THE JUMP OF A Σ_n-CUT

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Abstract. Let $n \geq 1$. We study the degree of the Turing jump of a Σ_n-cut I in a model $\mathcal{M}$ of weak Peano arithmetic (PA^-) endowed with Δ_n-induction ($I\Delta_n$) but not Σ_n-induction ($I\Sigma_n$). We show that I' is not recursive in $\emptyset^{(n)}$, and use this to investigate the jump hierarchy in these models. Taking $PA^- + I\Delta_n$ as the base theory, we show that the existence of a proper low$_n$ r.e. degree is equivalent to Σ_n-induction.

1. Introduction

In the study of Turing degrees in recursion theory (otherwise known as computability theory), the jump operator arguably occupies a central position, whether the degrees are viewed "locally" (below a given degree), or "globally" across the entire structure. In the case of recursively enumerable (r.e.) degrees, for example, the notions of (high$_n$ and low$_n$) jump classes have proved to be both very useful and important for the classification of "degree-theoretic" complexity of r.e. sets. Recent works by various authors have provided a much clearer picture of the basic properties of these classes. For example, Nies, Shore and Slaman [16] established the definability of low$_n$ degrees for $n \geq 2$ and high$_n$ degrees for $n \geq 1$. Jockusch, Li and Yang [11] showed that low$_1$ and low$_n$ ($n > 1$) r.e. degrees are not elementarily equivalent. This was generalized by Shore [17] to the non-elementary equivalence of the low$_n$ and low$_m$ r.e. degrees for $n \neq m$.

From the conceptual point of view, the jump operator and its associated jump classes are notions that arise naturally in any mathematical structure admitting a computation theory, e.g. models of full Peano arithmetic (PA). But it is possible to go further and consider models of fragments of PA, and investigate the existence as well as the degree-theoretic structure of a jump class. Proof-theoretically speaking, one is also interested in studying the strength of mathematical induction

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that is required to guarantee the existence of a jump class. More precisely, let PA^- denote the set of axioms in PA minus the induction scheme, but with the exponential function (in order to perform coding of "finite" sets in models of PA^-). For $n \geq 0$, let $I\Sigma_n$ and $I\Delta_n$ denote respectively the induction scheme for Σ_n- and Δ_n-formulas. By Kirby and Paris [12], and Slaman [18], there is a hierarchy of subtheories of PA in strictly increasing mathematical strength:

$$P^- + I\Delta_1 \Leftarrow P^- + I\Sigma_1 \Leftarrow P^- + I\Delta_2 \Leftarrow P^- + I\Sigma_2 \ldots .$$

Each theory in the hierarchy supports a system of computation. In particular, appropriate notions of Turing reducibility, degrees and jump operation may be defined for models of these theories (see the next section for details). In [18] it is shown that the Σ_n-bounding scheme $B\Sigma_n$, stating that the image of a "finite set" (in the sense of the model being considered) under a Σ_n-definable function is always bounded, is equivalent to $I\Delta_n$. The equivalence of $I\Delta_n$ and $B\Sigma_n$ will be used implicitly throughout this paper, and $B\Sigma_n$ will be mentioned specifically when it is used to bound the range of a Σ_n-function on a "finite set".

We are motivated by the following general question: What is the role played by the jump operator in the hierarchy of theories? In particular, does the jump hierarchy act as a differentiating agent between $I\Sigma_n$ and $I\Delta_n$ over a base theory?

Recall the definition of high$_n$ ($\mathbf{H_n}$) and low$_n$ ($\mathbf{L_n}$) jump classes for sets of natural numbers and their degrees. An r.e. degree $\mathbf{a}$ is in $\mathbf{H_n}$ if its n-th Turing jump is equal to $\mathbf{0^{(n+1)}}$, and is in $\mathbf{L_n}$ if this jump is $\mathbf{0^{(n)}}$. We say that an r.e. degree $\mathbf{a}$ is *proper* high$_n$ (resp. *proper* low$_n$) if it is in $\mathbf{H_n} \setminus \mathbf{H_{n-1}}$ (resp. $\mathbf{L_n} \setminus \mathbf{L_{n-1}}$). The priority method is an essential tool for the study of jump classes in the natural numbers. For example, a typical construction of a low (i.e. low$_1$) r.e. non-recursive set uses a finite injury priority argument, while those for proper high (i.e. high$_1$) and low$_2$ sets use infinite injury priority arguments. It is natural to ask whether the priority method is necessary for proving existence theorems concerning $\mathbf{H_n}$ and $\mathbf{L_n}$, and whether there is a connection between priority methods and proof-theoretic strengths of theorems on jump classes. In this regard, much is known about high$_n$ r.e. degrees. At the Σ_2 level, Groszek and Mytilinaios [8] proved the existence of an incomplete high r.e. degree assuming Σ_2-induction. Their method may be applied to show the existence of a proper high$_n$ r.e. degree under Σ_{n+1}-induction. In the opposite direction, we showed in [4] that over the base theory $I\Delta_2$, both the Sacks Jump Inversion Theorem and the existence of an incomplete high r.e. degree failed in the absence of Σ_2-induction. Again, the technique may be used to show that there is no proper high$_n$ r.e. degree in any model of $I\Delta_{n+1}$ without Σ_{n+1}-induction. It follows that over the base theory $I\Delta_{n+1}$, the existence of a proper high$_n$ r.e. degree is equivalent to Σ_{n+1}-induction.

Using a technique similar to that in Groszek and Mytilinaios [8], one can show that the Sacks Jump Inversion Theorem is also a theorem of $I\Sigma_2$. As a consequence, the existence of a proper low$_2$ r.e. degree follows from $I\Sigma_2$ as well. By relativization, one concludes that there is a proper low$_n$ r.e. degree in any model of $I\Sigma_n$.

The necessity of Σ_n-induction for the result, on the other hand, is quite a different problem. First of all, the priority method that worked so well fails completely in models without Σ_n-induction. This seems to suggest that Σ_n-induction is indeed necessary. On the other hand, there is the case of the Friedberg-Muchnik Theorem, whose classical proof uses a finite injury priority method similar to that of a proper low r.e. set, that remains valid in models of Δ_1-induction without $I\Sigma_1$ [2]. The proof is based on a priority-free construction that exploits the absence of Σ_1-induction. Therefore it is natural to ask if the technique extends to proper low r.e. sets.

More generally, while the link between $\mathbf{H_n}$- and Σ_{n+1}-induction has been established, the parallel question of $\mathbf{L_n}$ was left open. The problem we consider in this paper is therefore the existence of a proper low$_n$ r.e. degree in the absence of $I\Sigma_n$. We prove in §6 that for $n \geq 1$, over the base theory $PA^- + I\Delta_n$, the existence of a proper low$_n$ r.e. set is equivalent to Σ_n-induction. This implies, among other things, that the existence of a non-recursive low r.e. set is "inductively stronger" than the existence of a Friedberg-Muchnik pair since the latter is a theorem of Δ_1-induction. Thus even in the realm of theorems derived from the simplest of finite injury priority methods, there is a distinction among the theorems marked by their proof-theoretic strengths.

It turns out that the central notion in the study of jump classes in models of Δ_n-induction is that of a Σ_n-cut. In a rather striking way, the degree of a Σ_n-cut, and the degree of its jump, is the key to understanding the behavior of the jumps of degrees below $\mathbf{0}'$. For example, in the case of r.e. degrees in an $I\Delta_2$-model, the jump of an r.e. set may assume only one of the following three degrees ([6]): $\mathbf{0}'$, $\mathbf{0}''$ or $\deg(I) \vee \mathbf{0}'$, where I is a Σ_2-cut in the model. Our work here probes deeper into the degree-theoretic properties of a Σ_n-cut, and shows that many of the answers to the problems being investigated rest on a better grasp of this object and its jump.

