

A Π_1^1 -UNIFORMIZATION PRINCIPLE FOR REALS

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ABSTRACT. We introduce a Π_1^1 -uniformization principle and establish its equivalence with the set-theoretic hypothesis $(\omega_1)^L = \omega_1$. This principle is then applied to derive the equivalence, to suitable set-theoretic hypotheses, of the existence of Π_1^1 maximal chains and thin maximal antichains in the Turing degrees. We also use the Π_1^1 -uniformization principle to study Martin's conjectures on cones of Turing degrees, and show that under $V = L$ the conjectures fail for uniformly degree invariant Π_1^1 functions.

1. INTRODUCTION

We identify a real x with a subset of ω . A real is Π_1^1 if and only if it is definable or equivalent, over second order arithmetic, to a formula that begins with a universal set or function quantifier followed by an arithmetic relation. There is a notion of Π_1^1 inductive definitions for reals. Kleene's \mathcal{O} is perhaps the penultimate example of such an object.

One may introduce a parallel notion of inductive definition for sets of reals. If $Q : 2^{2^\omega} \rightarrow 2^{2^\omega}$ is a function, with the additional property that $Q(\emptyset) \subseteq 2^\omega$, then it is *progressive* (Sacks [15]) if $Q(X) \supseteq X$ for all $X \subseteq 2^\omega$. Q is *monotonic* if $Q(X) \subseteq Q(Y)$ whenever $X \subseteq Y$. Given an ordinal β , let $Q^0 = Q(\emptyset)$, $Q^{\beta+1} = Q(Q^\beta) \cup Q^\beta$, and, for a limit ordinal $\lambda > 0$, $Q^\lambda = \bigcup_{\beta < \lambda} Q^\beta$. Π_1^1 -monotonic relations have been studied by various authors, including Spector [18], Aczel and Richter [1] as well as Cenzer [4]. The main interest was in identifying the “ordinal of Q ”, i.e. the least ordinal β where $Q^{\beta+1} = Q^\beta$. However, the study of specific inductively defined relations has seen less activity.

The Kondo-Addison Uniformization Theorem states that given a Π_1^1 relation $P(x, y)$, there is a Π_1^1 relation P^* such that for all x , if there is a y satisfying $P(x, y)$, then there is a unique y satisfying $P^*(x, y)$. Conversely, for any x, y , if $P^*(x, y)$ then $P(x, y)$. Thus P^* is a Π_1^1 function that *uniformizes* P . There are Π_1^1 -progressive relations with very useful properties which may be applied to derive Π_1^1 -uniformization principles. Such principles may in turn be used to investigate interesting problems in descriptive set theory and recursion theory. In this paper we present an example to illustrate this point. More specifically, we introduce an inductively defined Π_1^1 -uniformization principle for a Π_1^1 relation on the hyperdegrees of reals, and show some of its applications.

There is a class of problems on the existence of definable maximal collections or families of reals with prescribed properties that are of interest to set theorists and recursion theorists. A naive approach, under $V = L$, is to proceed by induction over L_{ω_1} , where in a $\Sigma_1(L_{\omega_1})$ fashion a definable maximal family of reals is constructed. Such a family would typically yield a Σ_2^1 set of

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reals over second order arithmetic. Much harder work is required if one aims for a collection of lower complexity. Refining the naive construction, Kunen and Miller [13] have obtained a Π_1^1 maximal family of almost disjoint sets. In [5] and [6], we investigated the problem of the existence of Π_1^1 maximal chains and thin (i.e. without a perfect subset) antichains in the Turing degrees, and proved the equivalence of the existence of these sets with respective set theoretic hypotheses on ω_1 and 2^ω . Underlying the constructions were similar techniques and intuitions that pointed to a general principle applicable to different situations, perhaps at a deeper level. Our investigation leads to the formulation of a Π_1^1 -uniformization principle, denoted \mathfrak{J} , discussed below. The principle is applied to establish the key results in [5] and [6], and also to study Martin's conjectures on Turing degrees. It is worth noting that the idea of coding reals in the formulation of the principle \mathfrak{J} was also used in the work of Kunen and Miller to construct Π_1^1 maximal family of almost disjoint sets.

Let \mathcal{C} be the set of reals x such that $x \in L_{\omega_1^x}$, where as usual ω_1^x is the least admissible ordinal not recursive in x . It is well known that \mathcal{C} is Π_1^1 and that every Π_1^1 set of reals contains an element of \mathcal{C} (Sacks [15]). Elements of \mathcal{C} enjoy special properties that are not available to those in its complement. For example, if x is a master code in the sense of Jensen's fine structure theory [10] (or equivalently an "arithmetical copy" in the sense of Boolos and Putnam [3]), then $x \in \mathcal{C}$. In [5] and [6], the Π_1^1 maximal chain and thin antichain were constructed exploiting this fact, so that each element of these sets was chosen to be in \mathcal{C} with specific properties, including requiring it to have double Turing jump that is Turing equivalent to a master code. However, the double jump, while sufficient for the constructions of the sets in [5] and [6], is probably too restrictive and not necessary for a deeper understanding of the " Π_1^1 phenomenon", and the use of fine structure theory as well as Borel determinacy, while interesting and revealing, might also be avoided when viewed in a different light. This is the intuition behind our search for a general principle. The motivation is that the principle should not only prove theorems obtained in [5] and [6], but also be useful for studying other problems. The test problem we consider is Martin's conjectures on cones of Turing degrees for degree invariant functions. Since this is known to be false under AC , and the conjectures are posed under the assumption of the Axiom of Determinacy AD , and known to hold for uniformly degree invariant functions assuming AD (Steel [19], Slaman and Steel [16]), we focus our attention on uniformly definable degree invariant functions—in this case Π_1^1 —to learn how far AC and the conjectures may co-exist in a model of set theory without full AD .

