NONSTANDARD METHODS IN RAMSEY'S THEOREM FOR PAIRS

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Abstract. We discuss the use of nonstandard methods in the study of Ramsey type problems, and illustrate this with an example concerning the existence of definable solutions in models of $B\Sigma^0_2$ for the combinatorial principles of Ramsey's Theorem for pairs and cohesiveness.

1. Introduction

Ramsey's Theorem (RT^n_k) states that every partition of the n-element subsets of $\mathbb{N}$ into k classes has an infinite homogeneous set. The proof-theoretic strength of RT^n_k is an area of active research by recursion theorists in recent years.

Taking the system of second order arithmetic RCA_0 as the base theory, an earlier result of Jockusch [9] implies that for $n > 2$, the arithmetic comprehension axiom is a consequence of RT^n_k. This was shown not to hold under RT^2_2 by Seetapun and Slaman [12], so that over RCA_0, RT^2_2 is strictly weaker than RT^n_k for $n > 2$.

In a systematic study of RT^2_2, Cholak, Jockusch and Slaman [1] introduced two combinatorial principles related to RT^2_2, defined over a model $\mathcal{M} = \langle M, \mathbb{X}, +, \cdot, 0, 1\rangle$ of RCA_0, where $\mathbb{X}$ is the collection of second order objects of $\mathcal{M}$:

(1) The principle of stable Ramsey's Theorem for pairs (SRT^2_2): A partition f of $[M]^2$ into two classes (sometimes also called "two colors") is said to be *stable* if $\lim_s f(x,s)$ exists for all $x \in M$. SRT^2_2 states that every stable partition of $[M]^2$ into two classes has an infinite homogeneous set in $\mathcal{M}$.

1991 *Mathematics Subject Classification.* 03D20, 03F30, 03H15.

The author wishes to thank the Department of Mathematics, University of California, Berkeley, for its hospitality during his visit from October to December 2005, where part of the research was done. This paper is a revised version of a talk given in August 2005 at the Program in Computational Prospects of Infinity held at the Institute for Mathematical Sciences, National University of Singapore .

(2) The principle of cohesiveness (COH): Let Y be an M-infinite set in $\mathcal{M}$ and let $Y_s = \{t|(s,t) \in Y\}$. $\langle Y_s \rangle_{s\in M}$ is called a Y-array. Then a set C is Y-cohesive if for all s, either $C \cap Y_s$ is $\mathcal{M}$-finite or $C \cap \bar{Y}_s$ is $\mathcal{M}$-finite. COH states that every Y-array, where $Y \in \mathcal{M}$, has a Y-cohesive set $C \in \mathcal{M}$.

It is known ([1]) that over the base theory RCA_0, Ramsey's Theorem for pairs is equivalent to $\mathrm{SRT}^2_2 + \mathrm{COH}$. In view of the definition of stable 2-coloring, it is natural to conjecture that SRT^2_2 is strictly weaker than RT^2_2. Much effort has been expended on establishing this conjecture. First of all, it is not difficult to see that, given a stable partition $f \in \mathcal{M}$ of $[M]^2$ into two colors, there is an f-homogeneous set Δ^0_2 in the parameter defining f. Jockusch [9] exhibited a recursive partition f_J of $[\mathbb{N}]^2$ into two classes that has no Δ_2 homogeneous solution (thus the partition is necessarily nonstable). A first attempt to proving the relative strength of SRT^2_2 and RT^2_2 is to show that every stable recursive partition of $[\mathbb{N}]^2$ into two colors has a low homogeneous solution (i.e. whose jump is $\emptyset'$). An iteration of the construction will then produce a model of $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$ that does not include a homogeneous solution to f_J according to Jockusch's result. Downey, Hirschfeldt, Lempp and Solomon [6] showed that this approach failed by demonstrating the existence of a Δ_2 set A of integers with no infinite low subset in A or $\bar{A}$. Thus to produce a model $\mathcal{M} = \langle M, \mathbb{X}, +, \cdot, 0, 1\rangle$ of $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$ without RT^2_2 where $M = \mathbb{N}$ (i.e. an ω-model) requires a more sophisticated approach. In [7], Hirschfeldt, Jockusch, Kjos-Hanssen, Lempp and Slaman show that while a low homogeneous set is not always guaranteed for a stable 2-coloring of pairs, it is nevertheless true that an incomplete Δ_2 homogeneous solution exists. The authors also propose ways of attacking the SRT^2_2 problem by considering different possible ω-models that could be constructed to avoid RT^2_2.

