THE METAMATHEMATICS OF STABLE RAMSEY'S THEOREM FOR PAIRS

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Abstract. We show that, over the base theory RCA_0, Stable Ramsey's Theorem for Pairs implies neither Ramsey's Theorem for Pairs nor Σ^0_2-induction.

1. Introduction

In this paper, we are motivated by the related questions "What are the implications between familiar infinitary mathematical principles?" and "What are the finitary consequences of these principles?" We consider principles, such as comprehension, compactness, measure or combinatorics, that assert the existence of sets of natural numbers, or similarly, real numbers. To give two examples, the compactness of the Cantor set says that every infinite subtree of the full binary tree has an infinite path and Ramsey's Theorem for Pairs says that for every partition of the pairs of natural numbers into finitely many pieces there is an infinite set all of whose pairs belong to the same piece. We want to precisely pose and answer questions such as "Does the infinite Ramsey Theorem for Pairs follow from the compactness of the Cantor set?" or "What consequences for the finite sets follow from the infinite Ramsey Theorem for Pairs?"

When we compare such existence principles P_1 and P_2, we investigate whether P_1 follows from P_2 by purely effective means. That is to say that any instance of P_1 can be verified by sets obtained by applications of P_2 and/or computation relative to sets already obtained. The comparison can be conducted directly by showing that every collection of sets closed under both application of P_2 and relative computation is also closed under application of P_1. Alternately, the comparison can be conducted formally by showing that any instance of P_1 is provable from P_2, expressed axiomatically, over the base theory which asserts that the subsets of the natural numbers are closed under relative computation. However, if we fix the structure of the numbers and hence number-theoretic truth in advance, then we cannot measure the finitary, or number-theoretic, consequences of the principles being studied. Consequently, the axiomatic approach is necessary to address instances of our second motivating question.

2010 *Mathematics Subject Classification.* Primary 03B30, 03F35, 03D80; Secondary 05D10.

Key words and phrases. Reverse mathematics, RCA_0, Σ^0_2-bounding, Ramsey's Theorem for Pairs, Stable Ramsey's Theorem for Pairs.

Chong was partially supported by NUS grant WBS C146-000-025-00 for this research. Slaman's research was partially supported by NSF award DMS-1001551 and by the John Templeton Foundation. Yang's research was partially supported by NUS research grant WBS 146-000-159-112 and NSFC (No. 11171031). All three authors wish to thank Denis Hirschfeldt and the anonymous referee for their comments on the preliminary version of this paper.

The most familiar of all such principles, usually taken for granted, is the Comprehension Principle, which states that for any property of numbers Φ there is a set whose elements are exactly those numbers which satisfy Φ. When Φ is a relatively computable property of n, such as n's being an even number or n's being an even element of a given set A, the existence of the set of such n's is an instance of Recursive Comprehension. When comparing infinitary principles, we allow unrestricted use of Recursive Comprehension but explicitly track use of more complicated comprehension principles. More precisely, we work with models of second order arithmetic, $\mathfrak{M} = \langle M, \mathcal{S}, +, \times, 0, 1, \in\rangle$. These structures consist of two parts: $\langle M, +, \times, 0, 1\rangle$ is a version of the natural numbers with addition and multiplication; $\mathcal{S}$ is a version of the power set of the natural numbers, whose elements are subsets of M. When the arithmetic structure is understood, we will abbreviate our notation to $\mathfrak{M}$, and let the type of $\mathfrak{M}$ be clear from context. Our base theory RCA_0 is the mathematical system that incorporates the basic rules of the arithmetical operations, closure of sets under Turing reducibility and join, and mathematical induction for existential formulas, $I\Sigma^0_1$ (see Simpson, 2009). There are two canonical models of RCA_0: take the arithmetic part to be the natural numbers $\mathbb{N}$ with the structure of arithmetic, and let $\mathcal{S}$ either be the set of recursive subsets of $\mathbb{N}$ or be the set of all subsets of $\mathbb{N}$. Ultimately, we are attempting to understand the relationships between closure properties of $2^{\mathbb{N}}$, so we prefer the direct comparison when possible. Thus, models of second order arithmetic of the form $\langle\mathbb{N}, \mathcal{S}\rangle$, so-called ω-models, are particularly important.

Compactness is well understood in these terms and makes a good example. Let WKL_0 denote the principle that every infinite subtree of the full binary tree has an infinite path. First, consider WKL_0 from the infinitary perspective. There is an infinite recursive subtree of the full binary tree with no infinite recursive path, hence Recursive Comprehension is insufficient to prove WKL_0. Kleene (1943) showed that every such tree has an arithmetically definable infinite path and so there is a proof of WKL_0 that uses only Arithmetic Comprehension. In a sharpening of Kleene's theorem, Jockusch and Soare (1972) showed that every infinite subtree T of the full binary tree has an infinite path P such that P' is computable from $\emptyset'$, i.e. the Halting Problem relative to P is computable from the Halting Problem. It follows from this that there is a collection $\mathcal{S}$ of subsets of $\mathbb{N}$ such that $\mathcal{S}$ is closed under relative computability and under the existence of infinite paths through infinite binary trees but $\emptyset' \notin \mathcal{S}$. Thus, WKL_0 does not imply Arithmetic Comprehension.

For the finitary consequences of compactness, Harrington (see Simpson, 2009) adapted the Jockusch and Soare argument and showed that for any sentence ϕ in first order arithmetic, i.e. a finitary sentence, if ϕ follows from WKL_0 over RCA_0, then ϕ follows from RCA_0 alone, without appeal to compactness. (In fact, Harrington proved a substantially stronger theorem.) Thus, compactness does imply the existence of infinite sets beyond the computable ones but does not have any number theoretic consequences that go beyond those of Recursive Comprehension.

In this paper, we look at infinitary combinatorics, where the picture is much less clear, and we resolve two questions about the strength of Stable Ramsey's Theorem for Pairs. As stated above, Ramsey's Theorem for Pairs states that if f is a coloring of the set of pairs of natural numbers by two colors, then there is an infinite set H all of whose pairs of elements have the same color under f. Such an H is said to be f-homogeneous. Closely related to Ramsey's Theorem for Pairs, and intuitively a more controlled coloring scheme, is Stable Ramsey's Theorem for Pairs, which asserts the existence of an infinite f-homogeneous set for stable colorings: i.e. those f's such that for every x, all but finitely many y's are assigned the same color by f.

We let RT^2_2 be the formal assertion of Ramsey's Theorem for Pairs and let SRT^2_2 be the assertion restricted to stable colorings. Both can be expressed in the language of second order arithmetic. An early recursion theoretic theorem of Jockusch (1972) states that there is a recursive coloring of pairs with no infinite homogeneous set recursive in the halting set $\emptyset'$, or equivalently with no infinite homogeneous set that is Δ^0_2-definable. In particular, this coloring has no infinite recursive homogeneous set, so Jockusch's Theorem implies the earlier theorem of Specker (1971) that $RCA_0 \nvdash RT^2_2$. Though stable colorings do have Δ^0_2 infinite homogeneous sets, another recursion theoretic argument shows that there is a recursive stable coloring with no infinite recursive homogeneous set, so the stronger $RCA_0 \nvdash SRT^2_2$ also holds.

The strength of these two combinatorial principles, RT^2_2 and SRT^2_2, has been a subject of considerable interest. Strengthening Hirst (1987) for RT^2_2, Cholak, Jockusch, and Slaman (2001) showed that SRT^2_2 implies the Σ^0_2-bounding principle, $B\Sigma^0_2$, an induction scheme equivalent to Δ^0_2-induction (see Slaman, 2004), whose strength is known to lie strictly between Σ^0_1 and Σ^0_2-induction (see Paris and Kirby, 1978). It is also shown in Cholak et al. (2001) that RT^2_2 is Π^1_1-conservative over RCA_0+ the Σ^0_2-induction scheme $I\Sigma^0_2$, i.e. any Π^1_1-statement that is provable in $RT^2_2 + RCA_0 + I\Sigma^0_2$ is already provable in the system $RCA_0 + I\Sigma^0_2$. It follows immediately that any subsystem of $RT^2_2 + RCA_0 + I\Sigma^0_2$ (such as replacing RT^2_2 by SRT^2_2) is Π^1_1-conservative over $RCA_0 + I\Sigma^0_2$.

Three problems relating to RT^2_2 and SRT^2_2 are of particular interest: (1) whether over RCA_0, RT^2_2 is strictly stronger than SRT^2_2; (2) whether RT^2_2 or even SRT^2_2 proves $I\Sigma^0_2$, given that they already imply $B\Sigma^0_2$; and (3) whether RT^2_2, or even SRT^2_2, is Π^1_1-conservative over $RCA_0 + B\Sigma^0_2$. Of course, a positive answer to (3) would provide a negative answer to (2).

It has been generally believed that RT^2_2 is stronger than SRT^2_2, and the approach to establishing this as fact has been to look for a collection of subsets of $\mathbb{N}$ satisfying Ramsey's Theorem for Pairs for stable colorings and not for general ones. Historically, using ω-models to study Ramsey type problems has been fruitful, as witnessed by further work of Jockusch (1972) which, when cast in the language of subsystems of second-order arithmetic, shows that Ramsey's Theorem for triples implies arithmetic comprehension, the results presented in Cholak et al. (2001), Seetapun's theorem (see Seetapun and Slaman, 1995) separating RT^2_2 from Arithmetic Comprehension, and recent work of Liu (2012) showing WKL_0 to be independent of RT^2_2. However, the search for an ω-model separating RT^2_2 from SRT^2_2 has been unsuccessful.

The most direct approach to separating SRT^2_2 was that suggested by Cholak et al. (2001). SRT^2_2 is equivalent to the condition that, for every Δ^0_2-predicate P on the numbers, there is an infinite set G such that either all of the elements of G satisfy P or none of the elements of G satisfy P.[1] The suggestion was that if for

[1] The equivalence requires $B\Sigma^0_2$ in the proof. However it is known that each of the statements implies $B\Sigma^0_2$. Hence such a condition does not impose additional assumption to the base theory (see Chong, Lempp, and Yang, 2010).

every Δ^0_2-predicate there were such a set G which is also low (i.e. $G' = \emptyset'$), then by an iterative argument one could produce an ω-model of SRT^2_2 in which every set was low, and hence Δ^0_2. By the result of Jockusch (1972) mentioned above, this model would not satisfy RT^2_2. However, this approach was ruled out by Downey, Hirschfeldt, Lempp, and Solomon (2001), who exhibited a Δ^0_2 predicate for which there is no appropriate G that is also low.