The organization of the paper is as follows. In §2 we recall some standard notions, definitions and results in hyperarithmetic theory, highlighting in particular the Spector-Gandy Theorem which characterizes Π_1^1 sets of reals. In §3 we introduce a Π_1^1 -progressive relation and use it to define the Π_1^1 -uniformization principle \mathfrak{J} . We show the validity of this principle under the set-theoretic hypothesis $(\omega_1)^L = \omega_1$, and demonstrate that \mathfrak{J} and the hypothesis are equivalent. In §4 we show that the existence of a Π_1^1 maximal chain and a thin Π_1^1 maximal antichain is a consequence of, and equivalent to, \mathfrak{J} under appropriate set-theoretic axioms. In §5 we apply the uniformization principle to study Martin's conjectures, and show that it fails with Π_1^1 degree invariant functions under $V = L$. In particular, we exhibit Π_1^1 uniform degree invariant functions f on the reals such that $f(x) \equiv_T x$ on cofinally (in the sense of Turing reducibility) many x 's but $f(x)$ is not Turing equivalent to x on any cone of Turing degrees. Our final result states that $<_M$, the partial ordering defined on degree invariant functions (see §5), is not a prewellordering on Π_1^1 uniform degree invariant functions. Thus both parts of Martin's conjectures do not hold for uniformly degree invariant functions definable at Π_1^1 level of the analytic hierarchy.

2. PRELIMINARIES

We adopt set-theoretic notations throughout this paper. In particular, small Roman letters x, y, z, \dots refer to reals, while Greek letters α, β, \dots denote ordinals. The collection of paths of a perfect tree T is denoted $[T]$, and $\langle \rangle$ is a recursive bijection from $\omega \times \omega$ to ω . Finally, $<_L$ denotes the Δ_2^1 well ordering of the reals in Gödel's constructible universe L .

The reader is assumed to be familiar with hyperarithmetic theory as presented in Sacks [15] which is used as the standard reference in this paper. Here we recall some notions that may be considered to be at the intersection of set theory and recursion theory and which are needed throughout the paper.

For each real x , ω_1^x denotes the least ordinal which is not an order type of an x -recursive well ordering of ω (the least x -admissible ordinal). Kleene constructed a $\Pi_1^1(x)$ complete set \mathcal{O}^x with a $\Pi_1^1(x)$ well founded relation $<_{\mathcal{O}^x}$ on \mathcal{O}^x . \mathcal{O}^x is the hyperjump of x and if $x \in L$ then $\mathcal{O}^x \in \mathcal{C}$, where \mathcal{C} is the set defined in §1, and $\omega_1^{\mathcal{O}^x}$ is the least admissible ordinal after ω_1^x . The height of the ordering $<_{\mathcal{O}^x}$ on \mathcal{O}^x is exactly ω_1^x (see [15]). Furthermore, Kleene's construction of \mathcal{O}^x is uniform. In other words, the relation $\{(x, \mathcal{O}^x) \mid x \in 2^\omega\}$ is Π_1^1 . A fact that will be used implicitly is that given reals x and y , x is hyperarithmetic in y (written $x \leq_h y$) if and only if x is Δ_1^1 in y , and this is in turn equivalent to $x \in L_{\omega_1^y}[y]$.

A result of central importance to this paper, due to Spector and Gandy, is an application of Kleene's theory in the characterization of Π_1^1 sets of reals. This states that $A \subseteq 2^\omega$ is Π_1^1 if and only if there is an arithmetical relation $R(x, y)$ such that

$$x \in A \Leftrightarrow \exists y \leq_h x (R(x, y)).$$

Boolos and Putnam [3] also studied the relationship between Kleene's theory and Gödel's constructible universe L . They proved that hyperarithmetic reals are exactly the reals in $L_{\omega_1^{\text{CK}}}$. Moreover ω_1^x is the least ordinal $\alpha > \omega$ such that $L_\alpha[x]$ is admissible (see [2] and [15]). By their result, the Spector-Gandy Theorem may be restated as follows:

Theorem 2.1 (Spector, Gandy). *A set A of reals is Π_1^1 if and only if there is a Σ_0 -formula φ (in the language of ZF set theory) such that*

$$y \in A \Leftrightarrow (\exists x \in L_{\omega_1^y}[y])(L_{\omega_1^y}[y] \models \varphi(x, y)).$$

Recall that $\mathcal{C} = \{x \mid x \in L_{\omega_1^x}\}$. Every nonempty Π_1^1 set contains an element in \mathcal{C} ([15] III. Lemma 9.3). A relativized version of this result, whose proof is a straightforward adaptation of the original proof, is the following (we will denote in the rest of the paper $\omega^{(x, y)}$ for $\omega_1^{x \oplus y}$):

Proposition 2.2. *If $A \neq \emptyset$ is $\Pi_1^1(x)$ for some real x , then there is a $y \in A$ such that $y \in L_{\omega_1^{(y, x)}}[x]$.*

A particularly useful characterization of Π_1^1 sets of reals is due to Mansfield and Solovay ([15] III. Theorem 9.5, Mansfield [11], Solovay [17]):

Theorem 2.3. (Mansfield and Solovay). *Let A be a Π_1^1 set of reals. The following are equivalent:*

- (i) *A contains a constructibly coded perfect subset;*
- (ii) *There is an $x \in A$ such that $x \notin \mathcal{C}$.*

3. A Π_1^1 -UNIFORMIZATION PRINCIPLE

Definition 3.1. A binary relation $P(x, y)$ on $2^\omega \times 2^\omega$ is cofinally progressive if for every real x , the set $\{y | P(x, y)\}$ is cofinal in the hyperdegrees, i.e. for each z , there is a real $y \geq_h z$ such that $P(x, y)$ holds.