This paper is organized as follows. In §2, we recall some basic facts of recursion theory in arithmetic that will be used in the sequel. In §3, we prove a version of the Limit Lemma that gives a characterization and a better approximation of sets weakly recursive in $\emptyset^{(n)}$ in an $I\Delta_n$-model. In §4 we compute the jump of a Σ_n-cut and analyze the jump hierarchy in §5 using the notion of a critical pair which we introduce in that section. These results are then applied in §6 to establish the main theorems, including a characterization of the existence of a proper low$_n$ r.e. degree. The paper ends with a list of open problems in §7.

2. Preliminaries

The reader is referred to Chong and Yang [5] and [7] for more of what is summarized below. Standard notations in recursion theory follow Soare [19] closely.

All models $\mathcal{M}$ in this paper satisfy $PA^- + I\Sigma_0$. We will often assume $\mathcal{M}$ to be a model of $I\Delta_n$, for some $n \geq 1$, but not $I\Sigma_n$. We call such a model an *$I\Delta_n$-model*. In an $I\Delta_n$-model, the most interesting phenomenon is the existence of a *Σ_n-cut I*, which is a bounded non-empty Σ_n-subset of $\mathcal{M}$, closed downward as well as under the successor function.

Lemma 2.1. *Let $\mathcal{M}$ be an $I\Delta_n$-model. Then there is a Σ_n-cut I and a Σ_n-map $f : I \to M$ whose range is unbounded in $\mathcal{M}$.*

In any model $\mathcal{M}$, a set is *$\mathcal{M}$-finite* if and only if it has a code in $\mathcal{M}$. We will not distinguish an $\mathcal{M}$-finite set from its code if there is no possible confusion. We also follow the set-theoretic convention of identifying an element a in a model with its set of predecessors. A set $A \subseteq \mathcal{M}$ is *regular* if for every m in $\mathcal{M}$, $A\restriction m$ is $\mathcal{M}$-finite.

Lemma 2.2. *Let $\mathcal{M}$ be a model of $PA^- + I\Sigma_n$, where $n \geq 1$.*

1. *If A is Σ_n in $\mathcal{M}$, then A is regular.*
2. *If f is a partial Σ_n-function whose domain is bounded, then the range of f is also bounded.*

The following definitions and lemma capture the essence of coding in $I\Delta_n$-models. The reader may refer to [2] for details.

Definition 2.1. *Let A be a subset of $\mathcal{M}$. A set $X \subseteq A$ is* coded on A *if there is an $\mathcal{M}$-finite set $\hat{X}$ such that $\hat{X} \cap A = X$.*

For example, if X is coded on a cut I then X is an initial segment of an $\mathcal{M}$-finite set.

Definition 2.2. *Let A be a subset of $\mathcal{M}$. We say that a set X is Δ_n* on A *if both $A \cap X$ and $A \cap \overline{X}$ are Σ_n.*

Lemma 2.3 (Chong and Mourad [1]). *Let $\mathcal{M}$ be a model of $PA^- + I\Delta_n$ ($n \geq 1$) and let A be an arbitrary subset of $\mathcal{M}$. Then every bounded set that is Δ_n on A is coded on A.*

Consequently, if $\mathcal{M}$ is a model of $PA^- + I\Delta_n$, then any Δ_n-subset of $\mathcal{M}$ is regular.

We now turn to the notion of Turing reducibility and Turing degrees in an $I\Delta_n$-model $\mathcal{M}$.

A *Turing functional* is an r.e. set Φ of quadruples, $\langle x, y, P, N\rangle$, where P and N are disjoint $\mathcal{M}$-finite sets and x, y are numbers in $\mathcal{M}$. We assume that Turing functionals are monotone and well-defined, i.e.,

$$\begin{aligned}&[\langle x, y, P, N\rangle \in \Phi \ \wedge P' \supseteq P \ \wedge N' \supseteq N \ \wedge P' \cap N' = \emptyset] \rightarrow \\ &\quad [\langle x, y', P', N'\rangle \in \Phi \leftrightarrow y = y']\end{aligned}$$

We say that $\Phi^A(x) = y$ if there are $\mathcal{M}$-finite sets P and N (P is called a *positive neighborhood condition*, and N a *negative neighborhood condition*, of A) such that $P \subseteq A$, $N \subseteq \overline{A}$ and $\langle x, y, P, N\rangle \in \Phi$. In this paper, the letters P and N always denote $\mathcal{M}$-finite sets, unless indicated otherwise.

If the oracle A in Φ^A is a cut, then we may assume that the positive and negative conditions P and N are singletons $\{p\}$ and $\{n\}$ (by taking the maximum and minimum element respectively). We write $\langle x, y, p, n\rangle$ instead of $\langle x, y, \{p\}, \{n\}\rangle$. If the oracle A is a Σ_n-set and the model satisfies $I\Delta_n$, then we may omit the positive condition P. The reason is that under the Σ_n-collection scheme $B\Sigma_n$, we can choose another enumeration Ψ such that $\langle x, y, N\rangle \in \Psi$ if and only if $\exists P \langle x, y, P, N\rangle \in \Phi$ and $P \subset A$.

In a model of restricted induction, the notion "recursive in" requires some care to define. A set B is said to be *weakly recursive* in A if for some Turing functional Φ, $\Phi^A = B$. B is *recursive* in A if both $\{P | P \text{ is } \mathcal{M}\text{-finite and } P \subseteq B\}$ and $\{N | N \text{ is } \mathcal{M}\text{-finite and } N \subseteq \overline{B}\}$ are weakly recursive in A. Groszek and Slaman [9] showed that "weakly recursive in" is not a transitive relation among r.e. sets in some models of $I\Sigma_1$, whereas "recursive in" is a transitive relation on sets. Thus the notion of a Turing degree is well-defined only under the relation "recursive in". If A and B are sets in the model $\mathcal{M}$ and A is weakly recursive in B, then we write $A \leq_w B$. If A is recursive in B, we write $A \leq_T B$. We write $A <_T B$ (resp. $A <_w B$) if $A \leq_T B$ and $B \not\leq_T A$ (resp. $A \leq_w B$ and $B \not\leq_w A$). Also, let $A \equiv_T B$ denote $A \leq_T B$ and $B \leq_T A$. We say in this case that A and B have the same Turing degree.

The lemma below summarizes the basic properties concerning reducibilities whose proofs either follow from definition or can be found elsewhere (see [9]).

Lemma 2.4. *Let $\mathcal{M}$ be a model of $PA^- + I\Sigma_0$.*

1. *If $A \leq_w B$ and $B \leq_T C$ then $A \leq_w C$.*
2. *If A is many-one reducible to B, i.e. there is a total recursive function k such that $x \in A$ if and only if $k(x) \in B$, and $B \leq_w C$ then $A \leq_w C$.*

Next we recall some notions of classical recursion theory in the context of restricted induction.

- Let f be a function and X be a set. We say that f is *weakly recursive in* X if the graph of f is $\Sigma_1(X)$ (i,e, weakly recursive in X).
- We use $\{W_e^A : e \in \mathcal{M}\}$ to denote a fixed enumeration of all $\Sigma_1(A)$-subsets of $\mathcal{M}$.
- The *jump of* A is defined to be the set $A' = \{\langle e, x\rangle : x \in W_e^A\}$. Also $A^{(n+1)}$ is defined to be $(A^{(n)})'$ for all $n \geq 1$. Notice that any $\Sigma_1(A)$-set is many-one reducible to A'.
- Let X be a Σ_n-set. We say that X is Σ_n*-complete* if any Σ_n-set is many-one reducible to X.
- $\emptyset^{(n)}$ is Σ_n-complete, and has the same Turing degree as any other Σ_n-complete set.
- For $n \geq 0$, let $\mathbf{0^{(n)}}$ denote the degree of $\emptyset^{(n)}$. We say that A is *low*$_n$ if $A^{(n)}$ has degree $\mathbf{0^{(n)}}$, and A is *high*$_n$ if $A^{(n)}$ has degree $\mathbf{0^{(n+1)}}$. A low$_n$ (resp. high$_n$) set A is *proper* low$_n$ (resp. high$_n$) if it is not low$_{n-1}$ (resp. high$_{n-1}$).