Here, we exhibit a model $\mathfrak{M}$ of $RCA_0 + B\Sigma^0_2 + \neg I\Sigma_2$, hence not an ω-model, that is a model of SRT^2_2 but not RT^2_2. Thus, we have a positive answer to the first question and partial negative answer to the second. While Downey et al. (2001) demonstrated an insurmountable obstruction to the low-set proposal in the context of ω-models, quite the contrary is true in the realm of nonstandard models. We are able to make use of the customized features of its first-order part and bring the original proposal to fruition in $\mathfrak{M}$: all of the sets in the $\mathcal{S}$ of $\mathfrak{M}$ are low in the sense of $\mathfrak{M}$. The existence of $\mathfrak{M}$ is a prima facie demonstration that Stable Ramsey's Theorem for Pairs does not imply Σ^0_2-induction over the base theory RCA_0. Finally, by observing that Jockusch's theorem is provable in $RCA_0 + B\Sigma^0_2$, we conclude that $\mathfrak{M}$ is not a model of RT^2_2.

The paper is organized as follows. In Section 2, we review the basic facts about subsystems of first and second order arithmetic, and state the main results. In Section 3, we construct the first order model $\mathfrak{M}_0$. In Section 4, we show how to solve the one-step problem, given a Δ^0_2-predicate P there is a low set G either contained in or disjoint from P. In Section 5, we construct the collection of subsets of M_0 used to satisfy SRT^2_2. This is where we establish the results already mentioned. We also extend the method to show that $SRT^2_2 + WKL_0 \not\vdash RT^2_2$, so the assertion that $2^{\mathbb{N}}$ is compact does not strengthen SRT^2_2 sufficiently to prove RT^2_2. We raise some questions in Section 6.

2. Subsystems of Arithmetic

We recall some basic facts and definitions in subsystems of first and second order arithmetic. Σ^0_n and Σ^1_n-formulas are defined as usual. Unless indicated otherwise, all formulas are allowed to mention parameters. All first order variables and parameters are interpreted as natural numbers (in a given model of a subsystem), and all second order variables and parameters are interpreted as subsets of the set of natural numbers. A reference for basic facts about the arithmetical and analytical hierarchies is Rogers (1987).

2.1. **First Order Arithmetic.** Let P^- denote the standard Peano axioms without mathematical induction. For $n \geq 0$, let $I\Sigma^0_n$ denote the induction scheme for Σ^0_n-formulas. Suppose $\mathfrak{M} = \langle M, +, \times, 0, 1\rangle$ is a model of $P^- + I\Sigma^0_1$. A bounded set S in $\mathfrak{M}$ is $\mathfrak{M}$-*finite* if it is coded in $\mathfrak{M}$, i.e., there is an $a \in M$ which $\mathfrak{M}$ interprets as a Gödel number for a set with exactly the elements of S. It is known (Paris and Kirby (1978)) that $I\Sigma^0_n$ is equivalent to the assertion that every Σ^0_n-definable set has a least element. We will use this fact implicitly throughout the paper.

$B\Sigma^0_n$ denotes the scheme given by the universal closures of

$$(\forall x < a)(\exists y)\varphi(x, y) \to (\exists b)(\forall x < a)(\exists y < b)\varphi(x, y),$$

in which $\varphi(x, y)$ is a Σ^0_n-formula, possibly with other free variables. Intuitively, $B\Sigma^0_n$ asserts that every Σ^0_n-definable function with $\mathfrak{M}$-finite domain has $\mathfrak{M}$-bounded range. In Paris and Kirby (1978), it was also shown that for all $n \geq 1$,

$$\cdots \Rightarrow I\Sigma^0_{n+1} \Rightarrow B\Sigma^0_{n+1} \Rightarrow I\Sigma^0_n \Rightarrow B\Sigma^0_n \Rightarrow \cdots,$$

and that the implications are strict. Our interest here concerns the hierarchy up to level $n = 2$.

A *cut* $I \subset M$ is a set that is closed downwards and under the successor function. I is a Σ^0_n-cut if it is Σ^0_n-definable over $\mathfrak{M}$. The next proposition is well-known and we state it without proof.

Proposition 2.1. *If $\mathfrak{M} \models P^- + I\Sigma^0_1$, then $\mathfrak{M} \models I\Sigma^0_n$ if and only if every bounded Σ^0_n-set is $\mathfrak{M}$-finite. If $I\Sigma^0_n$ fails, then in $\mathfrak{M}$ there is a Σ^0_n-cut I and a Σ^0_n-definable function that maps I cofinally into $\mathfrak{M}$.*

We next turn our attention to sequences and trees. By a *sequence*, we mean an element of $M^{<M}$, as defined in $\mathfrak{M}$ by way of a standard Gödel numbering. We use $\sigma \prec \tau$ to mean σ is an initial segment of τ and use $\tau_0 * \tau_1$ to denote the concatenation of the two sequences in the indicated order. We refer to a subset of the numbers which appear in the range of τ simply as a subset of τ. A *tree* T is a subset of the $\mathfrak{M}$-finite sequences from $\mathfrak{M}$, such that T is closed under $\mathfrak{M}$-finite initial segments. T is *binary* or *increasing* if each sequence in T is binary or increasing, respectively. T is *recursively bounded* if there is a function f which is recursive in the sense of $\mathfrak{M}$ such that for all $s \in M$, there are at most $f(s)$ many elements in T of length s. Such trees will be important later when considered in the context of compactness arguments.

Sequences in $\mathfrak{M}$ are also used in connection with defining subsets of ω. We say that $X \subseteq \omega$ is *coded on ω in $\mathfrak{M}$* if there is a binary sequence $\sigma \in \mathfrak{M}$ such that for every $i \in \omega$, $i \in X$ if and only if $\sigma(i) = 1$. In this case, we say that σ is a code for X on ω. Nonstandard models of PA have an abundance of coded sets. In this paper we will work with a special model for which, among other things, the intersections of its definable sets with ω are coded on ω within it.

Finally, a set $X \subseteq M$ is *amenable* if its intersection with any $\mathfrak{M}$-finite set is $\mathfrak{M}$-finite. If $\mathfrak{M} \models B\Sigma^0_n$, then every X that is Δ^0_n-definable in $\mathfrak{M}$ (that is, both X and $M \setminus X$ are Σ^0_n-definable in $\mathfrak{M}$) is amenable.

2.2. **Second Order Arithmetic.** RCA_0 is the system consisting of P^-, $I\Sigma^0_1$ and the second order recursive comprehension scheme

$$(\forall x)[\varphi(x) \leftrightarrow \neg\psi(x)] \rightarrow (\exists X)(\forall x)[x \in X \leftrightarrow \varphi(x)],$$

where φ and ψ are Σ^0_1-formulas possibly with parameters (we refer to such formulas as Δ^0_1-formulas). Let $\mathfrak{M} = \langle M, \mathcal{S}, +, \times, 0, 1\rangle$ be a model of RCA_0.

There is a well-developed theory of computation for structures that satisfy RCA_0, possibly strengthened by either $B\Sigma^0_n$ or $I\Sigma^0_n$. In particular, one may define notions of computability and Turing reducibility over $\mathfrak{M}$. Thus, a set is recursively (computably) enumerable (r.e.) if and only if it is Σ^0_1-definable. A set is recursive (computable) if both it and its complement are recursively enumerable. If X and Y are subsets of M, then $X \leq_T Y$ ("X is *recursive in* Y" or "X is *Turing reducible to* Y") if there is an e such that for any $\mathfrak{M}$-finite o, there exist $\mathfrak{M}$-finite sets $P \subset Y$ and $N \subset \overline{Y}$ satisfying

$$o \subseteq X \leftrightarrow \langle o, 1, P, N\rangle \in \Phi_e$$

and

$$o \subseteq \overline{X} \leftrightarrow \langle o, 0, P, N\rangle \in \Phi_e,$$

where Φ_e is the eth r.e. set of quadruples in $\mathfrak{M}$'s enumeration of such sets. Two subsets of M (note that it is not required that they belong to $\mathcal{S}$) have the same Turing degree if each is reducible to the other. If $n \geq 1$ and $\mathfrak{M} \models B\Sigma^0_n$, then as in classical recursion theory there is a complete Σ^0_i-set $\emptyset^{(i)}$ for $1 \leq i < n$, and Post's Theorem holds: $X \subset M$ is Δ^0_{i+1} if and only if $X \leq_T \emptyset^{(i)}$. A set in $\mathfrak{M}$ is *low* if its Σ^0_1-theory (otherwise called its jump) is recursive in $\emptyset'$. From the point of view of recursion theory, a structure $\mathfrak{M}$ is a model of RCA_0 if and only if $\mathcal{S}$ is closed under Turing reducibility and join and $\mathfrak{M}$ satisfies $P^- + I\Sigma^0_1$.

We include set variables $\check{G}$ and $\check{G}_i$, where $i < \omega$, in the language of second order arithmetic which will be used to denote the generic homogeneous sets to be constructed. We let $\psi(\check{G})$ denote a Σ^0_1-formula of the form $\exists s\varphi(s, \check{G})$ where φ is a bounded formula possibly with first and second order parameters. We adopt the notational convention that the syntactic relationship between ψ and φ will always be as shown above and will be assumed without further mention. We will often not distinguish between a set and its characteristic function unless there is possibility of confusion. If $\psi(\check{G})$ is a Σ^0_1-formula and o is an $\mathfrak{M}$-finite set, then we adopt the convention that $\mathfrak{M} \models \psi(o)$ (or "$\psi(o)$ holds") means $(\exists s \leq \max o)\varphi(s, o)$ is true in $\mathfrak{M}$. If $G \subset M$, then $\mathfrak{M}[G]$ is the structure having the same first order universe M, and containing G as well as all the subsets of M recursive in G.