In this paper we discuss the use of nonstandard models in investigations of Ramsey type problems. From the proof-theoretic point of view, there is no reason to restrict oneself to ω-models in studying subsystems of second order arithmetic. Since there exist nonstandard models of RCA_0 that exhibit simpler structures in the Turing degrees, Ramsey type problems may be analyzed from a different perspective and this could shed light on the problems themselves. For example, there exist models of $\mathrm{RCA}_0 + B\Sigma^0_2$ in which every incomplete Turing degree is low. This means that the counterexample given in [6] does not apply to these models. It will be useful to study the problem of SRT^2_2 in such a situation. This is discussed further later in the paper.

Our main result gives an illustration of the link between Ramsey type problems and nonstandard models. Namely, we are concerned with the question of existence of solutions for RT^2_2 or COH that is recursive in the double Turing jump of the set parameter defining the 2-coloring or array. Classical results of Jockusch and others show that this is true for ω-models. In general, however, the answer is determined by the underlying first order inductive strength of the model being considered (Corollaries 3.1, 3.2 and 3.3).

We assume that the reader is familiar with recursion theory in models of fragments of Peano arithmetic as well as subsystems of second order arithmetic (cf. [4] and [14] for details). Here we will only present a summary of the facts that are relevant to the discussion below.

2. $B\Sigma^0_2$ MODELS

Let $I\Sigma^0_n$ denote the induction scheme for Σ^0_n formulas (with number and set parameters), where $n \geq 0$. All models $\mathcal{M}$ considered in this paper satisfy at least $I\Sigma^0_1$ ($I\Sigma^0_n$ is denoted Σ^0_n-IND in [14]). A bounded set in $\mathcal{M}$ is $\mathcal{M}$-finite if it is coded in $\mathcal{M}$. Otherwise it is called $\mathcal{M}$-infinite. An unbounded set in $\mathcal{M}$ is necessarily $\mathcal{M}$-infinite, although the converse is not always true (for example $\mathbb{N}$ is not $\mathcal{M}$-finite in any nonstandard model $\mathcal{M}$).

Let $B\Sigma^0_n$ denote the scheme which states that every Σ^0_n definable function maps an $\mathcal{M}$-finite set onto an $\mathcal{M}$-finite set. The result of Kirby and Paris [10] for first order formulas may be generalized to show that for all $n \geq 0$, $I\Sigma^0_{n+1}$ is strictly stronger than $B\Sigma^0_{n+1}$, which is in turn strictly stronger than $I\Sigma^0_n$. Indeed, the results stated without proof in this section were originally shown for first order fragments of Peano arithmetic. Their generalization to subsystems of second order arithmetic is immediate.

Proposition 2.1. *If $\mathcal{M} \models I\Sigma^0_n$, then every bounded $\Sigma^0_n(\mathcal{M})$ set is $\mathcal{M}$-finite.*

If $\mathcal{M} \models B\Sigma^0_n$, then a cut $I \subset M$ is a set that is closed downwards as well as under the successor function. I is a Σ^0_n cut if it is Σ^0_n definable over $\mathcal{M}$. Proposition 2.1 implies that $\mathcal{M} \models I\Sigma^0_n$ if and only if there is no proper (i.e. bounded) Σ^0_n cut. We consider only proper Σ^0_n cuts in this paper.

If $\mathcal{M} \models B\Sigma^0_n$ but not $I\Sigma^0_n$, we call it a $B\Sigma^0_n$ model. In this case, there is a $\Sigma^0_n(\mathcal{M})$ function mapping a Σ^0_n cut cofinally into M.