Lemma 3.2. Assume $(2^\omega)^L = 2^\omega$. If a Π_1^1 relation P is cofinally progressive, then for every real x and z there is a real $y \geq_h z$ such that $y \in \mathcal{C}$ and $P(x, y)$.

Proof. Suppose $(2^\omega)^L = 2^\omega$ and P is cofinally progressive. Then for any reals x, z , the set $A = \{y | x \oplus z \in L_{\omega_1^y} \wedge P(x, y)\}$ is a non-empty $\Pi_1^1(x \oplus z)$ set. Thus, by Proposition 2.2, there is a real $y \in A$ for which $y \in L_{\omega_1^{(x, z, y)}}[x \oplus z]$. Since $x \oplus z \in L_{\omega_1^y}$, we have $x \oplus z \leq_h y$ and so $\omega_1^{(x, z, y)} = \omega_1^y$. Then $y \in L_{\omega_1^{(x, z, y)}}[x \oplus z] = L_{\omega_1^y}$ and $P(x, y)$. \square

Given a countable set A and a real x , we say that x codes A if $\{(x)_n | n \in \omega\} = A$ where $(x)_n = \{m | m \in \omega \wedge \langle n, m \rangle \in x\}$.

A cofinally progressive relation generates a progressive relation (in the sense defined in §1) as follows: Let $P(x, y)$ be cofinally progressive. For each $X \subseteq 2^\omega$, let $Q(X) = X \cup \{y | \exists x \in X P(x, y)\}$. Then it is immediate that $Q(X) \supseteq X$ for all X . However, an operation such as Q may be too general to be useful and this motivates a refinement of Q to something close to a uniformization function for well-defined sets. Now if $P(x, y)$ is Π_1^1 , the Kondo-Addison Theorem guarantees a uniformization function for P . However, our interest is in *strong* uniformization, i.e. a function that uniformizes “initial segments of reals” that does roughly the following: If x codes a set of reals in \mathcal{C} such that any two elements $w <_L z$ in x satisfy $P(w, z)$, then there is a “least” $y \in \mathcal{C}$ such that $x \leq y$ and $P(x, y)$ holds. This is made precise by the Π_1^1 -uniformization principle \mathfrak{J} :

Π_1^1 -uniformization principle \mathfrak{J} . If a binary relation $P(x, y)$ is Π_1^1 and cofinally progressive, then there is a Π_1^1 set $A \subseteq \mathcal{C}$ such that:

- (i) $| \leq_L \upharpoonright A |$, the height of \leq_L on A , is ω_1 ;
- (ii) $\forall y (y \in A \implies \exists x (x \in L_{\omega_1^y} \wedge x \text{ codes the set } \{z | z \in A \wedge z <_L y\} \wedge P(x, y)))$.

Theorem 3.3. $V = L$ implies \mathfrak{J} .

Proof. Suppose $V = L$, and P is Π_1^1 and cofinally progressive. Note that for each constructibly countable β , there is an $\alpha > \beta$ such that $(L_{\alpha+1} \setminus L_\alpha) \cap 2^\omega \neq \emptyset$. For a given ordinal α and $X \subseteq \alpha \times \omega$, denote by $X[\beta]$ the real $\{n \in \omega | \langle \beta, n \rangle \in X\}$. We may regard X as a sequence of reals of length α .

Since P is Π_1^1 , by the Spector-Gandy Theorem there is a Σ_0 formula $\psi(x, y, s)$ such that

$$P(x, y) \Leftrightarrow L_{\omega_1^{(x, y)}}[x, y] \models \exists s \psi(x, y, s)$$

Assuming $V = L$, we define a partial function F , which we will show to be total on $\omega_1 \times \bigcup_{\alpha < \omega_1} 2^{\alpha \times \omega}$, as follows:

For each $\alpha < \omega_1$ and set $X \subseteq \alpha \times \omega$ with $\alpha < \omega_1$, we define $F(\alpha, X)$ to be the real z such that $L_{\omega_1^{(x, z)}}[X, z]$ satisfies the following properties:

- (1) There is a $\beta \geq \alpha$ so that $z \in L_{\beta+1} \setminus L_\beta$;
- (2) There is a real $x \in L_{\omega_1^{(x, z)}}$ coding X ;
- (3) There is a limit ordinal γ and an element $s \in L_\gamma$ so that $L_\gamma[X, z] \models \psi(x, z, s)$;
- (4) If $(t, y, \mu) <_L (s, z, \gamma)$, then $L_\mu[X, y] \models \psi(x, y, t) \implies \omega_1^{(X, y)} < \beta$;

- (5) If $(t, y, \mu) <_L (s, z, \gamma)$ and $L_\mu[X, y] \models \psi(x, y, t)$, then either no real $r \in L_{\omega_1^{(x, y)}}$ codes X or $y \notin L_{\omega_1^{(x, y)}}$.