Let $\mathfrak{M} \models RCA_0$. We list two combinatorial principles which are central to the subject matter of this paper. The first is D^2_2 (the second, WKL_0, will be introduced subsequently):

- D^2_2: Every Δ^0_2-set or its complement contains an infinite subset.

As mentioned earlier, D^2_2 is equivalent to SRT^2_2 over RCA_0. The main technical theorem we will establish is the following:

Theorem 2.2 (Main Theorem)**.** *There is a model $\mathfrak{M} = \langle M, \mathcal{S}, +, \times, 0, 1, \in\rangle$ of $RCA_0 + B\Sigma^0_2$ but not $I\Sigma^0_2$ such that every $G \in \mathcal{S}$ is low and $\mathfrak{M} \models D^2_2$.*

Corollary 2.3. *The statement "There is a Δ^0_2-set with no infinite low subset contained in it or its complement" is not provable in $P^- + B\Sigma^0_2$.*

The results in Jockusch (1972), appropriately adapted to the setting of second order arithmetic, yields Corollary 2.5 from Theorem 2.2:

Proposition 2.4. *Let $\mathfrak{M} = \langle M, \mathcal{S}\rangle \models RCA_0 + B\Sigma^0_2$ and $X \in \mathcal{S}$. There is an X-recursive two coloring of pairs with no X'-recursive infinite homogeneous set in $\mathfrak{M}$.*

Proof. We repeat here the argument for Theorem 3.1 of Jockusch (1972). Define an X-recursive two-coloring r and b (for *red* and *blue* respectively) of pairs of numbers in M for which no $\Delta^0_2(X)$-set is homogeneous.

Since $\mathfrak{M} \models B\Sigma^0_2$, every $\Delta^0_2(X)$-set is amenable. Furthermore, A is $\Delta^0_2(X)$ if and only if $A \leq_T X'$. Now there is a uniformly recursive collection of X-recursive functions f_e such that $\lim_s f_e(s, x) = A_e(x)$ for all x if and only if A_e is $\Delta^0_2(X)$. Furthermore, if A_e is such a set, then by $B\Sigma^0_2$ again, for each a, the "$\Delta^0_2(X)$ convergence of f_e to A_e" is tame, i.e. there is an s_a such that for all $s \geq s_a$, $f_e(s, x) = A_e(x)$ whenever $x \leq a$. For each e and s, let $D_e[s]$ be the set with $2e + 2$

numbers that appear to be the first $2e+2$ members of A_e at stage s. There are two possible reasons for the guess to be wrong: The correct stage s has not yet been reached, or A_e has less than $2e+2$ elements. If A_e has at least $2e+2$ elements, then by the tameness of $\Delta^0_2(X)$-sets, a correct s_e exists such that $D_e[s] = D_e[s]$ for all $s \geq s_e$. Define the coloring C as follows: (i) At stage s, in increasing order of $e \leq s$, if $D_e[s]$ is not defined, skip to the next e. Otherwise, there must be at least two (least) numbers x and y in $D_e[s]$ such that no colors have been assigned to (x, s) and (y, s). Color one r and the other b; (ii) For all (x, s), $x \leq s$, not colored following the above scheme, let $C(x, s) = r$. This diagonalization procedure ensures that no $\Delta^0_2(X)$-set is homogeneous for C.

We note that no priority argument is involved and the coloring C requires only $B\Sigma^0_2$ for the desired conclusion to hold. □

Corollary 2.5. *SRT^2_2 does not imply RT^2_2.*

Corollary 2.6. *SRT^2_2 does not imply $I\Sigma^0_2$.*

Let T be a tree in $\mathfrak{M}$. A *path* on T is a maximal compatible set of strings in T. A Π^0_1-class is the collection of paths on a recursively bounded recursive tree T. Note that not all paths on T have to be in $\mathfrak{M}$. The next combinatorial principle is known to be independent of RT^2_2 (Liu (2012)).

- WKL_0 (Weak Kőnig's Lemma): If T is an infinite subtree of the full binary tree, then T contains an infinite path.

Theorem 2.7. *There is a model $\mathfrak{M}$ of $RCA_0 + SRT^2_2 + WKL_0 + B\Sigma^0_2$ in which RT^2_2 fails.*

Corollary 2.8. *$SRT^2_2 + WKL_0$ does not prove RT^2_2 over $RCA_0 + B\Sigma^0_2$.*

Definition 2.9. Given two models $\mathfrak{M}_0 = \langle M_0, \mathcal{S}_0\rangle$ and $\mathfrak{M} = \langle M, \mathcal{S}\rangle$ of RCA_0, we say that $\mathfrak{M}$ is an *M_0-extension of* $\mathfrak{M}_0$ if $M_0 = M$ and $\mathcal{S}_0 \subseteq \mathcal{S}$, i.e. only subsets of M_0 are added to form $\mathfrak{M}$.

In the next section, we exhibit a (first order) model $\mathfrak{M}_0 \models P^- + B\Sigma^0_2$ that satisfies a bounding principle called *BME*. By adding the recursive (in $\mathfrak{M}_0$) sets as a second order part, one can convert $\mathfrak{M}_0$ into a second order model which will again be called $\mathfrak{M}_0$. This $\mathfrak{M}_0$ is then a model of $RCA_0 + B\Sigma^0_2$. The models for Theorems 2.2 and 2.7 will be M_0-extensions of $\mathfrak{M}_0$.

3. The First Order Part of a Model of SRT^2_2

3.1. **A Σ_1-Reflecting Model.** We now describe the first order part of our model $\mathfrak{M}_0$ of SRT^2_2. As indicated in Proposition 3.1, $\mathfrak{M}_0$ has three major features which will be essential to what follows. The first is that it is a union of Σ_1-reflecting initial segments $(\mathfrak{I}_k : k \in \omega)$, such that each $\mathfrak{I}_k$ is a model of PA. The use of Σ_1-reflection has precedents in higher recursion theory. For example, in α-recursion theory one uses α-stable ordinals to bound existential quantifiers in Σ_1-formulas for which there is no *a priori* bound (see Sacks, 1990). We will see a similar application of Σ_1-reflection here. The second feature is that the failure of $I\Sigma^0_2$ in $\mathfrak{M}_0$ is realized in ω's being a Σ^0_2-cut. This gives not only the obvious conclusion that $\mathfrak{M}_0$ has definable cofinality ω, but also that SRT^2_2 does not imply $I\Sigma^0_2$. The third feature of $\mathfrak{M}_0$ is that it is arithmetically saturated, which we will apply to produce parameters for controlling the complexity of the sets being constructed.

Proposition 3.1. *There is a countable model $\mathfrak{M}_0 = \langle M_0, +, \times, 0, 1\rangle$ of $P^- + B\Sigma^0_2$ with a Σ^0_2-function g with the following properties:*

(1) *$\mathfrak{M}_0$ is the union of a sequence of Σ_1-elementary end-extensions of models of PA:*
$$\mathcal{I}_0 \prec_{\Sigma_1,e} \mathcal{I}_1 \prec_{\Sigma_1,e} \mathcal{I}_2 \prec_{\Sigma_1,e} \cdots \prec_{\Sigma_1,e} \mathfrak{M}_0$$
(2) *For each $i \in \omega$, $g(i) \in \mathcal{I}_i$, and for $i > 0$, $g(i) \notin \mathcal{I}_{i-1}$, and hence $\mathfrak{M}_0 \not\models I\Sigma^0_2$.*
(3) *Every $\mathfrak{M}_0$-arithmetical subset of ω is coded on ω.*

Proof. We will give a direct, though metamathematically inefficient, proof of the existence of the desired model.

We begin with an uncountable model $\mathcal{V}$ of set theory such that $\mathbb{N}^{\mathcal{V}}$, the natural numbers of $\mathcal{V}$, is nonstandard and such that every subset of ω is coded in $\mathcal{V}$ on ω. For example, $\mathcal{V}$ could be any ω_1-saturated model of a large fragment of *ZFC*. Fix b to be a nonstandard element of $\mathbb{N}^{\mathcal{V}}$.

Working in $\mathcal{V}$, our second step is to define a sequence of theories T_i. We will use $S_{\Pi^0_1}$ to indicate the set of Π^0_1 sentences with parameters true in a model of *PA* as determined by that model's definition of Π^0_1-satisfaction. For a definable theory T, $CON(T)$ is the assertion that T is consistent, expressed in the usual way using Gödel numbering. Let
$$T_0 = PA + S_{\Pi^0_1},$$
$$T_{i+1} = T_i + CON(T_i).$$

Our third step is to define a length b sequence of Σ_1-elementary end-extensions:
$$\mathbb{N}^{\mathcal{V}} = \mathcal{I}_0 \prec_{\Sigma_1,e} \mathcal{I}_1 \prec_{\Sigma_1,e} \mathcal{I}_2 \prec_{\Sigma_1,e} \mathcal{I}_3 \prec_{\Sigma_1,e} \cdots \prec_{\Sigma_1,e} \mathcal{I}_b.$$
In $\mathcal{V}$, we will appear to be constructing a finite Σ_1-elementary sequence of models by injecting inconsistencies (see below) while unfolding the iterated consistency statements used to define the theories T_i, for $i < b$. We begin by setting $\mathcal{I}_0 = \mathbb{N}^{\mathcal{V}}$ and noting that $\mathcal{I}_0$ satisfies $PA + CON(T_{b-1})$, since it is the standard model of arithmetic in $\mathcal{V}$. Thus, from $\mathcal{I}_0$'s perspective, T_{b-1} is consistent. However, by the Gödel second incompleteness theorem, which is provable in *PA* and thereby holds in $\mathcal{I}_0$, $\mathcal{I}_0$ satisfies that T_{b-1} cannot prove $CON(T_{b-1})$. Finally, by the arithmetical completeness theorem, there is an $\mathcal{I}_1$ such that $\mathcal{I}_0 \prec_{\Sigma_1,e} \mathcal{I}_1$,
$$\mathcal{I}_1 \models T_{b-1} + \neg CON(T_{b-1}),$$
and $\mathcal{I}_1$ is definable in $\mathcal{I}_0$. (McAloon (1978) gives more details on applications of the arithmetical completeness theorem.) We could even take $\mathcal{I}_1$ to be defined in $\mathcal{I}_0$ as a low predicate relative to $0'$. Note, by the definition of T_{b-1}, $\mathcal{I}_1 \models PA + CON(T_{b-2})$.