We identify a number in M with the set of its predecessors.

Definition 2.1. *Let $\mathcal{M}$ be a model of* RCA_0. *A set $X \subset M$ is regular if $X \upharpoonright a$ is $\mathcal{M}$-finite for each $a \in M$.*

Proposition 2.2. *Let $\mathcal{M}$ be a $B\Sigma^0_2$ model of* RCA_0. *Then*

(i) *Every $\Delta^0_2(\mathcal{M})$ set is regular;*
(ii) *X is $\Delta^0_2(\mathcal{M})$ if and only if X is recursive in Y', where Y is the set parameter occurring in the definition of X.*
(iii) *Every $X \in \mathbb{X}$ is regular.*

Proposition 2.3. *Let $\mathcal{M}$ be a model of* RCA_0 *and $X \in \mathbb{X}$. Then $\mathcal{M} \models B\Sigma^0_2(X)$ if and only if $\mathcal{M} \models B\Sigma^0_1(X')$.*

Definition 2.2. *Let $\mathcal{M} \models$* RCA_0 *and $X \subset M$. Then X is hyperregular if the image of every bounded set under a function that is weakly recursive in X (i.e. $\Delta_1(X)$) is bounded.*

Given a model $\mathcal{M}$ of RCA_0, and $A \subset M$, let $\mathcal{M}[A]$ denote the structure generated from A over $\mathcal{M}$ by closing under functions recursive in $A \oplus X$, where $X \in \mathbb{X}$.

Proposition 2.4. (Mytilinaios and Slaman [11]) *Let $\mathcal{M}$ be a $B\Sigma^0_2$ model of* RCA_0 *and $A \subset M$. The following are equivalent:*

(i) *A is hyperregular;*
(ii) *$\mathcal{M}[A] \models I\Sigma^0_1$.*

The notion of coding in $B\Sigma^0_2$ models of RCA_0 is central to the analysis of Ramsey type problems in this paper. Although the original definition was given for first order structures, it carries over to second order structures in an obvious way.

Definition 2.3. *Let A be a subset of M, where $\mathcal{M} \models$* RCA_0. *A set $X \subseteq A$ is* coded on A *if there is an $\mathcal{M}$-finite set $\hat{X}$ such that $\hat{X} \cap A = X$.*

Definition 2.4. *Let A be a subset of $\mathcal{M}$. We say that a set X is Δ^0_n* on A *if both $A \cap X$ and $A \cap \overline{X}$ are $\Sigma^0_n(\mathcal{M})$.*

The following result of Chong and Mourad [2] will be used repeatedly in the sequel:

Proposition 2.5. *If $\mathcal{M}$ is a $B\Sigma^0_n$ model of* RCA_0*, then every bounded set that is $\Delta^0_n(\mathcal{M})$ on a given set A is coded on A.*

3. Ramsey's Theorem in $B\Sigma^0_2$ models of RCA_0

We first prove a general theorem for $B\Sigma^0_2$ models of RCA_0, and then use it to study Ramsey's Theorem for pairs and COH in such models. From now on, let $\mathcal{M} = \langle M, \mathbb{X}, +, \times, 0, 1\rangle$ be a model of $\mathrm{RCA}_0 + B\Sigma^0_2$ with a Σ^0_2 cut I. Also fix $g : I \to M$ to be an increasing Σ^0_2 cofinal function, with set parameter $Y \in \mathbb{X}$.

Theorem 3.1. *If $C \leq_T Y''$ is in $\mathbb{X}$, then $C' \leq_T Y'$.*

We first prove a lemma.

Lemma 3.1. *If $C \leq_T Y''$ is in $\mathbb{X}$, then $(C \oplus Y)' \leq_T I \oplus Y'$.*

Proof. Let $\Phi^{Y''} = C$. We identify a set with its characteristic function. By Proposition 2.2 (iii), C is regular. Hence for each $i \in I$, there is a neighborhood condition $\langle P_i, N_i\rangle$ of Y'' such that $\langle C \restriction g(i), P_i, N_i\rangle \in \Phi$.