Since $L_{\omega_1^{(x, z)}}[X, z] = L_{\omega_1^{(x, z)}}$, $<_L$ is uniformly Δ_1 in $L_{\omega_1^{(x, z)}}[X, z]$. Hence (1)–(5) are uniformly Σ_1 in $L_{\omega_1^{(x, z)}}[X, z]$ (see [8]).

We show that $F(\alpha, X)$ is defined for each $\alpha < \omega_1$ if X is countable.

Fix (α, X) . Since $V = L$, there is a countable $\gamma' > \alpha$ with a real $x \in L_{\gamma'}$ coding X . Since $V = L$ and P is cofinally progressive, by Lemma 3.2, there is a $\beta > \gamma'$ so that $(L_{\beta+1} \setminus L_\beta) \cap 2^\omega \neq \emptyset$ and a real $z \in L_{\beta+1} \setminus L_\beta$ so that $x \oplus z \in L_{\omega_1^z}$ and $L_\gamma \models \psi(x, z, s)$ for some limit ordinal $\gamma < \omega_1^z$ and $s \in L_\gamma$. Note that $L_{\omega_1^z} = L_{\omega_1^{(x, z)}}[X, z]$ and $L_\gamma = L_\gamma[X, z]$ for all sufficiently large $\gamma < \omega_1^z$. So (1)–(3) are satisfied. Obviously we can assume that (s, z, γ) is the $<_L$ -least satisfying these properties. So (4)–(5) are satisfied. By the absoluteness of $<_L$, one concludes that F is a well-defined function defined on each (α, X) where $\alpha < \omega_1$ and X is countable.

Observe that (1)–(4) are Σ_1 statements. By (4), one verifies that (5) is a Δ_1 -statement. Hence there is a Σ_0 formula $\varphi(\alpha, X, z, s)$ such that $F(\alpha, X) = z$ if and only if $L_{\omega_1^{(x, z)}}[X, z] \models (\exists s)\varphi(\alpha, X, z, s)$.

Thus we may perform transfinite induction on α to define a sequence $\{z_\alpha \mid \alpha < \omega_1\}$ using F . However, the resulting sequence in general yields a set of reals that is Σ_1 over L_{ω_1} , i.e. Σ_2^1 but not necessarily Π_1^1 over second order arithmetic. To ensure that Π_1^1 -ness is achieved, we refine the approach as follows.

Define $G(\alpha) = z$ if and only if $\alpha < \omega_1^z$ and there is a function $f : \alpha + 1 \rightarrow 2^\omega$ with $f \in L_{\omega_1^z}[z]$ so that for all $\beta \leq \alpha$, $f(\beta) = F(\beta, \{(\gamma, n) \mid n \in f(\gamma) \wedge \gamma < \beta\})$ and $f(\alpha) = z$. Since $L_{\omega_1^z}[z]$ is admissible, $\{f(\gamma) \mid \gamma \leq \alpha\} \in L_{\omega_1^z}[z]$. So $G(\alpha) = z$ if and only if there is a function $f : \alpha + 1 \rightarrow 2^\omega$ with $f \in L_{\omega_1^z}[z]$ such that

$$L_{\omega_1^z}[z] \models ((\forall \beta \leq \alpha)(\exists s)\varphi(\beta, \{(\gamma, n) \mid n \in f(\gamma) \wedge \gamma < \beta\}, s, f(\beta))) \wedge f(\alpha) = z.$$

Since $L_{\omega_1^z}[z]$ is admissible, G is Σ_1 -definable. In other words, $G(\alpha) = z$ if and only if there is a function $f : \alpha + 1 \rightarrow 2^\omega$ with $f \in L_{\omega_1^z}[z]$ such that

$$L_{\omega_1^z}[z] \models ((\exists t)(\forall \beta \leq \alpha)(\exists s \in t)\varphi(\beta, \{(\gamma, n) \mid n \in f(\gamma) \wedge \gamma < \beta\}, s, f(\beta))) \wedge f(\alpha) = z.$$

Define the range of G to be A . Then $z \in A$ if and only if there exists an ordinal $\alpha < \omega_1^y$ and a function $f : \alpha + 1 \rightarrow 2^\omega$ with $f \in L_{\omega_1^z}[z]$ such that

$$L_{\omega_1^z}[z] \models ((\exists t)(\forall \beta \leq \alpha)(\exists s \in t)\varphi(\beta, \{(\gamma, n) \mid n \in f(\gamma) \wedge \gamma < \beta\}, s, f(\beta))) \wedge f(\alpha) = z.$$

So A is Π_1^1 .

All that remains is to show that G is a well-defined total function on ω_1 . This can be done using the same argument as that for showing the recursion theorem over admissible structures (see Chapter I, Theorem 6.4 of Barwise [2]). The only non-trivial part is to argue that the function f defined above exists. This can be done by induction. Let $\alpha < \omega_1$ be given and suppose that $\{f_\beta\}_{\beta < \alpha}$ is a uniformly Σ_1 increasing (i.e. $f_\beta \supset f_\gamma$ for $\beta > \gamma$) sequence such that each f_β is an f at stage $\beta < \alpha$ as required, Let f_α be such that $f_\alpha \in L_{\omega_1^z}[z]$ and for all $\beta \leq \alpha$, $f_\alpha(\beta) = F(\beta, \{(\gamma, n) \mid n \in f_\beta(\gamma) \wedge \gamma < \beta\})$ and $f_\alpha(\alpha) = z$, where $F(\alpha, X) = z$ and $X = \{(\gamma, n) \mid \exists \beta[\gamma \leq \beta < \alpha \ \& \ n \in f_\beta(\gamma)]\}$. Note that $X = \{(\gamma, n) \mid \gamma \leq \alpha \ \& \ n \in f_\gamma(\gamma)\}$ by the absoluteness. Then $f_\alpha \supset f_\beta$ for all $\beta < \alpha$ and satisfies the requirement of such an f at stage α .