Working in $\mathcal{V}$, we can iterate this step b many times. For $0 < i < b$, we define $\mathcal{I}_{i+1}$ to be an end-extension of $\mathcal{I}_i$ such that $\mathcal{I}_{i+1}$ is a definable low in $0'$ model in $\mathcal{I}_i$ and
$$\mathcal{I}_{i+1} \models T_{b-(i+1)} + \neg CON(T_{b-(i+1)}).$$
The only difference between the initial and the general inductive step is that we are required to find an end-extension of $\mathcal{I}_i$, which $\mathcal{V}$ sees to be nonstandard. It is for this reason that we invoke the fact that $\mathcal{I}_i \models PA + CON(T_{b-(i+1)})$ and then apply the Gödel second incompleteness theorem (as a consequence of *PA*) and the arithmetical completeness theorem in $\mathcal{I}_i$ to obtain an $\mathcal{I}_i$-definable model of $T_{b-(i+1)} + \neg CON(T_{b-(i+1)})$.

Now, we prove the proposition. For each $n \in \omega$, define $I_n^{\mathcal{Y}}$ to be the universe of $\mathcal{I}_n$ and define $M_0^{\mathcal{Y}} = \cup_{n\in\omega} I_n^{\mathcal{Y}}$. Define $g(0) = 0$. For $n > 0$, define $g(n)$ to be the least number coding in $\mathfrak{M}_0^{\mathcal{Y}}$ a proof of $\neg CON(T_{b-n})$ from the axioms of T_{b-n}. Whether a formula belongs to T_{b-n} is a Π^0_1-property of that formula as evaluated in $\mathcal{I}_n$. Moreover, since Π^0_1-properties are absolute between all the models being discussed, T_{b-n} is uniformly Π^0_1 in $\mathfrak{M}_0^{\mathcal{Y}}$. Hence, the function g is Σ^0_2 in $\mathfrak{M}_0^{\mathcal{Y}}$. Finally, since $\mathfrak{M}_0^{\mathcal{Y}} \prec_{\Sigma_1,e} \mathcal{I}_b$, $\mathfrak{M}_0^{\mathcal{Y}}$ is a model of $B\Sigma^0_2$ (see Kaye, 1991, Chapter 10). Thus, $\mathfrak{M}_0^{\mathcal{Y}}$, g, and the initial segments $\mathcal{I}_n^{\mathcal{Y}}$ satisfy the first two conditions of the proposition.

To finish, let $\mathfrak{M}_0$ be a countable substructure of $\mathfrak{M}_0^{\mathcal{Y}}$ such that the following conditions hold.

(1) $b \in \mathfrak{M}_0$.
(2) $\mathfrak{M}_0$ with predicates for the $I_n^{\mathcal{Y}} \cap \mathfrak{M}_0$ is an elementary substructure of $\mathfrak{M}_0^{\mathcal{Y}}$ with predicates for the $I_n^{\mathcal{Y}}$.
(3) Every $\mathfrak{M}_0$-arithmetical subset of ω is coded on ω.

We obtain $\mathfrak{M}_0$ by closing under the usual Skolem functions for first order elementarity and also under the additional Skolem function that for each definable predicate adds a parameter coding the restriction of that predicate to ω. We let $I_n = I_n^{\mathcal{Y}} \cap \mathfrak{M}_0$ and let g be defined in $\mathfrak{M}_0$ as in $\mathfrak{M}_0^{\mathcal{Y}}$.

Then $\mathfrak{M}_0$, g, and the I_n's satisfy the first two conditions of the proposition by elementarity. They satisfy the third condition of the proposition by construction. □

Notation 3.2. We use $\mathfrak{M}_0$, $\{\mathcal{I}_n : n < \omega\}$ and g henceforth to refer to the model, collection of cuts and function constructed in Proposition 3.1.

3.2. **Monotone Enumerations.** We will have two notational conventions in this subsection, to be interpreted in the model $\mathfrak{M}_0$. To motivate the discussion to follow, we give some intuition here, which is inevitably less than precise. There are known ways to construct an infinite homogeneous set for a given partition (coloring) by recursion, where each step of the recursion specifies finitely many elements of the homogeneous set together with an infinite set P from which the remaining elements of the homogenous set are to be chosen. However, we did not find this approach adequate for our application. Instead, we found it necessary specify a tree V of possible such sets P whose properties are further controlled during the construction by auxiliary trees $E(P)$. The situation is further complicated by the fact that the structures in which we are working may not contain these infinite paths as elements, so we are constrained to working only with indices for the trees and approximations for their elements.

(1) When written with no argument, V will denote a procedure to compute a recursively-bounded recursive tree. Then, $V(X)$ will denote the procedure applied relative to X to compute an X-recursively-bounded X-recursive tree. In the context of relativizing V, we use τ to denote a finite string. Then $V(\tau)$ will be the finite tree that can be computed from τ according to V. We follow the usual convention that if m is the maximum of the length of τ and its greatest element, then $V(\tau)$ is defined only for arguments less than m such that the evaluation of V relative to τ takes less than m steps and τ is queried only at arguments for which it is defined.
(2) When written with no argument, E will denote a procedure to recursively enumerate a finitely-branching enumerable tree. We will use σ to denote

a finite string in the context of relativizing E, with $E(X)$ and $E(\sigma)$ interpreted as above.

(3) When clear from context, we will also use V or E to refer to the recursive or recursively enumerable trees defined by them.

Definition 3.3. We say that E is a *monotone enumeration* if and only if the following conditions apply to its stage-by-stage behavior.

(1) The empty sequence is enumerated by E during stage 0.
(2) Only $\mathfrak{M}$-finitely many sequences are enumerated by E during any stage.
(3) Suppose that τ is enumerated by E during stage s and let τ_0 be the longest initial segment of τ that had been enumerated by E at a stage earlier than s. Then,
    (i) τ_0 had no extensions enumerated by E prior to stage s and
    (ii) all the sequences enumerated by E during stage s are extensions of τ_0.

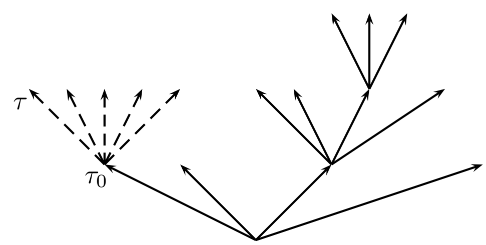

Figure 1. Monotone Enumeration

Let $E[s]$ denote the set of sequences that have been enumerated by E by the end of stage s. Condition (3) above asserts that if $E[s+1]\setminus E[s]$ is not empty, then there is a maximal path τ_0 in $E[s]$ such that for every element τ of $E[s+1]\setminus E[s]$, $\tau_0 \prec \tau$, i.e. $\tau = \tau_0 * \tau_1$, for some nontrivial sequence τ_1. Here, $\prec$ and $*$ indicate initial segment and concatenation according to the conventions of Section 2.1. We display this situation in Figure 1, where the nodes enumerated by E during that stage are indicated by dashed lines. Note that by speeding up the enumeration, one can extend more than one leaf in a single stage, as we do in the proof of Proposition 3.6 below. The key point of a monotone enumeration is that one never directly extends any node which is not a leaf.

Similarly, we can define E's being a monotone enumeration relative to a predicate X, or even relative to all strings σ in a recursive tree V.

Definition 3.4. Suppose that E is a monotone enumeration.

(1) For an element τ enumerated by E, let k be the number of stages in the enumeration by E during which τ or an initial segment of τ is enumerated. Let $(\tau_i : i < k)$ be the stage-by-stage sequence of the maximal initial segments of τ associated with those stages.
(2) We say that E's enumeration is bounded by b if for each τ in E, its stage-by-stage sequence has length less than or equal to b.

Proposition 3.5. *Suppose that $\mathfrak{M} \models P^- + I\Sigma^0_2$ and that E is a monotone enumeration procedure in $\mathfrak{M}$ which is bounded by b. Then $\mathfrak{M} \models$ "E is finite".*

Proof. Work in $\mathfrak{M}$ to show by induction on ℓ that there are only $\mathfrak{M}$-finitely many τ such that the stage-by-stage sequence associated with E's enumeration of τ has length ℓ. □

By Proposition 3.5, $I\Sigma^0_2$ is sufficient to show that bounded monotone enumerations are $\mathfrak{M}$-finite. However, that is not the case for $B\Sigma^0_2$.

Proposition 3.6. *There is a model $\mathfrak{M} \models P^- + B\Sigma^0_2$ such that in $\mathfrak{M}$ there is a monotone enumeration E which is bounded by b, but yet the enumeration of E is not finite in $\mathfrak{M}$.*

Proof. Let $\mathfrak{N}$ be a nonstandard model of PA and let b be a nonstandard element of $\mathfrak{N}$. To fix some notation, let $\emptyset'$ denote the universal Σ^0_1-predicate in $\mathfrak{N}$ and let $\emptyset'[s]$ denote the recursive approximation to it given by bounding the existential quantifier in its definition by s.

Define the function $t : \mathfrak{N} \to \mathfrak{N}$ by recursion: Let $t(0) = 0$, let $t(1) = b$ and let $t(x + 1)$ be the least s such that $\emptyset'[s] \restriction t(x) = \emptyset' \restriction t(x)$. Define $\mathfrak{M}$ to be the substructure of $\mathfrak{N}$ with elements given by

$$x \in \mathfrak{M} \iff (\exists n \in \omega)\mathfrak{N} \models x < t(n).$$

Then, $\mathfrak{M}$ is a Σ_1-substructure of $\mathfrak{N}$. Further, since $\mathfrak{N}$ is an end-extension of $\mathfrak{M}$, $\mathfrak{M}$ satisfies $B\Sigma^0_2$, an implication that we also noted in the construction of $\mathfrak{M}_0$.