Claim 1. $(C \oplus Y)'$ is regular.

Proof of Claim 1. Since C and Y are in $\mathbb{X}$, $B\Sigma^0_2(C \oplus Y)$ holds in $\mathcal{M}$. Fix $i \in I$. For $x \in g(i) \cap (C \oplus Y)'$, let $h(x)$ be the least y such that x is in $(C \oplus Y)'_y$, the set enumerated into $(C \oplus Y)'$ by stage y using $C \oplus Y$ as an oracle (recall that $(C \oplus Y)'$ is $\Sigma_1(C \oplus Y)$), and let it be 0 otherwise. Then by $B\Sigma^0_2(C \oplus Y)$, the image of $g(i)$ under h is $\mathcal{M}$-finite and has a largest element denoted y_0. Then for $x < g(i)$, $x \in (C \oplus Y)'$ if and only if $x \in (C \oplus Y)'_{y_0}$.

Claim 2. If $D \subset Y''$ is $\mathcal{M}$-finite, then there is a j_D such that $D \subset Y''_{g(j_D)}$, where $Y''_{g(j_D)}$ is the set of elements enumerated into Y'' by stage $g(j_D)$ using Y' as oracle.

Proof of Claim 2. By Proposition 2.3, $B\Sigma^0_1(Y')$ holds. The map taking each $x \in D$ to the least y such that $x \in Y''_y$ is Y'-recursive, hence $\mathcal{M}$-finite and bounded below a y_0. Let j_D be the least $j \in I$ such that $g(j) > y_0$.

Claim 3. If $X \leq_T Y''$ is regular, then $X \leq_T I \oplus Y'$.

Proof of Claim 3. Suppose $\Psi^{Y''} = X$. Then for each $i \in I$, there exists $\langle P_i, N_i\rangle$ such that $P_i \subset Y''$ and $N_i \subset \bar{Y}''$ with $\langle X \restriction g(i), P_i, N_i\rangle \in \Psi$. By Claim 2, there is a $j \in I$ such that $P_i \subset Y'_{g(j)}$ and $N_i \subset \bar{Y}''_{g(j)}$. Hence the set

$$Z = \{(i,j,j') \in I \times I \times I | \forall \langle P,N\rangle [P \subset Y''_{g(j)} \ \& \ N \subset \bar{Y}''_{g(j)} \ \& \\ \Psi^{\langle P,N\rangle} \restriction g(i) \text{ total} \to (N \cap Y''_{g(j')} \neq \emptyset]\} \tag{1}$$

satisfies the property that for each i, the set $\{(j,j')|(i,j,j') \in Z\}$ is bounded in $I \times I$. Furthermore, Z is $\Delta^0_2(Y)$ on $I \times I \times I$ and therefore coded on $I \times I \times I$ by an $\mathcal{M}$-finite set $\hat{Z}$ according to Proposition 2.5. We may assume that for each (i,j), j' is the least such that $(i,j,j') \in \hat{Z}$. Let j_i be the greatest j for $(i,j,j') \in Z$. Then for j_i+1, $(j_i+1)' \in \bar{I}$. In particular, there is a $\langle P,N\rangle \subset Y''_{g(j_i+1)} \times \bar{Y}''_{g(j_i+1)}$ such that $N \subset \bar{Y}''$ and $\Psi^{\langle P,N\rangle} \restriction g(i)$ is total, and so equal to $X \restriction g(i)$. Furthermore, since $X \restriction g(i)$ is $\mathcal{M}$-finite, all the computations in $\Psi_{g(j_i+1)}$ that disagree with $X \restriction g(i)$ will be identified at stage $g(j^*)$ for some $j^* > j_i$. The least such j^*, denoted j^*_i, is where there is only one output for $\Psi^{\langle P,N\rangle}_{g(j^*)} \restriction g(i)$ for any $\langle P,N\rangle \in Y''_{g(j_i+1)}, \bar{Y}''_{g(j_i+1)}$.