Our ultimate goal is to show that in the absence of Σ_n-induction, there is no proper low$_n$ r.e. degree. For this to make sense, we need to verify that in an $I\Delta_n$-model, the jump operator is well-defined, i.e. it still respects the Turing reducibility relation. The proof follows from the classical one, except that care must be exercised to ensure (strong) Turing reducibility.

Lemma 2.5. *Let $\mathcal{M}$ be an $I\Delta_n$-model. If $A \leq_T B$ then $A' \leq_1 B'$. Consequently $A' \leq_T B'$.*

Proof. For any given pair $\langle e, x\rangle$,

$$\langle e, x\rangle \in A' \Leftrightarrow (\exists E, F)[\langle x, 1, E, F\rangle \in W_e \wedge E \subseteq A \wedge F \subseteq \bar{A}].$$

Suppose that $A \leq_T B$ via Ψ. We wish to find a computation Φ such that $\Phi^B(\langle e, x\rangle) = 1$ only when the fact $\langle e, x\rangle \in A'$ is known through the computations from A. Formally, we enumerate $\langle\langle e, x\rangle, 1, P, N\rangle$ in Φ if there exist $\mathcal{M}$-finite sets E, F, P_1, P_2, N_1 and N_2 such that $P_1 \cup P_2 = P$, $N_1 \cup N_2 = N$ and

$$\langle x, 1, E, F\rangle \in W_e \;\wedge \langle E, 1, P_1, N_1\rangle \in \Psi \wedge \langle F, 0, P_2, N_2\rangle \in \Psi.$$

Let $\hat{e}$ be an index for the r.e. set Φ. Then

$$\langle e, x\rangle \in A' \Leftrightarrow \langle e, x\rangle \in \text{dom } \Phi^B \Leftrightarrow \langle\langle e, x\rangle, \hat{e}\rangle \in B'.$$

Thus $A' \leq_1 B'$. □

The following facts hold for models of $I\Delta_n$:

- $\emptyset^{(n)}$ is Σ_n. Indeed $I\Delta_{n-1}$ guarantees that $\emptyset^{(n-1)}$ is Σ_{n-1} and $\emptyset^{(n)}$, being $\Sigma_1(\emptyset^{(n-1)})$, is Σ_n by an application of $B\Sigma_{n-1}$ on $\mathcal{M}$-finite subsets of $\emptyset^{(n-1)}$.
- $A \leq_w \emptyset^{(n)}$ if and only if A is Δ_{n+1} (This is proved in Corollary 3.1). It is known that the relation $\leq_w$ cannot be strengthened to $\leq_T$.

The next result, whose proof may be found in [5] (Theorem 2.2), implies that the degree of a Σ_n-cut in any $I\Delta_n$-model $\mathcal{M}$ is independent of our choice of the underlying set.

Lemma 2.6. *Any two bounded non-regular Σ_n-sets in $\mathcal{M}$ have the same Turing degree. Hence, any two Σ_n-cuts have the same Turing degree,*

3. A LIMIT LEMMA IN $I\Delta_n$-MODELS

Let $\mathcal{M}$ be an $I\Delta_n$-model ($n \geq 1$). We fix henceforth a Σ_n-complete set U and a Σ_{n-1}-complete set K. Notice that K is regular and U is not. Since U is not regular, approximations to a computation of Φ^U is harder to manage. However, the following version of the Limit Lemma holds for U.

Theorem 3.1 (Limit Lemma)**.** *If $A = \Phi^U$ for some Turing functional Φ, then there is a K-recursive function $g(\_,\_)$, together with a Σ_n-cut J, such that for all $x \in \mathcal{M}$,*

$$A(x) = \lim_{j \in J} g(x, j).$$

The idea of the proof is to decompose Φ into "I many" pieces for some Σ_n-cut I, and then filter out the pieces with bad computations one by one, and re-assemble the remainder to obtain J.

Proof. Let I be a Σ_n-cut and $f : I \to \mathcal{M}$ be the associated Σ_n-map which is unbounded in $\mathcal{M}$. We will rearrange Φ into I many $\mathcal{M}$-finite pieces in a K-recursive way.

Lemma 3.1. *Let Φ, U and K be as above. Then there exists a sequence $\{\Phi_{f(i)} : i \in I\}$ such that*

1. *For any i in I, $\Phi_{f(i)}$ is $\mathcal{M}$-finite.*
2. *$\Phi^U(x) = y$ if and only if*

   $$(\exists i \in I)(\exists P)(\exists N)[\langle x, y, P, N\rangle \in \Phi_{f(i)} \wedge P \subset U \wedge N \cap U = \emptyset].$$

   (Observe that $B\Sigma_n$ actually allows us to incorporate P in Φ so that one may ignore the positive neighborhood condition in the definition.)
3. *The function $i \mapsto \Phi_{f(i)}$ is $\Sigma_1(K)$.*

Proof. First we divide Φ into I many $\mathcal{M}$-finite pieces Ψ_i as follows. If $n > 1$, then Φ is regular, and we define Ψ_i to be $\Phi \upharpoonright f(i)$. If $n = 1$, then we define Ψ_i to be the set enumerated in Φ up to the stage $f(i)$. Notice that the function mapping i to Ψ_i is $\Sigma_1(K)$: If $n > 1$, we first use K to enumerate $f(i)$. When $f(i)$ is found, the question whether a number c is the code of $\Phi \upharpoonright f(i)$ is a boolean combination of Σ_1- and Π_1-questions, which can be answered by $\emptyset'$, thus by K. If $n = 1$, then it follows from the fact that f itself is Σ_1.

We now deal with the positive neighborhood conditions. As the set U is Σ_n, let $\varphi(x, s)$ be a Π_{n-1} formula such that

$$U = \{x : (\exists s)\varphi(x, s)\}.$$

Define

$$U_i = \{x < f(i) : (\exists s < f(i))\varphi(x, s)\}.$$

By $I\Delta_n$, U_i is a bounded Π_{n-1}-set, hence $\mathcal{M}$-finite. Moreover, to decide if a number c is the code of U_i is a "$\Sigma_{n-1} \vee \Pi_{n-1}$"-question that can be answered by K. Hence the function $i \mapsto U_i$ is $\Sigma_1(K)$.

Let $\Phi_{f(i)}$ be the set of quadruples

$$\{\langle x, y, P, N\rangle \in \Psi_i : P \subseteq U_i\}.$$

It remains for us to verify (2). If $\Phi^U(x) = y$ then there is a quadruple $\langle x, y, P, N\rangle \in \Phi$, (hence in $\Phi_{f(i_0)}$ for some $i_0 \in I$), where $P \subseteq U$ and $N \cap U = \emptyset$. By $I\Delta_n$, $P \subseteq U_{i_1}$ for some $i_1 \in I$. Let i be the maximum of i_0 and i_1. We have $\langle x, y, P, N\rangle \in \Phi_{f(i)}$. The converse is straightforward. This establishes Lemma 3.1. □

We now continue with the proof of the Limit Lemma by examining the $\mathcal{M}$-finite collection $\Phi_{f(i)}$ one by one starting with $\Phi_{f(0)}$. First of all, we may assume that A is not recursive, since otherwise the proof of the Limit Lemma is immediate. With A non-recursive, use K to find the least i such that the set X of x's satisfying

$$(\exists P_1, P_2, N_1, N_2)(\exists y_1 \neq y_2)[\langle x, y_1, P_1, N_1\rangle \in \Phi_{f(i)} \wedge \langle x, y_2, P_2, N_2\rangle \in \Phi_{f(i)}]$$

is nonempty (the non-recursiveness of A guarantees that there is such an $i \geq 0$). Denote the least i as ν_0 and the corresponding set X as X_0.

