b}(\gamma)|$ be a bijection. Let $t(\gamma)$ be the least $\delta < \lambda$ such that $c_\gamma(h_A(\gamma)) < f(\delta)$. Then $t(\gamma) < \gamma$ on stationary many $\gamma < \lambda$. By Fodor's Theorem there is a stationary set $W \subset \lambda$ and a $\delta_0 < \lambda$ such that $t(\gamma) = \delta_0$ for all $\gamma \in W$. Then W is κ-finite and so using $\emptyset'$ (which computes all cardinals in κ) one is able to retrieve the set $\{h_A(\gamma) | \gamma \in W\}$ using $\{e_\gamma(h_A(\gamma)) | \gamma \in W\}$ as the parameter set. Then $A \leq_\kappa h_A \leq_\kappa \bar{b} \leq_\kappa \emptyset'$.

By Lemma 2.3, A is κ-recursive. $\square$

Corollary 9.4. *Let κ be singular of uncountable cofinality λ. Let $f : \lambda \to \kappa$ be normal such that $f(\gamma)$ is a cardinal for all $\gamma < \lambda$. Then there are at most κ many HIF degrees, and each of these is recursive in the degree of f.*

Proof. Similar to the proof of the above theorem, except to note that $A \leq_\kappa h_A \leq_\kappa f$. $\square$

For an arbitrary admissible α and $X \subset \alpha$, the function g_X introduced above need not be total since X may not be regular. However, since we only consider regular sets with regard to hyperimmune-free degrees, such a situation does not arise. The idea introduced in the previous theorem may now be adapted to derive additional results bounding the size of hyperimmune-free degrees for a wider class of admissible ordinals.

Theorem 9.5. *If $cf(\alpha) \neq cf(|\alpha|)$ then there are at most $|\alpha|$ many HIF subsets $X \subset \alpha$, and all of these are recursive in the cofinality function. In particular, if α is not Σ_2 admissible and $\rho^2_\alpha = |\alpha| > \sigma 2cf\alpha)$, then $cf(\alpha) \neq cf(|\alpha|)$ implies that there is no nontrivial hyperimmune-free degree.*

Proof. Let $\lambda = cf(\alpha)$ and $f : \lambda \to \alpha$ be a cofinal map. For $X \subset \alpha$ regular, define g_X as in Theorem 9.1. Let $h_X = g_X \circ f : \lambda \to \alpha$. Note that in both cases $\lambda \leq |\alpha|$.

If A is HIF then there exists a recursive $b : \alpha \to \alpha$ majorizing g_A. Thus $\bar{b} = b \circ f : \lambda \to \alpha$ majorizes h_A. For each $\gamma < \lambda$, let $c_\gamma : \bar{b}(\gamma) \to |\alpha|$ be an injection. There are two cases to consider.

Case 1. $cf(\alpha) < cf(|\alpha|)$. Then the set $(c_\gamma \circ h_A)''\lambda$ is bounded in $|\alpha|$ and thus $|\alpha|$-finite. But A can be α-recursively retrieved from f, $(c_\gamma : \gamma < \lambda)$ and $(c_\gamma \circ h_A)''\lambda$.

It follows that each α-recursive function can majorize g_A for at most $|\alpha|$ many A. Hence there are at most $|\alpha|$ many HIF subsets. Since $(c_\gamma : \gamma < \lambda) \leq_\alpha f$ and $(c_\gamma \circ h_A)''\lambda$ is α-finite, every HIF is recursive in f.

Case 2. $cf(\alpha) > cf(|\alpha|)$. Then $|\alpha| > \lambda$ and there is a $\delta_0 < |\alpha|$ such that unboundedly many $\gamma < \lambda$ satisfies $c_\gamma \circ h_A(\gamma) < \delta_0$. The set W of such γ's is $|\alpha|$-finite. Then $A \leq_\alpha f$ using W and δ_0 as parameters through the functions $\langle c_\gamma | \gamma \in W \rangle$.

If α is not Σ_2 admissible and $\sigma 2cf(\alpha) < |\alpha|$, then $cf(\alpha) = \sigma 2cf(\alpha)$ and f may be taken as a Σ_2 cofinal function. In such a situation, any HIF set A is recursive in f, and thus recursive by Lemma 2.3. □

Corollary 9.6. *Let $n \geq 1$. Then ω_{n+1} is the limit of admissible ordinals α with no HIF sets.*

Proof. Note that ω_{n+1} is the limit of admissible ordinals with Σ_2-cofinality equal to ω_n. □

We end this paper with some open problems, still assuming $V = L$ for the first three:

1. If κ is a singular cardinal of countable Σ_2-cofinality, does κ have a hyperimmune-free degree? We take $\aleph_\omega$ as the test case. The case for countable cofinality is particularly interesting because the tools used in uncountable cofinality, such as those related to Fodor's theorem on regressive functions, are no longer available, while the problems associated with Σ_2-inadmissibility persist.
2. It is not difficult to show that if α is countable, then there are 2^ω many hyperimmune-free degrees. Is there any uncountable α such that $|\alpha| < \alpha$ with $2^{|\alpha|}$ many hyperimmune-free degrees?
3. What is the ordering of the hyperimmune-free degrees in the upper semilattice of α-degrees for admissible α?
4. Investigate a theory of *HIF* sets in the absence of $V = L$.

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