Now we may compute C from $I \oplus Y'$ as follows: To decide $C \restriction y$, use Y' to choose an i such that $g(i) > y$. Use I to find the least j such that $(i,j,j') \in \hat{Z}$ and $j' \in \bar{I}$. This is j_i+1. Then $C \restriction g(i)$ is the set D such that there is an $N \subset \bar{Y}''_{g(j^*_i)}$ with $\langle D, Y''_{g(j_i+1)}, N\rangle \in \Psi_{g(j_i+1)} \cap \Psi_{g(j^*_i)}$.

By Claim 1 and Claim 3, $C \oplus Y \leq_T I \oplus Y'$. Since $B\Sigma^0_2(C \oplus Y)$ holds, $I\Sigma^0_1(C \oplus Y)$ is true and therefore by Proposition 2.4, $C \oplus Y$ is hyperregular. This means that for all i, there is a least j, denoted $\hat{j}_i$, such that $(C \oplus Y)' \restriction g(i) = (C \oplus Y)'_{g(j)} \restriction g(i)$. Now the set

$$V = \{(i,j)|(C \oplus Y)'_{g(j)} \restriction g(i) \neq (C \oplus Y)'_{g(j+1)} \restriction g(i)\}$$

is $\Delta^0_2(C \oplus Y)$ on $I \times I$ and therefore coded on $I \times I$ by an $\mathcal{M}$-finite set $\hat{V}$ according to Proposition 2.5. Furthermore, $\hat{j}_i$ is the least $j \in I$ such that $(i,j) \notin V$. Then using $\hat{V}$ as a parameter, we may compute $(C \oplus Y)'$ from $I \oplus Y'$ as follows: Given i, use I to obtain $\hat{j}_i$ and then use Y' to compute $(C \oplus Y)'_{g(\hat{j}_i))}$. Then $x \in (C \oplus Y)' \restriction g(i)$ if and only if $x \in (C \oplus Y)'_{g(\hat{j}_i)}$. This completes the proof of the lemma. □

Proof of Theorem 3.1

Proof. First note that just like Claim 1 of Lemma 3.1, one can show that Y' is regular. By Lemma 3.1 let $\Psi^{I\oplus Y'} = (C\oplus Y)'$. A neighborhood condition $\langle P, N\rangle$ of I may be identified with a pair (c, d) where $c \in I$ and $d \in \bar{I}$ (let c be the maximum of P and d be the minimum of N). We make the following claim.

Claim. For each $i \in I$ there exist $(j, j', j'') \in I \times I \times I$ such that

(i) *Existence.* There is a (c, d) with $c < j < j' \leq d$ and $\Psi^{(c,d)\oplus Y'\upharpoonright g(j'')} \upharpoonright g(i) = \Psi^{I\oplus Y'} \upharpoonright g(i)$ and

(ii) *Consistency.* There is no $x < g(i)$ and (c', d') with $c' \leq j < j' \leq d'$ such that $\Psi^{(c',d')\oplus Y'\upharpoonright g(j'')}(x) \downarrow\neq \Psi^{I\oplus Y'\upharpoonright g(j'')}(x)$.

Proof of Claim. Otherwise, there is an $i \in I$ such that for all $(j, j', j'') \in I \times I \times I$, either (i) or (ii) is false. More precisely, there is an $i \in I$ such that for all $(j, j', j'') \in I \times I \times I$, either

(iii) For any (c, d) with $c \leq j < j' \leq d$, if $\Psi^{(c,d)\oplus Y'\upharpoonright g(j'')} \upharpoonright g(i) \downarrow$, then there is an $x < g(i)$ such that $\Psi^{(c,d)\oplus Y'\upharpoonright g(j'')}(x) \neq \Psi^{I\oplus Y'\upharpoonright g(j'')}(x)$, or

(iv) There is an $x < g(i)$ and a (c, d) with $c \leq j < j' \leq d$ such that $\Psi^{(c,d)\oplus Y'\upharpoonright g(j'')}(x) \downarrow\neq \Psi^{I\oplus Y'\upharpoonright g(j'')}(x)$ (in this case d has to be in I).