As $\Phi_{f(\nu_0)}$ is $\mathcal{M}$-finite, so is X_0.

Claim 1. For any $x \in X_0$, there is $j \in I$ such that for $y \neq A(x)$, $(\exists P, N)[\langle x, y, P, N\rangle \in \Phi_{f(\nu_0)} \to N \cap U_j \neq \emptyset]$.

Proof of Claim 1. Consider the set

$$S = \{N : (\exists P)[\langle x, y, P, N\rangle \in \Phi_{f(\nu_0)} \wedge A(x) \neq y]\},$$

which is $\mathcal{M}$-finite since x is fixed. Since the positive neighborhood condition P is correct, $N \cap U \neq \emptyset$. Thus

$$(\forall N \in S)(\exists i)[i \in I \wedge (\exists n)(n \in N \cap U_i)].$$

By $B\Sigma_n$, there is a uniform bound $j \in I$ such that for all $N \in S$, $N \cap U_j \neq \emptyset$. Thus the claim holds.

By Claim 1, we have the following weaker statement:

$$\begin{aligned}&(\forall x \in X_0)(\exists j \in I)(\forall P_1, P_2, N_1, N_2, y_1, y_2)[(\langle x, y_1, P_1, N_1\rangle \in \Phi_{f(\nu_0)}\\ &\quad \wedge \langle x, y_2, P_2, N_2\rangle \in \Phi_{f(\nu_0)} \wedge N_1 \cap U_j = \emptyset \wedge N_2 \cap U_j = \emptyset) \to y_1 = y_2].\end{aligned}$$

As U_j is Π_{n-1} and the second universal quantifier is actually bounded, we can apply $B\Sigma_n$ again to obtain a uniform bound $i_0 \in I$ such that for any $x \in X_0$ there is at most one y such that

$$(\exists P, N)[\langle x, y, P, N\rangle \in \Phi_{f(\nu_0)} \wedge N \cap U_{i_0} = \emptyset].$$

Moreover this i_0 can be enumerated from K. We thus set the first approximation as follows:

$$g(x, 0) = \begin{cases} \text{the unique } y \text{ in } \Phi_{f(\nu_0)} \text{ described above,} & \text{if such a } y \text{ exists;} \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $g(x, 0)$ is defined for all x. We now continue the process inductively. Suppose that i_k and $g(x, k)$ are defined. We obtain i_{k+1} and $g(x, k+1)$ by essentially the same procedure (in fact, simply replace the index 0 by $k + 1$ and obtain ν_{k+1} and X_{k+1} using the non-recursiveness of A)). Let J be the set of k's such that i_k is defined.

Claim 2. J is a Σ_n-cut and the set $\{i_k : k \in J\}$ is unbounded in I.

Proof of Claim 2. By construction, if i_k is defined then i_{k+1} is also defined. Thus J is a cut. Since J is defined by recursion (with K as oracle), J is Σ_n. If $\{i_k : k \in J\}$ is bounded in I, say by i^*, then the construction will end before the stage $f(i^*)$. This can happen only if for all $i \geq i^*$, for all x, every quadruple $\langle x, y, P, N\rangle \in \Phi_{f(i)}$ agrees with every other on the second coordinate (i.e. y). This implies that A is recursive, which is a contradiction. Thus Claim 2 holds.

Observe that g is a $\Sigma_1(K)$-function.

It remains for us to verify that for all $x \in \mathcal{M}$,

$$\lim_{j \in J} g(x, j) = \Phi^U(x).$$

Fix x^*. Since $\Phi^U(x^*)$ is defined,

$$(\exists P, N)[\langle x^*, y, P, N\rangle \in \Phi \wedge P \subseteq U \wedge N \cap U = \emptyset].$$

Consider one fixed pair of correct neighborhood conditions P^* and N^*. Let $i \in I$ be such that $\langle x^*, y, P^*, N^*\rangle \in \Phi_{f(i)}$ (such an i exists because we did not discard any correct pair of neighborhood conditions when building $\Phi_{f(i)}$). Let $j^* \in J$ be such that $i_{j^*} > i$. Then for all $j > j^*$, $\langle x^*, y, P^*, N^*\rangle \in \Phi_{f(j)}$ and $N^* \cap U_j = \emptyset$. Hence $g(x, j) = y$, which establishes the Limit Lemma. □

Corollary 3.1. *Let $\mathcal{M}$ be a model of $P^- + I\Delta_n$, and let U be a complete Σ_n-set. Then A is weakly recursive in U if and only if A is Δ_{n+1}.*

Proof. Suppose that A is weakly recursive in U. By the Limit Lemma, there is a Σ_n-cut J and a K-recursive (thus Δ_n)-function g such that $A(x) = \lim_{j \in J} g(x, j)$. Hence $A(x) = k$ if and only if

$$(\exists j^* \in J)(\forall j > j^*)[j \notin J \vee g(x, j) = k].$$

Thus both A and $\overline{A}$ are Σ_{n+1}. Therefore A is Δ_{n+1}.

Conversely let A be a Δ_{n+1} set and let $\varphi(x, s)$ and $\psi(x, s)$ be Π_n formulas such that $A = \{x : (\exists s)\varphi(x, s)\}$ and $\overline{A} = \{x : (\exists s)\psi(x, s)\}$. Let e_φ and e_ψ be the canonical indices for $\neg\varphi$ and $\neg\psi$ respectively. Since U is Σ_n-complete, we have $A = \{x : \exists s((x, s), e_\varphi) \in \overline{U}\}$ and $\overline{A} = \{x : (\exists s)[((x, s), e_\psi) \in \overline{U}\}$. Let Γ be the Turing functional

$$\{\langle (x, s), 1, \emptyset, ((x, s), e_\varphi)\rangle : s \in \mathcal{M}\} \cup \{\langle (x, s), 0, \emptyset, ((x, s), e_\psi)\rangle : s \in \mathcal{M}\}.$$

Then $\Gamma^U = A$. □

4. Σ_n-CUT AND ITS JUMP

In this section we study the jump of a Σ_n-cut and show that it occupies a fairly high Turing degree. Let $\mathcal{M}$ be an $I\Delta_n$-model. As before let I be a Σ_n-cut in $\mathcal{M}$ and let f be an increasing, unbounded Σ_n-function from I into $\mathcal{M}$.

Fix an upper bound a of I. It is helpful if we visualize the discussion below on a two dimensional XY-plane. We will focus our attention on the square $[0,a] \times [0,a]$, where $[0,a]$ is the closed interval $\{x : 0 \leq x \leq a\}$. Let $\mathcal{F}$ be the collection of all non-decreasing $\mathcal{M}$-finite functions from $[0,a]$ to $[0,a]$.

We split the $\mathcal{M}$-finite collection $\mathcal{F}$ into two disjoint subsets $\mathcal{F}_1$ and $\mathcal{F}_2$ as follows.

$$\mathcal{F}_1 = \{g \in \mathcal{F} : (\exists i \in I)[g(i) \notin I]\},$$

and

$$\mathcal{F}_2 = \mathcal{F} \setminus \mathcal{F}_1 = \{g \in \mathcal{F} : (\forall i \in I)[g(i) \in I]\}.$$

It turns out that the study of the jumps of Σ_n-sets, and indeed Δ_{n+1}-sets in general (see for example Theorem **??**), hinges on computing the degrees of $\mathcal{F}_1$ and $\mathcal{F}_2$. As a first step, we consider the relation between I' and $\mathcal{F}_1$.