Now, we give a monotone enumeration in $\mathfrak{M}$ of a tree whose height is bounded by b but which is not $\mathfrak{M}$ finite. Again, we let $E[s]$ denote the set of sequences that have been enumerated by E by the end of stage s. In our enumeration at stage $s+1$, we will enumerate $\mathfrak{M}$-finitely many extensions of $\mathfrak{M}$-finitely many terminal nodes in $E[s]$ observe that it is possible to enumerate the same tree more slowly so that at most one terminal node is extended during each stage. At stage 0, E enumerates the empty sequence. So, $E[0]$ is the singleton set consisting of the empty sequence. At stage 1, E enumerates all the sequences $\langle x\rangle$ of length one such that $x \leq b$. At stage $s + 1$, we let m be the largest number that appears in any sequence in $E[s]$. If there is an $x \leq m$ such that $x \in \emptyset'[s + 1] \setminus \emptyset'[s]$, then for each such x, for each sequence $\tau \in E[s]$ such that x is the last element of τ, τ is maximal in $E[s]$ and τ has length less than b (if any), and for each $y \leq s + 1$, E enumerates $\tau * \langle y\rangle$. That concludes stage $s + 1$.

By construction, our enumeration of E is monotone. It remains to show that the enumeration by E is not finite in $\mathfrak{M}$. For this, note that for each $n \in \omega$, if n is greater than 0, then $t(n)$ appears on the nth level of E. We prove this by induction on n. It is true for $t(1)$, since $t(1)$ is b and the first level enumerated by E consists of all numbers less than or equal to b. Assume that E enumerates $t(n)$ on level n. When E enumerates the sequence $\tau_0 * \langle t(n)\rangle$ of length n, for each x less than $t(n)$ E also enumerates the sequence $\tau_0 * \langle x\rangle$. Now, $\mathfrak{M} \prec_{\Sigma_1} \mathfrak{N}$ and so the enumeration of $\emptyset' \restriction t(n)$ viewed within $\mathfrak{M}$ is completed exactly at stage $t(n + 1)$. Let x be an element less than $t(n)$ that is enumerated into $\emptyset'$ at stage $t(n + 1)$ and not before. The sequence $\tau_0 * \langle x\rangle$ will be a maximal element of $E[t(n+1)-1]$, since $x \notin E[t(n + 1) - 1]$, and of length n, which is less than b. By construction, E will enumerate $\tau_0 * \langle x\rangle * \langle t(n + 1)\rangle$ at stage $t(n + 1)$. □

Ultimately, we will need to consider iterated applications of instances of the Stable Ramsey Theorem. In our construction, we will require the analogous iterated version of the above, which we now develop.

Definition 3.7. Suppose that V is the index for a recursively bounded recursive tree and suppose that E is a monotone enumeration procedure. For σ in the tree computed by V, say that σ is *E-expansionary* if in the enumeration of $E(\sigma)$ some new element is enumerated at stage $|\sigma|$. We say that a level ℓ in the tree computed by V is *E-expansionary* if there is an n such that ℓ is the least level in the tree computed by V at which every σ in that tree with $|\sigma| = \ell$ has at least n many E-expansionary initial segments.

Definition 3.8. A *k-iterated monotone enumeration* is a sequence $(V_i, E_i)_{1\leq i\leq k}$ with the following properties.

(1) Each V_i is an index for a relativized recursive recursively-bounded tree.
(2) Each E_i is an index for a monotone enumeration procedure.
(3) For each $1 \leq j \leq k$, if $\sigma \in V_j$ is E_j-expansionary, then for every new element τ enumerated in $E_j(\sigma)$, $V_{j+1}(\tau)$ is a proper E_{j+1}-expansionary extension of $V_{j+1}(\tau_0)$, where τ_0 is the longest initial segment of τ that had previously been enumerated in $E_j(\sigma)$, that is by a stage less than the length of σ.

Definition 3.9. A *k-path* of the k-iterated monotone enumeration $(V_i, E_i)_{1\leq i\leq k}$ is a sequence $(\sigma_i, \tau_i)_{1\leq i\leq k}$ such that $\sigma_1 \in V_1$ and τ_1 is a maximal sequence in $E_1(\sigma_1)$, and for each j with $1 < j \leq k$, σ_j is a maximal sequence in $V_j(\tau_{j-1})$ and τ_j is a maximal sequence in $E_j(\sigma_j)$.

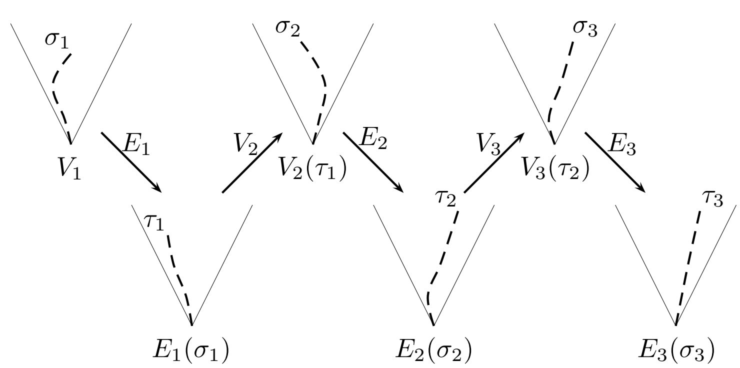

FIGURE 2. An example of k-path when $k = 3$

Figure 2 shows a 3-iterated monotone enumeration as realized by a particular 3-path.

Definition 3.10. (1) A k-iterated monotone enumeration is *b-bounded* if and only if for every sequence enumerated in $E_k(\sigma_k)$ by some k-path of the k-iterated enumeration, its stage-by-stage enumeration has length less than or equal to b.

(2) We say that $\mathfrak{M}$ satisfies *bounding for iterated monotone enumerations* (*BME*) if and only if for every $k \in \omega$, every b in $\mathfrak{M}$ and every b-bounded k-iterated monotone enumeration, there are only boundedly many E_1-expansionary levels in V_1.

(3) If we restrict our attention to k-iterated monotone enumerations, we say that $\mathfrak{M}$ satisfies BME_k.

Proposition 3.11. *$\mathfrak{M}_0$ satisfies BME.*

Proof. Suppose that $(V_i, E_i)_{i \leq k}$ is a k-iterated monotone enumeration and that in $\mathfrak{M}_0$ there are unboundedly many E_1-expansionary levels in V_1. We must show that there is no b which bounds the lengths of the stage-by-stage enumerations of elements of E_k on all k-paths of $(V_i, E_i)_{i \leq k}$.

Fix n so that b and the other parameters defining $(V_i, E_i)_{i \leq k}$ belong to $\mathfrak{I}_n$. Since $\mathfrak{I}_n \prec_{\Sigma_1, e} \mathfrak{M}_0$ and there are unboundedly many E_1-expansionary levels in V_1,

$$\mathfrak{I}_n \models \text{There are unboundedly many } E_1\text{-expansionary levels in } V_1.$$

In particular, since $\mathfrak{I}_n$ is a model of *PA*,

$$\mathfrak{I}_n \models V_1 \text{ is a recursively bounded infinite tree.}$$

Again, since $\mathfrak{I}_n \models PA$, let X_1 be an $\mathfrak{I}_n$-definable infinite path in V_1. Note that $\mathfrak{I}_n[X_1]$, obtained by adding X_1 as an additional predicate to $\mathfrak{I}_n$, still satisfies *PA* relativized to X_1. Since E_1 is a monotone enumeration and there are unboundedly many E_1-expansionary levels in V_1,

$$\mathfrak{I}_n[X_1] \models E_1(X_1) \text{ is a finitely branching unbounded tree.}$$

Now, we can let Y_1 be an $\mathfrak{I}_n[X_1]$-definable infinite path in $E_1(X_1)$, and note that $\mathfrak{I}_n[X_1, Y_1]$ satisfies *PA* relative to (X_1, Y_1). Further, because each sequence τ enumerated in E_1 exhibits a new E_2-expansionary level in $V_2(\tau)$,

$$\mathfrak{I}_n[X_1, Y_1] \models (V_i, E_i)_{1 \leq i \leq k} \text{ is a } (k-1)\text{-iterated monotone enumeration.}$$

By a k-length recursion, there is an $\mathfrak{I}_n$-definable sequence $(X_1, Y_1, \ldots, X_k, Y_k)$ extending (X_1, Y_1) such that for each i, X_i is an infinite path in $V_{i-1}(Y_{i-1})$ and Y_i is an infinite path in $E_i(X_i)$. Consequently, the stage-by-stage enumeration of the initial segments of Y_k in $\mathfrak{I}_n[X_1, Y_1, \ldots, X_k]$ is infinite, and there is no b which bounds the lengths of the stage-by-stage enumerations of elements of E_k on all k-paths of $(V_i, E_i)_{1 \leq i \leq k}$, as required. □

4. Low Homogeneous Sets

4.1. A Generic Instance of SRT^2_2.

Let $\mathfrak{M}_0$ be the model constructed in Proposition 3.1. This section is devoted to a proof of the following theorem.

Theorem 4.1. *Suppose that A is Δ^0_2. There is a pair of sets (G_r, G_b) with the following properties.*

(i) $G_r \subseteq A$ and $G_b \subseteq \overline{A}$.

(ii) At least one of G_r or G_b has unboundedly many elements in $\mathfrak{M}_0$. Call that set G.

(iii) G is low in $\mathfrak{M}_0$. Consequently, $\mathfrak{M}_0[G]$ satisfies $B\Sigma^0_2$.

Given a set A, we refer to the numbers in A and in $\overline{A}$ as *red* and *blue,* respectively. We first describe a way to select a homogeneous set (namely, a subset of A or $\overline{A}$) which decides one Σ^0_1-formula ψ (meaning to make either ψ or $\neg\psi$ true in the structure $\mathfrak{M}_0[G]$). The approach derives its inspiration from Seetapun and Slaman (1995) and is central to the techniques developed in this paper. Two key notions—that of *Seetapun disjunction* (to force a Σ^0_1-formula, see Definition 4.2) and that of *U-tree* (to force the negation of a Σ^0_1-formula, see Case 1 of the construction in §4.3)—will be introduced for this purpose.