By (1) and (2) above, $(x, z) \in L_\gamma$ and $L_\gamma \in L_{\omega_1^z}$. So $A \subset \mathcal{C}$.

Since P is cofinally progressive, (i) in \mathcal{J} is true.

For (ii): If $y \in A$ then $y = G(\alpha)$ for some $\alpha < \omega_1$. By the definition of G , there is a real $x \in L_{\omega_1^{G(\alpha)}}$ such that x codes the set $\{G(\beta) \mid \beta < \alpha\}$ and $P(x, G(\alpha))$ holds. But obviously $\beta < \alpha$ if and only if $G(\beta) <_L G(\alpha)$. Hence (ii) holds. \square

It turns out that the Π_1^1 -uniformization principle \mathfrak{J} is equivalent to the set-theoretic assumption $(\omega_1)^L = \omega_1$, as we now show.

Theorem 3.4. $(\omega_1)^L = \omega_1$ if and only if the Π_1^1 -uniformization principle \mathfrak{J} is true.

Proof. The “if” direction is immediate: Choose $P(x, y)$ to express $x \leq_T y$ (x is Turing reducible to y). Then P is cofinally progressive. Let A be a Π_1^1 set satisfying \mathfrak{J} for P , of $<_L$ -height ω_1 . Thus $(\omega_1)^L = \omega_1$.

For the other direction, suppose P is Π_1^1 and cofinally progressive. Then for any pair of reals $x, z \in L$, the set $U_{x,z} = \{y \mid y \geq_h z \wedge P(x, y)\}$ is a nonempty $\Pi_1^1(x, z)$ set. So there must be some $y \in U_{x,z} \cap L$. Then by the absoluteness of Π_1^1 relations,

$$L \models P \text{ is cofinally progressive.}$$

By Theorem 3.3, there is a Π_1^1 set A so that $(A)^L$ witnesses the correctness of the uniformization principle in L .

Since $(\omega_1)^L = \omega_1$, (i) in the uniformization principle is true in V .

Since the statement “ $x \in L_{\omega_1^x}$ ” is Π_1^1 and

$$L \models \forall x(x \in A \implies x \in L_{\omega_1^x}),$$

by the absoluteness of Π_2^1 statements, $V \models \forall x(x \in A \implies x \in L_{\omega_1^x})$. Hence $A \subset \mathcal{C}$.

Choose any $y \in A$. Since $A \subseteq L$, $y \in L$. Then there exists a real $x \in L_{\omega_1^y}$ so that

$$L \models x \text{ codes the set } \{z \mid z \in A \wedge z <_L y\} \wedge P(x, y).$$

Since $A \subset L$, x codes the set $\{z \mid z \in A \wedge z <_L y\}$. Since the relation P is Π_1^1 and $x, y \in L$, by the absoluteness of Π_1^1 relations, $P(x, y)$ holds. Hence (ii) follows.

Thus the Π_1^1 -uniformization principle \mathfrak{J} holds. \square

One may relativize the Π_1^1 -uniformization principle to admit real parameters to obtain the boldface version of \mathfrak{J} . Then Theorem 3.4 may be generalized to state: Boldface Π_1^1 -uniformization principle \mathfrak{J} fails if and only if there is no real x so that $(\omega_1)^{L[x]} = \omega_1$. This leads to the following result:

Corollary 3.5. *The statement “ $ZFC + \Pi_1^1$ -uniformization principle \mathfrak{J} is false” is consistent if and only if “ $ZFC +$ there exists an inaccessible cardinal” is consistent.*

Proof. If \mathfrak{J} fails, then $(\omega_1)^{L[x]} < \omega_1$ for all x , so that the latter is inaccessible in L . Conversely, suppose $ZFC +$ “There is an inaccessible cardinal” is consistent. Then, by Levy collapse, $ZFC +$ “ ω_1 is inaccessible in L ” is consistent. Now in a model say \mathcal{M} with such a property, $\omega_1 > (\omega_1)^{L[x]}$ for all x . So Π_1^1 -uniformization principle fails in \mathcal{M} . \square

4. MAXIMAL CHAINS AND THIN MAXIMAL ANTICHAINS IN THE TURING DEGREES

In this section, we apply the uniformization principle \mathfrak{J} to solve the existence problem of Π_1^1 maximal chains and thin Π_1^1 maximal antichains in the Turing degrees. These problems were studied in [5] and [6] using direct constructions involving the Boolos-Putnam analysis of constructible reals. Here we derive the results from the principle \mathfrak{J} .