Lemma 4.1. *$\mathcal{F}_1$ is many-one reducible to I'.*

Proof. By definition, I' is the set

$$\{\langle e, x\rangle : \Phi_e^I(x) \downarrow\}.$$

Consider the following collection of quadruples:

$$\Phi = \{\langle g, 1, \{p\}, \{n\}\rangle : p < n \leq a \wedge g \in \mathcal{F} \wedge g(p) = n\}.$$

Then $\Phi^I(g) \downarrow = 1$ if and only if $g \in \mathcal{F}_1$, and $\Phi^I(g) \uparrow$ if and only if $g \in \mathcal{F}_2$. Thus $g \in \mathcal{F}_1$ if and only if $g \in \operatorname{dom} \Phi^I$. Since Φ is a Σ_1 set, it has an index $\hat{e}$. Thus $g \in \mathcal{F}_1$ if and only if $\langle \hat{e}, g\rangle \in I'$. $\square$

Next we introduce a natural order on the set $\mathcal{F}$ which is inspired by the notion of domination. Let g and G be in $\mathcal{F}$. We write $g \ll G$, or $G \gg g$ if $g(i) \leq G(i)$ for all $i \leq a$. With respect to the order $\ll$, $\mathcal{F}_1$ is upward closed, i.e., if $g \in \mathcal{F}_1$ and $g \ll G$ then $G \in \mathcal{F}_1$. Similarly $\mathcal{F}_2$ is downward closed.

We now demonstrate a link between $\mathcal{F}_1$ and U. For each $n \leq a$, let c_n denote the constant function from $[0,a]$ to $[0,a]$ such that $c_n(x) = n$ for every $x \leq a$. Notice that c_n is in $\mathcal{F}$ for every $n \leq a$. The following lemma is the key step towards imposing a lower bound on the degree of I'.

Lemma 4.2. *Let $\mathcal{M}$ be an $I\Delta_n$-model. Then $\mathcal{F}_2 \not\leq_w U$.*

Proof. Suppose otherwise and that $\mathcal{F}_2 = \Phi^U$ for some Turing functional Φ. Let us assume that for any function $g \in \mathcal{F}$,

$$g \in \mathcal{F}_2 \Rightarrow (\exists P, N)[\langle g, 2, P, N\rangle \in \Phi \wedge P \subseteq U \wedge N \subseteq \overline{U}],$$

and

$$g \notin \mathcal{F}_2 \Rightarrow (\exists P, N)[\langle g, 1, P, N\rangle \in \Phi \wedge P \subseteq U \wedge N \subseteq \overline{U}].$$

By the downward closure of $\mathcal{F}_2$, we may assume that if $\langle g, 1, P, N\rangle$ enters Φ at stage s, then for all $G \gg g$ $\langle G, 1, P, N\rangle$ also enters Φ no later than stage s. A similar assumption applies to $\langle g, 2, P, N\rangle$, reversing $\gg$ appropriately.

The goal is to diagonalize against computations in Φ. We split Φ into I many $\mathcal{M}$-finite pieces $\Phi_{f(i)}$ as in Lemma 3.1. For notational convenience, let us assume that the cut J we obtained in Lemma 3.1 is still I. We diagonalize against computations in $\Phi_{f(i)}$ for each $i \in I$.

The informal plan, which may be regarded as a forcing argument, is as follows. For each $i \in I$, we enumerate an $\mathcal{M}$-finite string σ_i such that any "reasonable" extension of σ_i cannot be computed from $\Phi_{f(i)}$. Each σ_i is a constant function c_e restricted to an interval $[x_1, x_2)$ for some numbers e, x_1 and x_2 in I. In the end we obtain a function σ from I to I by taking the union $\sigma = \bigcup_{i\in I} \sigma_i$. The enumeration of σ_i will have certain effectiveness so that σ has an $\mathcal{M}$-finite extension g. Since σ is from I to I, g is in $\mathcal{F}_2$ and has to be decided by Φ^U at a certain stage. However, the construction of g ensures that this is not the case, yielding a contradiction.

We define σ_i by recursion on $i \in I$. Suppose that σ_i is defined; $\bigcup_{j\leq i} \sigma_j$ is a piecewise constant function from $[0, d]$ to $[0, e]$ for some d and e in I. In particular, the string σ_i has an endpoint with coordinates (d, e) in the two dimensional picture. Let us consider all extensions τ of σ_i in $\mathcal{F}$ whose graphs are located in the rectangle $(d, a] \times (e, a]$. We need to define σ_{i+1}.

For $d \in I$, define $\Phi_{f(d)}$ as in the proof of the Limit Lemma. As before, for all quadruples $\langle x, y, P, N\rangle \in \Phi_{f(d)}$ considered below, we may assume that, by $B\Sigma_n$, P is an $\mathcal{M}$-finite subset of U. The goal is to find a string ρ satisfying the following conditions:

- ρ extends σ_i via a constant function c_{e^*} on an interval $(d, d^*]$, where $d^* > d$ and $e^* > e$ are in I;
- $\Phi_{f(d)}(\tau)$ is undefined for any $\tau \in \mathcal{F}$ extending ρ;
- moreover the string ρ can be enumerated from K.

We first find the number e^*. Consider all constant functions defined on the interval $(d, a]$ with range in I but greater than e. For simplicity, we continue to use c_n $(n > e)$ to denote the function taking the constant value n on $(d, a]$, and undefined elsewhere. Note that by assumption we have $n \in I$ if and only if $n \leq e$ or $g_n = \sigma_i \cup c_n \in \mathcal{F}_2$.

Claim 1. There is an $n \in I$ such that

$$(\forall N)[\langle g_n, 2, P, N\rangle \in \Phi_{f(d)} \to N \cap U \neq \emptyset].$$

Proof of Claim 1. Suppose, for the sake of contradiction, that for all $n \in I$,

$$(\exists N)[\langle g_n, 2, P, N\rangle \in \Phi_{f(d)} \wedge N \subset \overline{U}].$$

Now since $\Phi^U = \mathcal{F}_2$, if

$$(\exists N)[(\exists P)(\langle g_n, 2, P, N\rangle \in \Phi_{f(d)}) \wedge N \subset \overline{U}],$$

then $g_n \in \mathcal{F}_2$, so $n \in I$. Therefore, $n \in I$ if and only if

$$(\exists N)[(\exists P)(\langle g_n, 2, P, N\rangle \in \Phi_{f(d)} \wedge N \subset \overline{U}].$$

As $\Phi_{f(d)}$ is an $\mathcal{M}$-finite set, the quantifier $\exists N$ is bounded. Observe that $N \subset \overline{U}$ is Π_n. By $I\Delta_n$, the formula on the right hand side is equivalent to a Π_n-formula. Thus I is a Δ_n-cut, contradicting Lemma 2.3. This establishes Claim 1.

We now argue that the number n in Claim 1 can be enumerated from K. This effective enumerability is crucial to the proof, because in the end we need to extend a function $\sigma : I \to I$ to a function in $\mathcal{F}$. Without the effectiveness, such an extension may not exist.

We can enumerate such an n by the following procedure. Fix a K-enumeration $\{I_s : s \in I\}$ of I. At stage s, if there is an $n \in I_s$ such that no quadruple $\langle g_n, 2, P, N\rangle$ is in $\Phi_{f(d)}$, then let e^* be the least such n. On the other hand, if for all $n \in I_s$, there is an N such that $\langle g_n, 2, P, N\rangle$ is in $\Phi_{f(d)}$, check if there is an $n \in I_s$ such that for all $\langle g_n, 2, P, N\rangle \in \Phi_{f(d)}$, $N \cap U_s \neq \emptyset$, where U_s is the set of elements in U enumerated by K at the end of stage $f(s)$. If the answer is no, go to stage $s + 1$. Otherwise, by Claim 1, there is an n such that for all $\langle g_n, 2, P, N\rangle \in \Phi_{f(d)}$, $N \cap U \neq \emptyset$. For this particular n, there are only $\mathcal{M}$-finitely many such N's. By $B\Sigma_n$, there is a stage d^* by which all N's where $N \cap U \neq \emptyset$ will have been enumerated. Let e^* be the least such n at stage d^*. Let σ^* be defined by $\sigma^*(x) = e^*$ for all x in $(d, d^*]$ and $\sigma_{i+1} = \sigma_i \cup \sigma^*$. Then σ_{i+1} is the desired extension.