We pause to give some intuition of the construction. We are building piece by piece finite initial segments of red and blue sets (at least one of which will turn out to be infinite and so be the desired homogenous set) along with a tree of "acceptable pools" of numbers. The finite parts are used to realize existential sentences and the tree is used to universal sentences. During the construction, the finite parts may be increased and the tree may be trimmed or thinned. We have two complications: first, we must leave both red and blue options open and accept that the final outcome (the choice of color) becomes clear only at the end of the construction; second, we must consider all subsets specified by a node on the tree, instead of the node on the tree. After analyzing the situation for a single Σ^0_1-formula, we will move to handling an $\mathfrak{M}_0$-finite set of formulas, leading to the definition of the notion of forcing in Definition 4.7, and then construct the desired low homogeneous set stated in Theorem 4.1.

We will generalize from the notion of a Seetapun disjunction to that of an exit tree, which is defined by a stage-by-stage enumeration. The enumeration of an exit tree is the origin of the abstract notion of a k-iterated monotone enumeration introduced in §3.2, and is key to our proof. Similarly, the notion of a U-tree used extensively in §4 and §5 is a concrete realization of the recursively bounded recursive tree V in §3.2. Note also that the construction in this section only requires the simplest version of the bounded monotone enumeration principle, namely BME_1. The k-iterated version is required in §5, where we will implement a scheme to perform iterations of a more complex construction in order to also preserve BME in the generic extension.

4.2. **Seetapun Disjunction for a Single Σ^0_1-formula.** We begin with some terminology. We will refer to a recursive sequence of $\mathfrak{M}_0$-finite sets $\vec{o}$ as a sequence of *blobs* if for each s less than the length of the sequence, $\max o_s < \min o_{s+1}$. Let $\vec{o}$ be an $\mathfrak{M}_0$-finite sequence of blobs, say of length h. Consider the set of all choice functions σ with domain h such that $\sigma(s) \in o_s$, together with their initial segments $\sigma \restriction h'$ for $h' < h$. By regarding them as strings and adding the empty string as root, the collection may be viewed naturally as a tree, called the *Seetapun tree* associated with $\vec{o}$.

Definition 4.2. Given a Σ^0_1-formula $\psi(\check{G})$, a *Seetapun disjunction* δ (or S-disjunction for short) for ψ is a pair $(\vec{o}, S)$, where $\vec{o}$ is a sequence of blobs of length $h > 0$ and S is the Seetapun tree associated with $\vec{o}$, such that:

(i) For each $s < h$, $\mathfrak{M}_0 \models \psi(o_s)$ in the sense of §2.2.
(ii) For each maximal branch τ of S, there exists an $\mathfrak{M}_0$-finite subset $\iota \subseteq \tau$ such that $\mathfrak{M}_0 \models \psi(\iota)$ (here again we identify a string with its range and

$\mathfrak{M}_0 \models \psi(\iota)$ is interpreted in the sense of §2.2. This convention will be followed throughout the paper). We refer to the set ι as a *thread* (in τ).

Figure 3 is an illustration of a Seetapun disjunction:

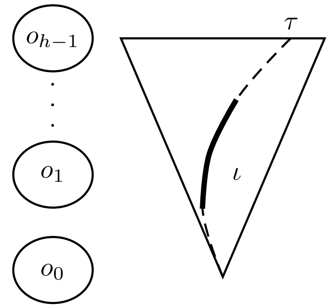

FIGURE 3. A Seetapun disjunction

Notice that an $\mathfrak{M}_0$-finite tree's being a Seetapun disjunction for a fixed Σ^0_1-formula ψ is a recursive property of that tree. The main feature of an S-disjunction is that it anticipates all possible amenable sets. Namely, if an S-disjunction for ψ is found, then for any amenable set A, ψ can be "forced" in a Σ^0_1-way by either a subset of A or a subset of $\overline{A}$. We isolate this fact in the following lemma, which also informally explains the meaning of a "disjunction" and the meaning of "forcing ψ":

Lemma 4.3. *Let $\psi(\check{G})$ be a Σ^0_1-formula and δ be an S-disjunction for ψ. Then for any amenable set A, one of the following applies:*

(i) There is an $\mathfrak{M}_0$-finite set $o \subseteq A$ such that $\psi(o)$ holds in $\mathfrak{M}_0$.
(ii) There is an $\mathfrak{M}_0$-finite set $\iota \subseteq \overline{A}$ such that $\psi(\iota)$ holds in $\mathfrak{M}_0$.

Proof. Assume that the S-disjunction δ is $(\vec{o}, S)$ with code c. For any amenable set A, let D and $\overline{D}$ be the $\mathfrak{M}_0$-finite sets $A \restriction (c+1)$ and $\overline{A} \restriction (c+1)$ respectively. If $D \supseteq o$ for some o in the sequence $\vec{o}$, then (i) holds. Otherwise, every o in $\vec{o}$ contains at least one element in $\overline{D}$. By induction for bounded formulas and the definition of δ, there exists a thread ι in some τ which is contained entirely in $\overline{D}$ such that $\psi(\iota)$ holds, which establishes (ii). □

Definition 4.4. We define the *exit taken by A from δ* to be the (canonically) least o or ι that satisfies Lemma 4.3.

4.3. **Forcing a Π^0_1-Formula.** Now assume that no S-disjunction for ψ exists. Then it is possible to "force $\neg\psi$" as follows. Begin with enumerating a sequence of blobs $\vec{o}$ by stages. (The sequence $\vec{o}$ of blobs may be either $\mathfrak{M}_0$-finite or $\mathfrak{M}_0$-infinite.)

At stage 0, the blob sequence $\vec{o}[0]$ is empty.

At stage $s+1$, suppose $\vec{o}[s]$ has been defined. Check if there exists an $\mathfrak{M}_0$-finite set o such that the code of o is less than $s+1$, $\min o >$ any number appearing in any blob in the sequence $\vec{o}[s]$ and $(\exists t < s+1)\varphi(t, o)$. If no such o exists, then let $\vec{o}[s+1] = \vec{o}[s]$; otherwise, take o^* to be the least (in a canonical order) such o. Define $\vec{o}[s+1] = \vec{o}[s] * o^*$ and proceed to the next stage.

The Seetapun tree associated with this blob sequence $\vec{o}$ which we defined previously may now be given a precise description as follows. Let $S[0] = \emptyset$. $S[s+1] = S[s] \cup \{\tau * x : \tau \in S[s] \wedge \ x \in o[s+1]\}$. Then $S = \bigcup_s S[s]$ is the Seetapun tree. Moreover S is $\mathfrak{M}_0$-finite if and only if $\vec{o}$ is $\mathfrak{M}_0$-finite. There are two possibilities to consider (corresponding to two possible ways of "forcing $\neg\psi$"):
Case 1. The Seetapun tree S is $\mathfrak{M}_0$-infinite. Then the *U-tree* for $\neg\psi$ which is defined as

$$U = \{\tau \in S : (\forall s < |\tau|)(\forall \iota \subseteq \tau)\neg\varphi(s, \iota)\}$$

is a recursively-bounded increasing infinite recursive tree due to the absence of a Seetapun disjunction. Then as long as one stays within U (meaning the numbers to be used at any stage in the rest of the construction are taken from one of its branches), $\neg\psi$ will always hold. We refer to this as *forcing $\neg\psi$ by thinning.*
Case 2. The Seetapun tree S is $\mathfrak{M}_0$-finite. Then by working with sets consisting only of numbers larger than (the code of) S, ψ will never be satisfied. Hence $\neg\psi$ is forced instead. We refer to this action as *forcing $\neg\psi$ by skipping.*

Notice that exactly how $\neg\psi$ is forced depends on whether the Seetapun tree S is $\mathfrak{M}_0$-finite or infinite, which is a two-quantifier question. In general, $\emptyset'$ is unable to answer this question. This is the reason that Seetapun's original argument could not produce low homogeneous sets. However, in $\mathfrak{M}_0$ we will exploit the presence of codes to reduce the complexity of the Π^0_2-question above by one quantifier. First though, we apply the *blocking method*, which is next discussed, to handle an $\mathfrak{M}_0$-finite block of Σ^0_1-formulas simultaneously.

4.4. **A Block of Requirements and Exit Trees.**

4.4.1. *Requirement Blocks.* Fix an enumeration $\{\psi_e(\check{G}) : e \in M_0\}$ of all Σ^0_1-formulas. Given an $\mathfrak{M}_0$-finite set B, we call the set of Σ^0_1-formulas $\{\psi_e : e \in B\}$ a *block* of formulas. We will identify a formula ψ_e with its index e and loosely say that ψ_e is in B when e is in B.

Given an $\mathfrak{M}_0$-finite set B of Σ^0_1-formulas, we first force in Σ^0_1-fashion as many formulas in B as possible using S-disjunctions. Each S-disjunction brings with it exits o and ι each of which forces at least one formula in B. Lemma 4.3 says that if A is amenable, then either there is an $o \subseteq A$ or an $\iota \subseteq \overline{A}$. Since different Δ^0_2-sets may take different exits, a situation which we cannot recursively decide, one assumes that each exit is a possible subset of A or $\overline{A}$, and use each exit as a *precondition* to search for a new S-disjunction that will force another formula in B.

This brings up the two main issues in this subsection. One is the organization of the exits as a tree, which we will call an *exit tree*; the other is the enumeration of Seetapun disjunctions using previously enumerated exits as preconditions. After clarifying these points, we will note that B's being $\mathfrak{M}_0$-finite implies that our enumeration is bounded, and we invoke *BME* to argue that our enumeration process eventually stops. When that happens, we will have completed the portion of forcing those formulas in B which can be decided in a Σ^0_1-way. The formulas in B not yet forced to be true by this stage will be forced negatively in a Π^0_1-fashion via a suitable recursively bounded recursive increasing tree.

We begin by introducing a modified version of the notion of a Seetapun disjunction.

Definition 4.5. Given two blocks B_r and B_b of Σ^0_1-formulas, and a pair of disjoint $\mathfrak{M}_0$-finite sets ρ and β, a *Seetapun disjunction* δ *for* (B_r, B_b) *with preconditions* (ρ, β) is a pair $(\vec{o}, S)$ as in Definition 4.2, such that:

(i) For each $s < h$, $\mathfrak{M}_0 \models \psi_e(\rho * o_s)$ for some $e \in B_r$.
(ii) For each maximal branch τ of S, there exists an $\mathfrak{M}_0$-finite subset $\iota \subseteq \tau$ such that $\mathfrak{M}_0 \models \psi_d(\beta * \iota)$ for some $d \in B_b$.