Theorem 4.1 (Chong and Yu [5]). *Assume $(\omega_1)^L = \omega_1$. There is a Π_1^1 maximal chain in the Turing degrees.*

Proof. Define a binary relation P as follows:

$$P(x, y) \Leftrightarrow y \text{ is a minimal cover of } \{(x)_n | n \in \omega\}.$$

It was shown in [5] that P is a Δ_1^1 cofinally progressive relation (in the language of the current paper). Hence by Theorem 3.4, there is a Π_1^1 set A as prescribed. \mathfrak{J} (ii) implies that A is a maximal chain. \square

In the case of a maximal antichain, a diagonal argument shows that under ZFC there is a thin maximal antichain in the Turing degrees. Hence the definability of a thin set with such a property becomes particularly interesting. Since every set of reals whose degrees form a maximal antichain has size 2^{\aleph_0} , it cannot be Σ_1^1 (else it would contain a perfect subset). The next level of definability is then Π_1^1 . We apply Theorem 3.4 to construct a thin Π_1^1 maximal antichain in the Turing degrees.

Theorem 4.2. *Assume $(2^\omega)^L = 2^\omega$. There exists a Π_1^1 thin maximal antichain in the Turing degrees.*

Proof. Define a binary relation $P(x, y)$ as follows:

- (1) $\{(x)_n | n \in \omega\}^1$ is not an antichain or;
- (2) $\{(x)_n | n \in \omega\} \cup \{y\}$ is an antichain and
 - (2a) $x \oplus (y)_0 \in L_{\omega_1^y}$;
 - (2b) $\{(x)_n | n \in \omega\} \cup \{(y)_0\}$ is an antichain;
 - (2c) for every $z <_L (y)_0$, $\{(x)_n | n \in \omega\} \cup \{z\}$ is not an antichain.

As shown in [6], P is a cofinally progressive relation. Note that (2c) is equivalent to

$$L_{\omega_1^{(x,y)}}[x, y] \models \exists \beta (y_0 \in L_\beta \wedge \forall z \in L_\beta (z <_L y \implies \exists n ((x)_n \geq_T y \vee y \geq_T (x)_n))).$$

Thus P is Π_1^1 . Hence by Theorem 3.4, there is a Π_1^1 set A with the prescribed properties. Using \mathfrak{J} (ii), we show by induction on $<_L$ that A is an antichain.

Suppose $x \in A$ and $\{y | y \in A \wedge y <_L x\}$ is an antichain. By \mathfrak{J} (ii), there is a z coding $\{y | y \in A \wedge y <_L x\}$ so that $P(z, y)$ holds. Then by (2), $\{(z)_n | n \in \omega\} \cup \{y\}$ is an antichain. So A is an antichain. Note that $A \subset \mathcal{C}$. This together with Theorem 2.3 implies that A is a thin set. By (2b) above, A is maximal. \square

Note that despite the use of an uniformization principle, the proofs of the above theorems still appeal to metamathematical assumptions to establish the results. One may wonder if this approach is at all avoidable. It turns out that this is a necessary route:

¹To rule out trivial cases: if there exist $m \neq n$ such that $(x)_m = (x)_n$, then we assume $(x)_m$ and $(x)_n$ are the same set.

Theorem 4.3. (*ZFC*)

- (i) *There is a thin Π_1^1 maximal antichain of Turing degrees if and only if $(2^\omega)^L = 2^\omega$.*
- (ii) *There is a thin $\mathbf{\Pi}_1^1$ (boldface) maximal antichain of Turing degrees if and only if $(2^\omega)^{L[x]} = 2^\omega$ for some real x .*

Proof. (i): Suppose A is a thin Π_1^1 maximal antichain. Then by Theorem 2.3, $A \subset L$. Now let x be a real. By a theorem of Cooper [7], there is a real y of minimal degree such that $x \leq_T y'$. Since A is a maximal antichain, there is a real $z \in A$ with $z \geq_T y$. So $x \leq_T z'$. Hence $x \in L$.

Conversely, suppose $(2^\omega)^L = 2^\omega$. Fix a Π_1^1 set G as in Lemma 4.2. Since the statement “ G is an antichain in the Turing degrees” is Π_2^1 and

$$L \models \text{“}G \text{ is an antichain in the Turing degrees”},$$

G is an antichain in the Turing degrees in V by absoluteness. Fix a real x . Since $(2^\omega)^L = 2^\omega$, $x \in L$. The statement $T(x)$: “there exists $y \in G$ so that y is Turing comparable with x ” is $\Sigma_2^1(x)$ and $L \models T(x)$. It follows that $T(x)$ is true. Thus G is a maximal antichain.

(ii): Relativize the proof of (i). □

A direct proof of Theorem 4.3 may be found in [6]. It follows that to construct a model in which there is no thin $\mathbf{\Pi}_1^1$ maximal antichain of Turing degrees, one only needs to refute *CH* in the model.

Theorem 3.4 explains why additional hypothesis is required to show Theorem 4.2, even with help from the uniformization principle \mathfrak{J} , since the assumption $(\omega_1)^L = \omega_1$ is not sufficient to construct a Π_1^1 thin maximal antichain. Nevertheless, the following theorem still holds:

Proposition 4.4. *If $(\omega_1)^L = \omega_1$, then there is a Π_1^1 thin antichain of size \aleph_1 in the Turing degrees.*

Proof. Define a binary relation $P(x, y)$:

- (1) $\{(x)_n \mid n \in \omega\}$ is not an antichain, or
- (2) $\{(x)_n \mid n \in \omega\} \cup \{y\}$ is an antichain.