Since $(C \oplus Y)' \leq_T I \oplus Y'$, there exist (j_0, j_0', j_0'') in $I \times I \times I$ and $c_0 \leq j_0 < j_0' < d_0 \in \bar{I}$ such that $\Psi^{(c_0,d_0)\oplus Y'\upharpoonright g(j_0'')} \upharpoonright g(i) = \Psi^{I\oplus Y'} \upharpoonright g(i)$. Hence for the triple (j_0, j_0', j_0''), (iii) is false and (iv) must hold. Now if the set $J = \{j | \text{(iv) holds for } (j_0, j, j_0'') \text{ with } c \leq j_0 < j \leq d\}$ is bounded in I, say by j^*, then it implies that for any (c, d) such that $c \leq j_0 < j^* \leq d$, if $\Psi^{(c,d)\oplus Y'\upharpoonright g(j_0'')}(x) \downarrow$ then it is equal to $\Psi^{I\oplus Y'\upharpoonright g(j_0'')}(x)$. Since $c_0 < j^* < d_0 \in \bar{I}$, we see that (j_0, j^*, j_0'') satisfies (i) and (ii), contradicting the assumption that i is a counterexample to the claim.

Hence J is unbounded in I. But then this implies that for $d < d_0$, $d \in \bar{I}$ if and only if for all d' where $d \leq d' \leq d_0$, there is no $c \leq j_0$ and $x < g(i)$ such that $\Psi^{(c,d')\oplus Y'\upharpoonright g(j_0'')}(x) \downarrow\neq \Psi^{(c_0,d_0)\oplus Y'\upharpoonright g(j_0'')}(x)$. This gives $\bar{I}$ a $\Pi^0_1(Y')$ definition (since $Y' \upharpoonright g(j_0'')$ and $(C \oplus Y)' \upharpoonright g(i)$ are both $\mathcal{M}$-finite), hence $\mathcal{M}$-finite, which is a contradiction, proving the Claim.

By the Claim, the set of quadruples (i, j, j', j'') satisfying (i) and (ii) is $\Delta^0_2(C\oplus Y)$ on $I\times I\times I\times I$, hence coded on $I\times I\times I\times I$ by an $\mathcal{M}$-finite set $\hat{h}$ which we may consider to be a function taking i to the least triple (j, j', j''). Then Y' computes $(C\oplus Y)'$ as follows: Given y, find an $i \in I$ using Y' such that $g(i) > y$. Now use $\langle \hat{h}_1(i), \hat{h}_2(i)\rangle \oplus Y' \upharpoonright g(\hat{h}_3(i))$ to enumerate $(C \oplus Y)' \upharpoonright g(i)$ via Ψ, where $\hat{h}(i) = \langle \hat{h}_1(i), \hat{h}_2(i), \hat{h}_3(i)\rangle$. This computation has to be correct by the choice of the function $\hat{h}$.

Since $(C \oplus Y)' \leq_T Y'$, it follows that $C' \leq_T Y'$. ∎

We now apply Theorem 3.1 to study Ramsey's Theorem for pairs. The proof of the following result follows essentially Jockusch [9]. The only point to note is that the construction works under the assumption of $B\Sigma^0_2$.

Proposition 3.1. *Let $\mathcal{M}$ be a $B\Sigma^0_2$ model of* RCA_0. *Let $Y \in \mathbb{X}$. Then there is a Y-recursive partition f_J of $[M]^2$ into two colors with no homogeneous solution recursive in Y'.*

Definition 3.1. *Let $\varphi(X,Y)$ be a second order formula with X and Y as free set variables. A model $\mathcal{M}$ of* RCA_0 *is a double-jump model of $\forall Y \exists X \varphi(X,Y)$ if for all $Y \in \mathcal{M}$, there is an $X \in \mathcal{M}$ such that $X \leq_T Y''$ and $\mathcal{M} \models \varphi(X,Y)$.*

Results of Stephan and Jockusch [**?**] and Cholak, Jockusch and Slaman [1] show that every ω-model of RCA_0 is an ω-submodel of one that is a double-jump model of RT^2_2, if $\forall Y \exists X \varphi(X,Y)$ is interpreted as asserting the existence of a homogeneous set X for a Y-recursive 2-coloring of pairs. The same conclusion also holds for COH. It turns out that the strength of first order induction plays a crucial role for this to be true.