We use the letters ρ and β to suggest red and blue, respectively. Given $\varepsilon = (\rho, \beta)$, define two blocks $B_r(\varepsilon)$ and $B_b(\varepsilon)$ to be the set of formulas in B yet to be forced by ρ and β, respectively. In other words, $B_r(\varepsilon) = B \setminus \{e : \mathfrak{M}_0 \models \psi_e(\rho)\}$ and $B_b(\varepsilon) = B \setminus \{d : \mathfrak{M}_0 \models \psi_d(\beta)\}$. Lemma 4.6 is the generalization of Lemma 4.3 to S-disjunctions with preconditions. The proof is similar and is omitted.

Lemma 4.6. *Let $\varepsilon = (\rho, \beta)$ be a pair of disjoint $\mathfrak{M}_0$-finite sets. Let $\delta = (\vec{o}, S)$ be an S-disjunction for $(B_r(\varepsilon), B_b(\varepsilon))$ with the pair of preconditions (ρ, β). Let A be amenable such that $\rho \subseteq A$ and $\beta \subseteq \overline{A}$. Then one of the following applies:*

(i) There is an $o \in \vec{o}$ such that $\rho * o \subseteq A$ and $\psi_e(\rho * o)$ holds for some $e \in B_r(\varepsilon)$;
(ii) There is a $\tau \in S$ and a thread $\iota \subseteq \tau$ such that $\beta * \iota \subseteq \overline{A}$ and $\psi_d(\beta * \iota)$ holds for some $d \in B_b(\varepsilon)$.

4.4.2. *Exit Trees.* We now enumerate the *exit tree E for B* as follows.

At stage 0, set $E[0]$ to be the code of the empty set (as root of the exit tree). Begin the search for a Seetapun disjunction δ for (B, B) with the pair of preconditions $(\emptyset, \emptyset)$.

We pause to explain the intuitive idea behind this enumeration procedure and introduce some terminology. First we describe how the exit tree will look once the first S-disjunction δ is enumerated. Assume that the exits in δ consist of blobs $o_0, o_1, \ldots, o_{s_0-1}$ and threads $\iota_0, \iota_1, \ldots, \iota_{t_0-1}$ (in the case of an ι appearing in multiple τ's, we simply ignore the repetitions). The sets o_s $(0 \leq s < s_0)$ and ι_t $(0 \leq t < t_0)$ are represented by their codes denoted by ρ_s and β_t. Let a node ε (on the first level of the exit tree) be a pair of codes (ρ, β), where ρ or β (but not both) is the code of the empty set. As in the case of an S-disjunction for a single ψ, given an amenable set A, either A is a superset of some o_s or $\overline{A}$ is a superset of some ι_t. Thus A must exit from some $\varepsilon = (\rho, \beta)$. The first level of the exit tree E may be visualized as the diagram below,

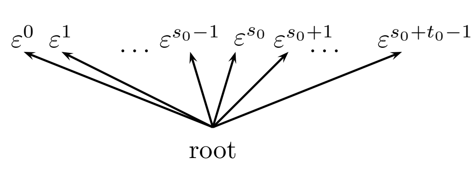

Figure 4. First level of an exit tree

where $\varepsilon^s = (\rho^s, \beta^s)$, ρ^s is the code of o_s for $s < s_0$ and the code of the empty set $\emptyset$ for $s \geq s_0$, and β^s is the code of $\emptyset$ for $s < s_0$ and the code of ι_{s-s_0} when $s \geq s_0$. The enumeration of future S-disjunction will have their own versions "over" each exit. In other words, future S-disjunctions will use (ρ, β) as a pair of preconditions. Therefore over certain preconditions, we may enumerate further S-disjunctions, and over others, we may enumerate no more. In general, we obtain a stack of Seetapun disjunctions which generates the exit tree. A typical node ε in an exit tree is of the form

$$(\langle \rho_1 * \rho_2 * \cdots * \rho_h \rangle, \langle \beta_1 * \beta_2 * \cdots * \beta_h \rangle),$$

where (ρ_1, β_1) is an exit taken from the first S-disjunction δ_1, followed by (ρ_2, β_2) which is an exit taken from the next S-disjunction δ_2 which uses (ρ_1, β_1) as precondition, and so on. Also for each i, one of ρ_i, β_i, but not both, may code the empty set $\emptyset$.

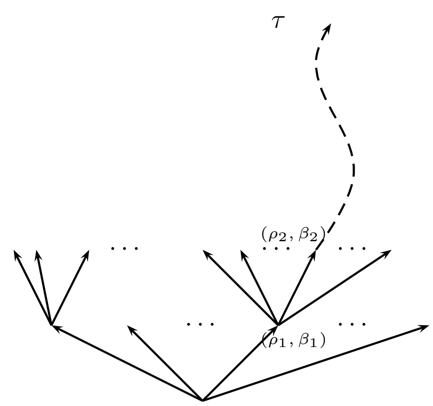

FIGURE 5. An example of an exit tree

For an exit ε of the above form, after discarding those that code $\emptyset$, we may assume that each ρ_s or β_t is the code of a blob o_s or a thread ι_t respectively. We let the sets *specified* by ε be $o_1 \cup o_2 \cup \cdots \cup o_h$ and $\iota_1 \cup \iota_2 \cup \cdots \cup \iota_h$, and denote them by ρ and β respectively, where we have abused the notations for the sake of simplicity.

We now return to the description of the enumeration of the exit tree E.

At stage $s+1$, suppose that the exit tree $E[s]$ is given. Following the canonical order of exits on the tree $E[s]$, check each maximal branch ε (with specified sets ρ and β) on $E[s]$ to see if there exists an S-disjunction δ for $(B_r(\varepsilon), B_b(\varepsilon))$ with (ρ, β) as a pair of preconditions, whose code is less than $s+1$. If no such δ is found, do nothing. Otherwise, without loss of generality, we may assume that only one S-disjunction is enumerated, say over ε. Concatenate with ε all of (the codes of) the exits of δ, and also concatenate with ε pairs of the form $(\rho, \emptyset)$ and $(\emptyset, \beta)$ where ρ and β are exits in δ. Let the resulting tree be $E[s+1]$. This ends the description of enumerating E.

The enumeration of E is clearly monotone. Since the height of the tree E is no more than $2|B|$ (each formula can be forced at most twice, once by red and once by blue), BME_1 implies that the enumeration process will stop at some stage s^*. In other words, after stage s^*, no new S-disjunctions for B on any ε of $E[s^*]$ will be enumerated.

Given an amenable set A, by Lemma 4.6 there is an exit $\varepsilon^* = (\rho^*, \beta^*)$ that A may take from this maximal stack of S-disjunctions. Then the formulas forced by ε^*

are exactly those that may be forced in a Σ^0_1-way using ε^* through the enumeration of E.

For the remaining formulas in B not yet forced by ε^*, we now show that their negations can be forced in a similar way as in §4.2.

First continue to enumerate the sequence of blobs *over* ε^*, i.e., those $\mathfrak{M}_0$-finite sets o with $\min o > \max \varepsilon^*$ such that $\mathfrak{M}_0 \models \psi_e(\rho^* * o)$ for some $e \in B_r(\varepsilon^*)$. Form the Seetapun tree S associated with this blob sequence $\vec{o}$. Then either by skipping the Seetapun tree S over ε^* (if it is $\mathfrak{M}_0$-finite) or by thinning through the U-tree U_b for $B_b(\varepsilon^*)$,

$$U_b = \{\tau \in S : (\forall s < |\tau|)(\forall \iota \subseteq \tau)(\forall d \in B_b(\varepsilon^*))\neg\varphi_d(s, \beta^* * \iota)\},$$

we force the remaining formulas in a Π^0_1-way. This leads us to the formal definition of a notion of forcing which we next introduce.

4.5. **Forcing Formalized.**

Definition 4.7. The partial order $P = \langle p, \leq\rangle$ of forcing conditions p satisfies:

(1) $p = (\varepsilon, U)$ where $\varepsilon = (\rho, \beta)$ is a pair of $\mathfrak{M}_0$-finite increasing strings of the same length and U is an $\mathfrak{M}_0$-infinite recursively bounded recursive increasing tree such that the maximum number appearing in either ρ or β is less than the minimum number appearing in U.

(2) We say that $q = (\varepsilon_q, U_q)$ is *stronger* than $p = (\varepsilon_p, U_p)$ (written $p \geq q$) if and only if

   (i) If $\varepsilon_p = (\rho_p, \beta_p)$ and $\varepsilon_q = (\rho_q, \beta_q)$, then $\rho_p \preceq \rho_q$ and $\beta_p \preceq \beta_q$;

   (ii) $(\forall \sigma \in U_q)(\exists \tau \in U_p)(\mathit{range}(\sigma) \subseteq \mathit{range}(\tau))$.

Similarly, we could work in $\mathfrak{M}_0[X]$ and relativize Definition 4.7 to X.

Given a Σ^0_1-formula ψ with a free set variable $\check{G}$ of the form $\exists s \varphi(s, \check{G})$, we say that p *red forces* ψ (written $p \Vdash_r \psi$) if

$$\mathfrak{M}_0 \models \exists s \leq \max(\rho_p)\varphi(s, \rho_p).$$

Define *blue forcing* similarly, except that ρ_p is replaced by β_p and $\Vdash_r$ by $\Vdash_b$. Also we say that p *red forces* $\neg\psi$ (written $p \Vdash_r \neg\psi$) if for all $\tau \in U_p$, for all $o \subseteq \tau$,

$$(*) \quad \mathfrak{M}_0 \models \forall s \leq \max(\tau)\neg\varphi(s, \rho_p * o).$$

Define $p \Vdash_b \neg\psi$ similarly, replacing ρ_p by β_p. [For consistency of notation with that for an S-disjunction, we use ι in place of o in $(*)$ above for $p \Vdash_b \neg\psi$.]