Obviously $P(x, y)$ is Δ_1^1 and cofinally progressive. By Theorem 3.4, there is a Π_1^1 set A as described in the uniformization principle. One verifies that A is an antichain of size \aleph_1 . □

It should be pointed out that a number of results concerning Π_1^1 -ness—for example those considered in [13]—may be derived from Theorem 3.4. On the other hand, the existence of Π_1^1 maximal chains and antichains does not extend to all notions of reducibility. For example, there is no Π_1^1 set of reals whose hyperdegrees form a maximal chain. To see this, suppose that A is such a set. Then A does not contain a perfect subset since otherwise there will be two paths having incomparable hyperdegrees. By Theorem 2.3 of Mansfield and Solovay, this implies that $A \subset \mathcal{C}$. Let α be countable admissible that is also a limit of an increasing sequence of admissible ordinals $\{\alpha_n^{x_n} \mid n < \omega\}$ with $x_n \in A$. Then there is a real x such that $\omega_1^x = \alpha$ [14]. By the choice of α , $x \notin \mathcal{C}$ and therefore $x \notin A$. However, $x_n <_h x$ for all n and x may be chosen to be in $L_{\omega_1^y}$ for any y in \mathcal{C} that is an upper bound (in the hyperdegrees) of $\{x_n\}$. Then $x <_h y$ for any such y . Since A is maximal, this is a contradiction.

Observe also that there is no thin Π_1^1 set of reals whose hyperdegrees form an antichain. Indeed, if A is such a set, then as above $A \subset \mathcal{C}$. But then A is not an antichain.

5. MARTIN'S CONJECTURES AND \mathfrak{J}

In this section, we apply Theorem 3.4 to study Martin's conjectures under the assumption $V = L$. We begin by recalling some definitions.

Let $f : 2^\omega \rightarrow 2^\omega$.

- Definition 5.1.**
- (1) f is degree invariant if and only if $\forall x \forall y (x \equiv_T y \implies f(x) \equiv_T f(y))$.
 - (2) f is uniformly degree invariant if and only if there is a function $t : \omega \rightarrow \omega$ such that for all $i, j < \omega$ and reals x and y , $x = \Phi_i^y \wedge y = \Phi_j^x \implies f(x) = \Phi_{(t((i,j)))_0}^{f(y)} \wedge f(y) = \Phi_{(t((i,j)))_1}^{f(x)}$.
 - (3) f is increasing on a cone if and only if there is a real y such that for all $x \geq_T y$, $x \leq_T f(x)$.
 - (4) f is order preserving on a cone if and only if there is a real y such that for all $x, z \geq_T y$, $x \leq_T z \implies f(x) \leq_T f(z)$.
 - (5) f is constant on a cone if and only if there are reals y and y_0 such that for all $x \geq_T y$, $f(x) \equiv_T y_0$.

Given degree invariant functions f and g , write $f \geq_M g$ if $f(x) \geq_T g(x)$ on a cone. Martin's conjectures state that assuming ZF together with AD and Axiom of Dependent Choice (DC),

- (I) Every degree invariant function f that is not increasing on a cone is constant on a cone; and
- (II) $<_M$ is a prewellordering on degree invariant functions which are increasing on a cone. If f has $<_M$ -rank α in the prewellordering, then f' has $<_M$ -rank $\alpha + 1$, where $f'(x) = (f(x))'$ by definition (' denotes Turing jump).

Many results related to Martin's conjectures can be found in [19] and [16]. In general, since Borel determinacy is a theorem of $ZF + DC$ [12], Steel and Slaman-Steel's results remain true for all Δ_1^1 (and hence Σ_1^1) functions. Steel [19] proved that (II) is true if f is, in addition, a uniformly degree invariant and increasing function on a cone. Note that (II) implies that if f is degree invariant and $f(x) \equiv_T x$ cofinally (i.e. for every z , there is a real $x \geq_T z$ such that $f(x) \equiv_T x$), then $f(x) \equiv_T x$ on a cone. We prove that this fails at the Π_1^1 level under the assumption $V = L$.

Theorem 5.2.² *Assume $V = L$. There is a Π_1^1 uniformly degree invariant function g that is increasing and order preserving on a cone such that for all y , there are two reals $x_0, x_1 \geq_T y$ satisfying $g(x_0) \equiv_T x_0$ and $g(x_1) \geq_T \mathcal{O}^{x_1}$.*

Proof. Assume $V = L$. Define $P(x, y)$ as

$$y \in L_{\omega_1^y} \wedge \mathcal{O}^{\mathcal{O}^x} = (y)_0 \wedge (y)_1 = x \wedge (y)_2 \text{ is the } <_L \text{-least real so that } (y)_2 \not\leq_T (y)_0.$$

Since " $\mathcal{O}^{\mathcal{O}^x} = (y)_0$ " is Π_1^1 (see [15]), P is a Π_1^1 cofinally progressive relation. By Theorem 3.4, there is a Π_1^1 set A satisfying the prescribed properties in \mathfrak{J} .

By the definition of P , every real is Turing below some real in A . Moreover, P ensures that $<_T$ is a wellordering and consistent with $<_L$ on A .

Define $f(x) = y$ if

- (a) $y \in A$;
- (b) $x \leq_T y$;
- (c) $x >_T \emptyset \implies \forall n (x \not\leq_T ((y)_1)_n)$.

²Also proved independently by Slaman using a different argument.

Roughly speaking, y is the $<_T$ -least real in A such that $x \leq_T y$.