Corollary 3.1. *No $B\Sigma^0_2$ model of* RCA_0 *is a double-jump model of* RT^2_2.

Proof. Suppose otherwise. Let $\mathcal{M}$ be a $B\Sigma^0_2$ model of RCA_0 that is also a double-jump model for RT^2_2. Let $Y \in \mathcal{M}$ be chosen such that the $\Sigma^0_2(\mathcal{M})$ cut I is defined with parameter Y. Let f_J be the Y-recursive partition given in Proposition 3.1. Then by assumption there is a homogeneous solution $H_{f_J} \leq_T Y''$ for f_J. By Theorem 3.1, $H_J <_T Y'$. However, this contradicts Proposition 3.1. □

Let DB-RT^2_2 be the following statement: For any Y, any Y-recursive 2-coloring of pairs has a homogeneous set X such that $X \leq_T Y''$.

Corollary 3.2. *Let $\mathcal{M}$ be a model of* RCA_0. *The following are equivalent:*

(i) $\mathcal{M} \models I\Sigma^0_2$;
(ii) $\mathcal{M}$ *is an M-submodel of an* $\mathcal{M}^* \models$ DB-RT^2_2.

Proof. The proof of (i) ⇒ (ii) may be adapted from Theorem 4.2 of Jockusch [9], by carrying out the construction using Σ^0_2 induction. For the other direction, first note that Hirst [8] showed that every model of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ satisfies $B\Sigma^0_2$. Suppose that $\mathcal{M}$ is an M-submodel of $\mathcal{M}^* \models$ DB-RT$^2_2$ and $\mathcal{M}^*$ is a $B\Sigma^0_2$ model. Again let Y be such that there is a $\Sigma^0_2(Y)$ cut with a Σ^0_2 cofinal function from the cut into M. Let f_J be the Jockusch partition of pairs recursive in Y. If H_J is homogeneous for f_J and recursive in Y'', then by Theorem 3.1 it is recursive in Y', which is not possible. Thus $\mathcal{M}^*$, hence $\mathcal{M}$, satisfies $I\Sigma^0_2$. □

The proof of the next result is essentially in Theorem 12.4 of [1].

Proposition 3.2. *Let $\mathcal{M}$ be a $B\Sigma^0_2$ model of* RCA$_0$. *Let $g : I \to M$ be Σ^0_2, increasing and cofinal with parameter Y. Then there is a $Z \leq_T Y$ with a Z-array that has no Z-cohesive set C satisfying $C' \leq_T Y'$.*

Proof. Let e be such that $\Phi^{Y'}_e$ is a $\{0,1\}$-valued partial function that has no Y'-computable extension to a total function. Define $Z = \{(s,t) | \Phi^{Y'_t}_t(s) \downarrow= 0\}$. Let $f(s,t) = 0$ if $(s,t) \in Z$, and equal to 1 otherwise. Let $Z_s = \{t | (s,t) \in Z\}$. Then both f and Z are recursive in Y.

Suppose that C is cohesive for the array $\langle Z_s \rangle_{s \in M}$ and $C' \leq_T Y'$. Then $\hat{f}(s) = \lim_{t \in C} f(s,t)$ exists for each s and is computable in C', hence in Y' by assumption. Furthermore $\hat{f}$ is a total function extending $\Phi^{Y'}_e$, contradicting the choice of e. □

Let $\forall Y \exists X \varphi(X,Y)$ be COH: For every $Y \in \mathcal{M}$ and $Z \leq_T Y$, every Z-array has a Z-cohesive set $X \in \mathcal{M}$. Theorem 3.1 leads to the following corollary.

Corollary 3.3. *Let $\mathcal{M}$ be a $B\Sigma^0_2$ model of* RCA$_0$. *Then $\mathcal{M}$ is not a double-jump model of* COH.