Let J be the set $\{i \in I : \sigma_i \text{ is defined}\}$. For each $j \in J$, let the endpoint of σ_j be (d_j, e_j).

Claim 2. J is a cut and $\{d_j : j \in J\}$ is cofinal in I.

Proof of Claim 2. By the construction of σ_i, J is downward closed and closed under successor, hence a cut. Fix an i^* in I. Suppose that i^* is an upper bound for d_j ($j \in J$). Then the construction of σ_i will end before stage $f(i^*)$. This would imply that J is $\mathcal{M}$-finite, which is a contradiction. Thus Claim 2 holds.

Finally define $\sigma : I \to I$ by $\sigma(i) = e_j$ for all i in $(d_j, d_{j+1}]$. Then σ is a $\Sigma_1(K)$ function from I to I. Hence the graph of σ is Δ_n on $I \times I$. By Lemma 2.3, σ is coded on $I \times I$. We can modify the code to obtain a function $g \in \mathcal{F}$ such that $g \cap (I \times I) = \sigma$. By construction g is in $\mathcal{F}_2$. On the other hand, for $i \in I$, $\Phi_{f(i)}$ contains no quadruple $\langle g, 2, P, N\rangle$ with $N \cap U = \emptyset$ (because for each $i \in I$, g dominates an eventually constant function c_i such that $\Phi_{f(i)}$ contains no "correct" quadruple $\langle c_i, 2, P, N\rangle$) to compute g_i. Thus $\Phi^U(g)$ is undefined, a contradiction which establishes Lemma 4.2. ☐

The above lemma leads to the next theorem showing that a Σ_n-cut is a deceptively simple looking object with a highly non-trivial Turing jump.

Theorem 4.1. *In an $I\Delta_n$-model, $I' \not\leq_w U$.*

Proof. Clearly $\mathcal{F}_1$ is $\Sigma_1(I)$. ☐

In §6 we shall apply Theorem 4.1 to characterize the existence of proper low$_n$ r.e. degrees in the hierarchy of arithmetic theories. However, this requires a detour to an analysis of degrees below $\mathbf{0^{(n)}}$. We do this in the next section.

5. Critical Pairs and Moving Up the Jump Hierarchy

We begin our investigation of the structure of Δ_{n+1}-degrees (degrees that coincide with those weakly recursive in $\mathbf{0^{(n)}}$ by Corollary 3.1) with a closer look at the interplay between I and regular Δ_{n+1}-sets. As before, fix an $I\Delta_n$-model $\mathcal{M}$ throughout this section. Also let U be a complete Σ_n-set and K a complete Σ_{n-1}-set. I continues to denote a Σ_n-cut.

Let a be an upper bound of I, and D an $\mathcal{M}$-finite subset of $[0, a] \times [0, a]$. We say that two pairs (x, y_1) and (x, y_2) are *adjacent* if both are in D and for any y if $y_1 < y < y_2$ then $(x, y) \notin D$. We say that (x, y_1) and (x, y_2) form a *critical pair* in column x if they are adjacent and $y_1 \in I$ and $y_2 \notin I$. We assume, without loss of generality, that for all $x \leq a$, both $(x, 0)$ and (x, a) are in D. This assumption is for convenience and helps simplify the discussion below. Notice that if $D^{[x]} \cap I$ is unbounded in I then there is no critical pair in column x; if $D^{[x]} \cap I$ is bound in I then there is a unique critical pair in column x.

Lemma 5.1. *Let D and I be as above. Then the set C of critical pairs is recursive in I.*

Proof. The informal idea is just to go through every column looking for an adjacent pair. Using I one may decide if a given pair is critical. To get a formal proof we define a Turing functional Γ consisting of the following types of quadruples ($(x, y_1), (x, y_2)$ below are adjacent pairs):

- $\langle((x, y_1), (x, y_2)), 1, \{y_1\}, \{y_2\}\rangle$;
- $\langle((x, y_1), (x, y_2)), 0, \{y_1, y_2\}, \emptyset\rangle$;
- $\langle((x, y_1), (x, y_2)), 0, \emptyset, \{y_1, y_2\}\rangle$;
- $\langle(u, v), 0, \emptyset, \emptyset\rangle$ if $(u, v) \notin a \times a$ or they do not form an adjacent pair.

Thus C is weakly recursive in I via Γ. The prefix "weak" may be removed after a slight modification of Γ. □

Notice that the proof also shows that there is a Turing functional Δ such that $\Delta^I(x)$ is the critical pair in column x (if it exists).

As before, let f be a Σ_n-cofinal function from I into $\mathcal{M}$.

Theorem 5.1. *Let A be a regular set in $\mathcal{M}$. If A is recursive in U, then A is recursive in $I \oplus K$.*

The natural approach is to exploit the regularity of A and use $I \oplus K$ to calculate the "final" value of approximations of $A \restriction a$ for each a. However, there is no direct link between the regularity of A and its approximations when A is Δ_{n+1}. The problem is that even if A is very simple, it may have an erratic approximation function as derived from the definition of A. For example, if for every $i \in I$, i enters A at stage $f(i)$ and exits at $f(i+1)$, then A is the empty set with a Δ_{n+1}-definition, and yet the approximation of A does not settle down at any stage $f(i)$ on the initial segment $A \restriction a$ for any $a \notin I$. The same difficulty occurred with the proof of the Limit Lemma. Thus our solution adopts a strategy similar to the one used there. We will first define a better approximation function for A, using the fact that $A \leq_T U$. As was done for the Limit Lemma, we decompose the computation into I many $\mathcal{M}$-finite pieces, and tackle them individually.

Proof. As A is recursive in U, there is a Turing functional Φ such that $A = \Phi^U$. Let $\{\Phi_{f(i)} : i \in I\}$ be defined as in Lemma 3.1.

Fix a $k \in I$. Since A is regular, both $A_k = A \cap [0, f(k)]$ and $\overline{A}_k = \bar{A} \cap [0, f(k)]$ are $\mathcal{M}$-finite. They form a partition of the interval $[0, f(k)]$. We search $\Phi_{f(i)}$ for each $i \in I$ to look for A_k. Fix an $i \in I$. Consider the set $B_{k,i,j}$ of pairs $\langle E_0, E_1\rangle$ such that

- E_0 and E_1 are $\mathcal{M}$-finite.
- E_0 and E_1 form a partition of the interval $[0, f(k)]$, i.e., $E_0 \cup E_1 = [0, f(k)]$, and $E_0 \cap E_1 = \emptyset$.

- There are $\mathcal{M}$-finite sets P and N such that $\langle E_0, 0, P, N\rangle \in \Phi_{f(i)}$ and $N \cap U_j = \emptyset$.
- There are $\mathcal{M}$-finite sets P and N such that $\langle E_1, 1, P, N\rangle \in \Phi_{f(i)}$ and $N \cap U_j = \emptyset$.

$\langle E_0, E_1\rangle$ is intended to be an approximation of $\langle \overline{A}_k, A_k\rangle$.

Claim . For each pair k and i in I there is $j^* \geq i$ in I such that B_{k,i,j^*} has at most one element.

Proof of Claim . For any partition E_0 and E_1 in $B_{k,i,j}$, if $E_0 \neq \overline{A}_k$ and $E_1 \neq A_k$, then $N \cap U \neq \emptyset$ where N is a negative neighborhood condition to witness $\langle E_0, E_1\rangle$ in $B_{k,i,j}$. Thus there is a j such that $N \cap U_j \neq \emptyset$. Since $\Phi_{f(i)}$ is $\mathcal{M}$-finite, there are $\mathcal{M}$-finitely many such partitions $\langle E_0, E_1\rangle$. By $B\Sigma_n$, there is a uniform upper bound j^* for these j's. This proves the claim.