By (c) and the property of A , f is well-defined. Furthermore, f is a Π_1^1 increasing, order preserving, and uniformly degree invariant function ($x \equiv_T y$ implies $f(x) = f(y)$). Moreover, since $f(x) = x$ for every $x \in A$, we have $f(x) \equiv_T x$ cofinally.

For every real s , take a real $x_s \in A$ with $x_s \geq_T s$. Then for the $<_T$ least real $y \in A$ above x_s , there is a real z coding $\{x \mid x <_L y \wedge x \in A\} = \{x \mid x <_T y \wedge x \in A\}$ so that $P(z, y)$ holds. Obviously $x'_s \leq_T z' \leq_T \mathcal{O}^z$ (where x'_s is the Turing jump of x_s). By the definition of P ,

$$\mathcal{O}^{x'_s} \leq_T \mathcal{O}^z \leq_T y,$$

and $(y)_1 = z$. Thus

$$\{((y)_1)_n \mid n \in \omega\} = \{r \mid r \leq_T x \wedge r \in A\}.$$

Since $x'_s \not\leq_T r$ for all $r \in A$ with $r \leq_T x_s$, we have $x'_s \not\leq_T ((y)_1)_n$ for all $n \in \omega$. Hence $f(x'_s) = y$. In other words, $f(x'_s) \geq_T \mathcal{O}^{x'_s}$. \square

Slaman and Steel [16] proved that assuming AD , if f is uniformly degree invariant and not increasing on a cone, then f is constant on a cone. We show that this also fails for Π_1^1 functions under the Axiom of Constructibility.

Proposition 5.3. *Assume $V = L$. There is a uniform degree invariant, non-increasing Π_1^1 function f such that $f(x)|_T x$ (i.e. $f(x)$ and x have incomparable Turing degrees) on a cone. In particular, f is not constant on a cone.*

Proof. Let A be the Π_1^1 set as constructed in Theorem 5.2. Define $f(x) = y$ if and only if there is a real $z \equiv_T y'$ so that

- (a) $z \in A$;
- (b) y has a minimal degree;
- (c) $y = \Phi_e^z$ for some e so that for all $i < e$, if Φ_i^z has a minimal degree then $(\Phi_i^z)' \not\equiv_T z$; and
- (d) $x >_T \emptyset \implies \forall n (x \not\leq_T ((z)_1)_n)$.

We leave it to the reader to verify that f is the desired counterexample. \square

As a final application of \mathfrak{J} , we show that (II) of Martin's conjectures fails at the Π_1^1 level in a strong way, i.e. there is a $<_M$ -decreasing sequence, again under the assumption $V = L$.

Theorem 5.4. *Assume $V = L$. There is a sequence of uniform degree invariant, increasing, Π_1^1 functions $\{f_n\}_n$ such that $f_{n+1} <_M f_n$ for all $n \in \omega$. Thus $<_M$ is not a prewellordering.*

Proof. Assume $V = L$. Define $P(x, y)$ by

$$y \in L_{\omega_1^y} \wedge (y)_0 = x \wedge (y)_1 \text{ is } <_L \text{-least such that } (y)_1 \not\leq_T (y)_0 \wedge y \equiv_T ((y)_0 \oplus (y)_1 \oplus (y)_2)'$$

Obviously $P(x, y)$ is a Π_1^1 cofinally progressive relation. By Theorem 3.4, there is a Π_1^1 set with the prescribed properties.

By the definition of P , every real is Turing below some real in A . Moreover, by the definition of P , $<_T$ is a wellordering and consistent with $<_L$ on A .

It is not difficult to see that there is an arithmetical set $W = \{(m, n, z) \mid m, n \in \omega \wedge z \in 2^\omega\}$ such that for each real z , $\{W_n^z\}_{n \in \omega} = \{\{m \mid (m, n, z) \in W\}\}_{n \in \omega}$ is a sequence of z -r.e. sets so that for all n , $z <_T W_{n+1}^z <_T W_n^z$.

Define a function $f_n(x) = y$ if there exists a real $z <_T y$ such that

- (a) $z \in A$;
- (b) $z \geq_T x$;
- (c) $y = W_{n+1}^z$;
- (d) $x >_T \emptyset \implies \forall m(x \not\leq_T ((z)_0)_m)$.

Roughly speaking, y is the W_{n+1}^z for the $<_T$ -least real z in A with $z \geq_T x$.

Obviously $\{f_n\}_n$ is a sequence of Π_1^1 functions. Note that for any $x <_T y$ in A , $x' \leq_T y$. So for every n , f_n is degree invariant and increasing.

By (d) and the property of A , f is well-defined.

By the choice of $\{W_n^z\}_{n \in \omega}$, $\{f_n\}_{n \in \omega}$ is a $<_M$ -descending sequence. \square

We end this paper with an open question:

Results in this section for Π_1^1 -uniform degree invariant functions were proved under $V = L$. It means that Slaman-Steel's results in [16] may not hold even for Π_1^1 functions assuming a hypothesis different from AD (or Π_1^1 -determinacy). One may then wish to identify the consistency strength of (I) and (II) in Martin's conjectures. Thus, is the consistency of (I) and (II) (for uniform degree invariant functions, say) in the Π_1^1 case equivalent to a large cardinal hypothesis? Since Π_1^1 -determinacy, which implies the Slaman-Steel results, is equivconsistent with the existence of 0^\sharp [9], and the latter is stronger than a simple assumption of the existence of an inaccessible cardinal, the consistency strength of (I) and (II) for Π_1^1 -uniform degree invariant functions becomes particularly interesting.

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