It follows from Corollary 3.3 that starting with a $B\Sigma^0_2$ model $\mathcal{M}$ of RCA$_0$, it is not possible to add second order objects to $\mathcal{M}$ within double jump to obtain a model of RCA$_0$ + COH while preserving $B\Sigma^0_2$. This is expressed as follows. Let DB-COH denote the statement that every Y-array has a Y-cohesive set recursive in Y''.

Corollary 3.4. RCA$_0$ + COH *does not prove* DB-COH.

Chong, Slaman and Yang [3] have recently proved a Π^1_1-conservation theorem for RCA$_0$ + COH + $B\Sigma^0_2$ over RCA$_0$ + $B\Sigma^0_2$. A key step in the proof is to expand, for a given $\mathcal{M}$ and Y-array where $Y \in \mathcal{M}$, a Y-cohesive set in the generic extension that preserves $B\Sigma^0_2$. The Y-cohesive set thus obtained is not recursive in Y'' (in fact highly non-effective). Corollary 3.3 hints at the need for a non-Y''-effective approach.

Let $\forall Y \exists X \varphi(X, Y)$ be SRT^2_2. Theorem 3.1, when applied to SRT^2_2, says that any $B\Sigma^0_2$ double-jump model $\mathcal{M}$ of SRT^2_2 is necessarily one that contains only incomplete homogeneous sets (incomplete relative to the parameter defining the 2-coloring). Furthermore, $\mathcal{M}$ satisfies the following useful result whose origin (in the form of α-recursion theory) goes back to Shore [13] (See also Mytilinaios and Slaman [11] and Chong and Yang [4]):

Proposition 3.3. *Let $\mathcal{M}$ be a $B\Sigma^0_2$ model of* RCA_0. *Let I be a $\Sigma^9_2(\mathcal{M})$ cut with parameter $Y \in \mathcal{M}$. Then every $X <_T Y'$ in $\mathcal{M}$ is Y-low, i.e. $X' \leq_T Y'$.*

Proof. Let $\mathcal{M}$ and Y be as given, and let I be a $\Sigma^0_2(Y)$ cut. Let $X <_T Y'$ be in $\mathcal{M}$. Then $B\Sigma^0_2(X)$ holds in $\mathcal{M}$ and so X' is regular. Since $X' \leq_T Y''$, Lemma 3.1 implies that X' is recursive in $I \oplus Y'$. The argument in the proof of Theorem 3.1 (after Claim 3) now implies that $X' \leq_T Y'$. □

It is not difficult to see that every stable 2-coloring of pairs has a homogeneous solution Δ^0_2 in the parameter defining the coloring. In a $B\Sigma^0_2$ setting, the crucial question is whether every stable 2-coloring of pairs has a homogeneous solution that is low relative to the parameter that defines the coloring. If the answer is yes, then one may generate a $B\Sigma^0_2$ model of $\mathrm{RCA}_0 + \mathrm{SRT}^2_2$ as follows: Begin with a countable $B\Sigma^0_2$ model $\mathcal{M}_0$ (with no second order objects). Let this be the ground model. At stage $n + 1$, construct an incomplete homogeneous set for a stable 2-coloring defined over $\mathcal{M}_n$. One can arrange the stable 2-colorings in such a way that $\mathcal{M} = \bigcup_n \mathcal{M}_n$ is a model of SRT^2_2. Now $B\Sigma^0_2$ is preserved in every $\mathcal{M}_n$ since only low sets are added, guaranteeing that $\mathcal{M} \models B\Sigma^0_2$. Then Proposition 3.1 ensures that $\mathcal{M}$ is not a model of RT^2_2.

We end this paper with two questions.

Question 1. Is there a double-jump $B\Sigma^0_2$ model for SRT^2_2?

Question 2. Is there a $B\Sigma^0_2$ model of $\mathrm{RCA}_0 + \mathrm{RT}^2_2$ or COH obtained from a ground model by working within the nth jump, where $n > 2$?

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