Let $j_{k,i}$ be the least j^* for the Claim. Notice that the pair $\langle E_0, E_1\rangle$ in $B_{k,i,j_{k,i}}$ may not be $\langle \overline{A}_k, A_k\rangle$. Or it may be $\langle \overline{A}_k, A_k\rangle$ but with a wrong negative neighborhood condition. The important point here is that once $\langle \overline{A}_k, A_k\rangle$ shows up with a correct neighborhood condition, it will always be seen as correct.

We now code the information on approximations of A_k via triples $\langle k, i, j\rangle$. If $|B_{k,i,j}| = 1$, let $\langle E_0^{k,i,j}, E_1^{k,i,j}\rangle$ be the unique pair of $\langle E_0, E_1\rangle$ in $B_{k,i,j}$. Let $D \subseteq I^3$, defined as $I \times I \times I$, be the set of triples $\langle k, i, j\rangle$ such that

- $|B_{k,i,j}| = 1$ and $j = j_{k,i}$;
- If i' is the largest below i such that $|B_{k,i',j_{k,i'}}| = 1$, then

$$\langle E_0^{k,i',j_{k,i'}}, E_1^{k,i',j_{k,i'}}\rangle \neq \langle E_0^{k,i,j_{k,i}}, E_1^{k,i,j_{k,i}}\rangle.$$

D is Δ_n on I^3 and by Lemma 2.3, it is coded on I^3. Let $\hat{D}$ be an $\mathcal{M}$-finite set such that $\hat{D} \cap I^3 = D$. Suppose $|B_{k,i,j_{k,i}}| = 1$ and $(k, i, j_{k,i}) \in \hat{D}$. If $\langle E_0^{k,i,j_{k,i}}, E_1^{k,i,j_{k,i}}\rangle = \langle \overline{A}_k, A_k\rangle$ and is computed from $\Phi_{f(i)}$ using a correct neighborhood condition, then no $i' > i$ in I will have $\langle k, i', j_{k,i'}\rangle \in \hat{D}$. If $\langle E_0^{k,i,j_{k,i}}, E_1^{k,i,j_{k,i}}\rangle \neq \langle \overline{A}_k, A_k\rangle$, then for some $i' > i$ in I, $|B_{k,i',j_{k,i'}}| = 1$ and $\langle k, i', j_{k,i'}\rangle \in \hat{D}$. If $\langle E_0^{k,i,j_{k,i}}, E_1^{k,i,j_{k,i}}\rangle = \langle \overline{A}_k, A_k\rangle$ using a wrong neighborhood condition of U, such an $i' > i$ still need not exist. Indeed it may happen that whenever $|B_{k,i',j_{k,i'}}| = 1$ and $i' > i$, the unique partition on $f(k)$ is the correct partition of $A \restriction f(k)$, even though the computation is based on a wrong neighborhood condition.

We now use $\hat{D}$ as a parameter to determine, with $I \oplus K$ as the oracle, the $\mathcal{M}$-finite subsets of A and $\overline{A}$. First use K to calculate the value of $f(k)$ one by one. Then enumerate A_k and $\overline{A}_k$ as follows. Use I to search through $\hat{D}$ for a pair of triples $\langle k, i_0, j_{k,i_0}\rangle$ and $\langle k, i_1, j_{k,i_1}\rangle$ such that $j_{k,i_0} \in I$, $j_{k,i_1} \notin I$ and for all i, if $i_0 < i < i_1$, then $\langle k, i, j_{k,i}\rangle \notin \hat{D}$ (i.e. (k, j_{k,i_0}) and (k, j_{k,i_1}) form a critical pair in column k on the two dimensional plane $\{(k, j_{k,i}) | k, i \in I\}$). Use K to find the unique $\langle E_0, E_1\rangle \in B_{k,i_0,j_{k,i_0}}$. Then $E_0 = \overline{A}_k$ and $E_1 = A_k$.

□

The next result is essentially in Mourad [14]. We provide a different proof using the idea of critical pairs.

Theorem 5.2. *The degree of* $I \oplus K$ *is a minimal cover of* $\mathbf{0^{(n-1)}}$ *(i.e. the degree of* K*) for* Σ_n*-degrees. In other words, if* A *is* Σ_n *and* $K \leq_T A \leq_T I \oplus K$*, then either* $A \leq_T K$ *or* $I \oplus K \leq_T A$*.*

Proof. Since A is Σ_n, there is a $\Sigma_1(K)$-enumeration of A. Let A_j denote the set of elements enumerated into A by stage $f(j)$ using K as oracle.

First assume that A is regular. Let

$$X = \{(i, j) | i, j \in I \text{ and } (A_{j+1} \setminus A_j \restriction f(i) \neq \emptyset\}.$$

Then X is Δ_n on $I \times I$ and hence by Lemma 2.3 it is coded by an $\mathcal{M}$-finite set $\hat{X}$. Notice that this yields an algorithm to compute A from $I \oplus K$: By the regularity of A, for each $i \in I$ there is a least $j \in I$ such that $A \restriction f(i) = A_j \restriction f(i)$. Denote this by j_i. Then j_i is the largest $j \in I$ such that $(i, j) \in \hat{X}$, and can be identified by I through examining the critical pair of $\hat{X}$ on column i. Using j_i and K, the set $A \restriction f(i)$ may be computed.

There are now two cases to consider.

Case 1. There is a $b \in \bar{I}$ such that for all $i \in I$, if $(i, j_i), (i, j_i^*)$ is a critical pair on column i of $\hat{X}$, then $j \geq b$.

In this case, $j = j_i$ if and only if j is the largest number less than b with $(i, j) \in \hat{X}$. Thus $\{j_i | i \in I\}$ is coded on I and so K may use this set to compute A, showing that $A \leq_T K$.

Case 2. For all $b \in \bar{I}$, there is an $i \in I$ such that $j_i^* < b$ for the critical pair $(i, j_i), (i, j_i^*)$ on column i.

We use A to compute I as follows: First of all, since A computes K and I is $\Sigma_1(K)$, it is also $\Sigma_1(A)$. Now using $K \leq_T A$ it is possible to determine j_i for each $i \in I$: This is the least j such that $A \restriction f(i) = A_j \restriction f(i)$ (we are using the fact that A is $\Sigma_1(K)$, so that for any i, $A_j \restriction f(i) \subseteq A_{j+1} \restriction f(i)$ for all j). Then $b \in \bar{I}$ if and only if there is an $i \in I$ such that $j_i^* < b$. Hence $A \geq_T I \oplus K$.

Now suppose A is non-regular. Let $i \in I$ be fixed such that $A \upharpoonright f(i)$ is not $\mathcal{M}$-finite. Then Lemma 2.6 implies that this set, hence A, computes I. Thus A computes $I \oplus K$. $\square$

Recall the notion of hyperregularity which we give below. This will be used in Lemma 5.3.

Definition 5.1. *A set A is* hyperregular *if for every partial A-recursive function h, if the domain of h is bounded in $\mathcal{M}$ then so is its range.*

The following characterization of hyperregular sets was established in [15].

Lemma 5.2. *Let $\mathcal{M}$ be a model of $P^- + I\Sigma_0$ and suppose that A is a regular set on $\mathcal{M}$. The following are equivalent:*

1. *A is hyperregular.*
2. *$\mathcal{M} \models I\Sigma_1^A$.*
3. *For every e in $\mathcal{M}$, W_e^A is regular.*

We now evaluate the iterated jumps of an incomplete r.e. degree in an $I\Delta_n$-model $\mathcal{M}$ where $n \geq 2$.

Lemma 5.3. *Let R be an incomplete r.e. set in $\mathcal{M}$. Then $R^{(n-2)}$ is regular. Moreover, $R^{(n-2)}$ is either Turing equivalent to K, or it is hyperregular. Consequently, in the latter case, $R^{(n-1)}$ is regular.*

Proof. Let A denote the set $R^{(n-2)}$ for notational simplicity. Then A is at most a Σ_{n-1}-set, and so is regular using $I\Delta_n$. Furthermore, by Lemma 2.5, $A \leq_T K$.

Next assume that A is not hyperregular. We show that $K \leq_T A$. The proof is a relativized version of Lemma 11 in Chong and Yang [6]. We outline the main idea below.

If A is not hyperregular, then there is a $\Sigma_1(A)$-cut I_A and a $\Sigma_1(A)$-cofinal map f_A from I_A into $\mathcal{M}$. Let K_μ be the set of elements in K enumerated by stage $f_A(\mu)$. Set

$$X_r = \{(\nu, \mu) \in I_A \times I_A | K \upharpoonright f_A(\nu) = K_\mu \upharpoonright f_A(\nu)\}.$$

Since A is Σ_{n-1}, the set X_r is Δ_n on $I_A \times I_A$, hence by Lemma 2.3 it is coded on $I_A \times I_A$, say via an $\mathcal{M}$-finite set $\hat{X}_r$. But this gives an algorithm to compute K from A: $K \upharpoonright f_A(\nu) = K_\mu \upharpoonright f_A(\nu)$ if $(\nu, \mu) \in \hat{X}_r$ and $\nu, \mu \in I_A$. Finally, by Theorem 5.2 (3), $R^{(n-1)}$, which is A', is regular. $\square$

6. Applications

In this section, we apply the results of the previous sections to derive a number of consequences.

Theorem 6.1. *Let $\mathcal{M}$ be an $I\Delta_n$-model where $n \geq 1$. Then there is no proper low$_n$ r.e. set in $\mathcal{M}$.*

Proof. First consider $n = 1$ and suppose that R is a proper low r.e. set. If R is regular, then according to Theorem 5.1 (where U is now $\emptyset'$ and $K = \emptyset$), $R \leq_T I$. By Theorem 5.2, R is either recursive or computes I. If it is recursive it is not proper low. If it computes I then Lemma 2.5 implies that $I' \leq_T R' \leq_T \emptyset'$. However, Theorem 4.1 rules out this possibility. If R is not regular, then it has an initial segment which is not regular. But then R computes I by Lemma 2.6. As before, this shows that R is not low.

Now let $n \geq 2$ and suppose R is a nonlow$_{n-1}$ r.e. set. Then $R^{(n-1)}$ is Σ_n, hence recursive in the complete Σ_n-set U. $R^{(n-1)}$ also computes K by repeated applications of Lemma 2.5 beginning with $R' \geq_T \emptyset'$. If $R^{(n-2)}$ is not hyperregular, then by Lemma 5.3, $R^{(n-2)} \geq_T K$, hence $R^{(n-1)} \geq_T U$, and therefore R is not low$_n$. On the other hand, if $R^{(n-2)}$ is hyperregular, by Lemma 5.3 we know that $R^{(n-1)}$ is regular. Theorem 5.1 then says that $R^{(n-1)}$ is recursive in $I \oplus K$. Theorem 5.2 now implies that $R^{(n-1)} \equiv_T K$ or $R^{(n-1)} \equiv_T I \oplus K$. If $R^{(n-1)} \equiv_T K$, then R is low$_{n-1}$ which contradicts the assumption. On the other hand, if $R^{(n-1)} \equiv_T I \oplus K$, then Lemma 2.5 implies that $I' \leq_w R^{(n)}$. But if $R^{(n)} \leq_T U$, then $I' \leq_w U$ by Lemma 2.4 (1), which is not possible according to Theorem 4.1. Hence $R^{(n)} \not\leq_T U$, and this completes the proof of the theorem. □

Since the classical construction of a proper low$_n$ r.e. set may be carried out in any model of Σ_n-induction, we are led to the following corollary:

Corollary 6.1. *Over the base theory $P^- + I\Delta_n$, the existence of a proper low$_n$ r.e. degree is equivalent to $I\Sigma_n$.*

In particular, over the base theory $P^- + I\Delta_1$, the existence of a non-recursive low r.e. set is equivalent to $I\Sigma_1$.

It is proved in [2] that the Friedberg-Muchnik Theorrem is true in any model of $PA^- + I\Delta_1$. Hence combining Theorem 6.1 with known results yields the following:

Corollary 6.2. *Over the base theory $P^- + I\Delta_1$, the existence of a non-recursive low r.e. set is proof-theoretically stronger than the Friedberg-Muchnik Theorem on the existence of incomparable r.e. sets.*

Since the Sacks Splitting Theorem is equivalent to Σ_1-induction [13], the following corollary follows:

Corollary 6.3. *Over the base theory $P^- + I\Delta_1$, the existence of a non-recursive low r.e. set is equivalent to the Sacks Splitting Theorem.*

Groszek, Mytillinaios and Slaman [10] showed that the Sacks Density Theorem is provable in $P^- + I\Delta_2$. Hence

Corollary 6.4. *Over the base theory $P^- + I\Delta_2$, the existence of a proper low$_2$ r.e. set is strictly stronger than the Sacks Density Theorem.*

The next two corollaries follow from Theorem 6.1 and [6].

Corollary 6.5. *Over the base theory $P^- + I\Delta_2$, the existence of a proper low$_2$ r.e. set is equivalent to the existence of a high incomplete r.e. set.*

Corollary 6.6. *Over the base theory $P^- + I\Delta_n$ ($n \geq 2$), the existence of a proper low$_n$ r.e. set is equivalent to the existence of a proper high$_{n-1}$ r.e. set.*

7. Questions and Open Problems

We end this paper with some questions and remarks.

1. This paper studies the iterated jumps of r.e. degrees in models of weak fragments of arithmetic. What is the distribution of the iterated jumps of other degrees? In particular, which Δ_{n+1}-sets in $I\Delta_n$-models computing $\emptyset'$ can be realized as jumps? This question is related to the most general form of Friedberg Jump Inversion (FJI): In an $I\Delta_n$-model, is every Δ_{n+1}-degree above $\mathbf{0}'$ the jump of a (necessarily Δ_{n+1}) degree? Working with Δ_{n+1}-sets in the absence of $I\Sigma_n$ is always a challenge. We believe that the answer is in the negative. Chong, Qian and Yang [3] have produced a saturated model of PA in which FJI is false.
2. Identify a theory in subsystems of second order arithmetic that is equivalent to the FJI. It is clear that FJI is a consequence of ACA_0 but not RCA_0 (since the latter has a model consisting only of recursive sets).
3. Very little is known about nonregular sets and their degrees. Most of the techniques developed have been for regular sets which are more tractable. It will be an interesting program to study nonregular sets and the degrees of their jumps.

   It is natural to ask if the results on r.e. degrees discussed in this paper extend beyond finite n. Specifically,

(4) Lachlan, Martin and Sacks hasve shown the following intermediate jump theorem: There is an r.e. degree $\mathbf{c}$ such that $\mathbf{0^{(n)}} < \mathbf{c^{(n)}} < \mathbf{0^{(n+1)}}$ for all $n \in \omega$. What is the theory T for which every model $\mathcal{M}$ has an r.e. set C such that, for each $\nu \in \mathcal{M}$, $\emptyset^{(\nu)} <_T C^{(\nu)} <_T \emptyset^{(\nu+1)}$? It is not difficult to see that PA has to be a subtheory of T. However, PA by itself is not sufficient. In fact, there is a model $\mathcal{M}$ of PA in which $\emptyset^{(\nu)}$ is defined only for ν belonging to a proper cut in the model. Slaman (private communication) has shown that the second order system ACA_0 does not prove the intermediate jump theorem.

(5) We make the following general remark. While our original objective was to study the proof-theoretic strength of low$_n$ r.e. degrees, in the process we were led to investigate the structure of definable degrees in models of fragments of PA. These models exhibit many special and interesting properties, from both the recursion- and model-theoretic points of view. We believe that a closer look at these structures will be mathematically rewarding.

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