Let the Δ^0_2-set A be fixed and let $B_n = \{\psi_e(\check{G}) : e \leq g(n)\}$. The generic set G will be obtained from an ω-sequence of conditions $\langle p_n : n \in \omega\rangle$ which we now construct. The sequence will have the property that $p_{n+1} \leq p_n$, $p_n = \langle \varepsilon_n, U_n\rangle$, $\varepsilon_n = (\rho_n, \beta_n)$ with $\rho_n \subseteq A$ and $\beta_n \subseteq \overline{A}$. Furthermore, for each n, either (a) for each $\psi_e \in B_n$, p_n red forces ψ_e or its negation, or (b) for each $\psi_e \in B_n$, p_n blue forces ψ_e or its negation. The construction is carried out recursively in $\emptyset'$ modulo some parameters.

4.6. **Construction of a Generic Set.** Recursively in $\emptyset'$, we enumerate Σ^0_1-formulas in blocks $B_n = \{\psi_e : e < g(n)\}$ where $n \in \omega$. Let $B_{-1} = \emptyset$. The enumeration of the sets B_n relies on $\emptyset'$ to compute the sequence $\langle g(n) : n \in \omega\rangle$.

Let the initial recursively bounded recursive increasing tree U_{-1} be the tree version of the identity function, i.e., for any $\sigma \in U_{-1}$, $\sigma(i) = i$ for all $i < |\sigma|$. In particular, the (only) branch of U_{-1} has range M_0. Also let ε_{-1} be the pair of (codes of) empty strings and let the condition p_{-1} be $\langle \varepsilon_{-1}, U_{-1}\rangle$.

At stage $n+1$ ($n \geq -1$), suppose that we have defined conditions $p_i = \langle \varepsilon_i, U_i\rangle$ such that $\varepsilon_i = (\rho_i, \beta_i)$ with $\rho_i \subseteq A$ and $\beta_i \subseteq \overline{A}$, $p_{-1} \geq p_0 \geq \cdots \geq p_i \geq \cdots \geq p_n$ and either for each $\psi_e \in B_i$, $p_i \Vdash_r \psi_e$ or $p_i \Vdash_r \neg\psi_e$; or for each $\psi_e \in B_i$, $p_i \Vdash_b \psi_e$ or $p_i \Vdash_b \neg\psi_e$. Also, assume we have defined the sequence $\langle z(0), z(1), \ldots, z(n)\rangle$ where $z(i) = 0$ (for thinning) or > 0 (for skipping). We now consider the block B_{n+1}.

First apply the enumeration procedure E described in §4.4 along each $\mathfrak{M}_0$-finite branch of the tree U_n. Thus, instead of forming blobs by taking arbitrary numbers, we require the numbers to be drawn from (the range of) a node $\sigma \in U_n$. The procedure E will guarantee that $E(\sigma)$ will be an $\mathfrak{M}_0$-finite tree. If λ is an $\mathfrak{M}_0$-infinite path of U_n, then $E(\lambda)$ will be a tree which may or may not be $\mathfrak{M}_0$-finite. In this sense, what we did in §4.4 was to enumerate $E(M_0)$.

Now we are poised to apply BME_1. By construction, E specifies a monotone enumeration procedure. Since the height of any exit tree is uniformly bounded by $2|B_{n+1}| = 2g(n+1)$, there are only $\mathfrak{M}_0$-finitely many expansionary levels on U_n.

For $\sigma \in U_n$, let $\#_\delta\sigma$ be the number of S-disjunctions enumerated drawing numbers only from the range of σ, which is also equal to the number of nonterminal nodes in $E(\sigma)$. Here the subscript δ indicates the counting of Seetapun disjunctions, and the same applies to the superscript δ in T_a^δ below. Note that $\#_\delta\sigma$ is a particular instance of the number k in Definition 3.4 (1), when S-disjunctions are enumerated. For each $a \in M_0$, let T_a^δ be the subtree of U_n every node of which computes at most a many S-disjunctions. More precisely:

$$T_a^\delta = \{\sigma \in U_n : \#_\delta\sigma \leq a\}.$$

Then T_a^δ is a recursive subtree of U_n. Since there are only $\mathfrak{M}_0$-finitely many expansionary levels, it cannot be the case that for all $a \in M_0$, T_a^δ is $\mathfrak{M}_0$-finite. In other words, for some $a \in M_0$, T_a^δ is $\mathfrak{M}_0$-infinite.

Consider the set $\{a' : T_{a'}^\delta \text{ is } \mathfrak{M}_0\text{-finite}\}$, which is Σ_1^0. By assumption, it is bounded. Let a_δ be the largest such a' which can be found using $\emptyset'$. If the set is empty, let $a_\delta = -1$. Then $a_\delta + 1$ is the least number a such that T_a^δ is $\mathfrak{M}_0$-infinite.

Claim. There is a $\sigma_\delta \in T_{a_\delta+1}^\delta$ such that $\#_\delta\sigma_\delta = a_\delta + 1$ and σ_δ has $\mathfrak{M}_0$-infinitely many extensions in $T_{a_\delta+1}^\delta$.

Proof of Claim. Assume otherwise. Since $T_{a_\delta}^\delta$ is $\mathfrak{M}_0$-finite, there is an s such that every σ of length s has computed at least $(a_\delta + 1)$-many S-disjunctions along σ. If every $\sigma \in T_{a_\delta+1}^\delta$ has only $\mathfrak{M}_0$-finitely many extensions with $(a_\delta + 1)$-many S-disjunctions, let $s(\sigma)$ be the least bound for σ on the number of such extensions. Then $\sigma \mapsto s(\sigma)$ is recursive. By $B\Sigma_1^0$ there is a uniform bound on the set $\{s(\sigma) : \sigma \in T_{a_\delta+1}^\delta\}$. But this implies that $T_{a_\delta+1}^\delta$ is $\mathfrak{M}_0$-finite as well, a contradiction, proving this Claim.

Note that $\emptyset'$ is able to compute the σ_δ in the above claim. Once σ_δ is fixed, we may select an $\mathfrak{M}_0$-infinite recursively bounded recursive increasing tree $\hat{U}_n \subseteq T_{a_\delta+1}^\delta$:

$$\hat{U}_n = \{\sigma \in T_{a_\delta+1}^\delta : \sigma_\delta \preceq \sigma \text{ and } \sigma \text{ enumerates only } (a_\delta + 1) \text{ S-disjunctions over } \varepsilon_n\}.$$

In other words, every node in $\hat{U}_n$ enumerates the same $(a_\delta+1)$-many S-disjunctions over ε_n as enumerated by σ_δ. The collection of these S-disjunctions will be maximal as long as we work inside $\hat{U}_n$, i.e., as long as at any future stages in the construction, the numbers involved in any computation of blobs or S-disjunctions always form a subset of some nodes in $\hat{U}_n$. Let E_{n+1} be the exit tree corresponding to this maximal collection of S-disjunctions and let $\varepsilon_{n+1} = (\rho_{n+1}, \beta_{n+1})$ be the exit in E_{n+1} taken by A. In particular, $\rho_{n+1} \subseteq A$ and $\beta_{n+1} \subseteq \overline{A}$. This completes the construction for the "Σ^0_1-part" of forcing for the block B_{n+1}. [Note: *ε_{n+1} and $\hat{U}_n$ together handle the "Σ^0_1-part" of the block B_{n+1}.*]

We now take up the matter of forcing the negation of formulas in $B_{n+1,r}(\varepsilon_{n+1})$ and $B_{n+1,b}(\varepsilon_{n+1})$, i.e. formulas not yet "positively forced". This is resolved by a similar yet more delicate "T_a analysis" than the one given above.

First, given a $\sigma \in \hat{U}_n$, define a sequence of σ-blobs to be blobs $o \subseteq \sigma$. For each σ in $\hat{U}_n$, let the *$(n+1)$-blobs enumerated by* σ be the sequence of σ-blobs $\vec{o}$ such that for each $o \in \vec{o}$, $\mathfrak{M}_0 \models \psi_e(\rho_{n+1} * o)$ for some $e \in B_{n+1,r}(\varepsilon_{n+1})$. This enumeration can be carried out uniformly for any node σ in the recursive tree $\hat{U}_n$ in a coherent way, i.e., if $\sigma \preceq \sigma'$, then the sequence of $(n+1)$-blobs enumerated by σ' end-extends the one by σ. Let $\#_o\sigma$ denote the number of $(n+1)$-blobs enumerated by σ under such an enumeration.

Let $T^o_a = \{\sigma \in \hat{U}_n : \#_o\sigma \leq a\}$. Here the subscript and superscript o in $\#_o$ and T^o_a refer to the counting of blobs. We consider two cases. The case that we are in will be recorded by the $(n+1)$-st bit $z(n+1)$.

Case 1 (Skipping for $(n+1)$-blobs). There is an $a \in M_0$ for which T^o_a is $\mathfrak{M}_0$-infinite. Fix the least such a. Set $z(n+1) =$ the least l such that $g(l) \geq a$ and $l > \max\{z(i) : i \leq n\}$.

Applying a similar argument as in the case of S-disjunctions, we use $\emptyset'$ to find the number a_o and a node $\sigma_o \in \hat{U}_n$ such that $\#_o\sigma_o = a_o + 1$ and the tree

$$\tilde{U}_n = \{\sigma \in \hat{U}_n : \sigma_o \preceq \sigma \text{ and } \#_o\sigma = a_o + 1\}$$

is $\mathfrak{M}_0$-infinite. Since every node $\sigma \in \tilde{U}_n$ is of the form $\sigma_o * \tau$ for some τ, we may "discard the initial segment σ_o" and define $U_{n+1} = \{\tau : \sigma_o * \tau \in \tilde{U}_n\}$. It is clear that U_{n+1} is a recursively bounded recursive increasing tree since $\hat{U}_n$ is. Let $p_{n+1} = \langle \varepsilon_{n+1}, U_{n+1} \rangle$. We see that p_{n+1} red forces $\neg\psi_e$ for all $e \in B_{n+1,r}(\varepsilon_{n+1})$ in the sense that for any $\tau \in U_{n+1}$ no $o \subset \tau$ satisfies $\psi_e(\rho_{n+1} * o)$, and red forces ψ_e for all other $\psi_e \in B_{n+1}(\varepsilon_{n+1})$ through ρ_{n+1}. Notice that this form of skipping also involves some thinning of the tree $\hat{U}_n$.

Case 2 (Thinning for $(n+1)$-blobs). For all $a \in M_0$, T^o_a is $\mathfrak{M}_0$-finite. We set $z(n+1) = 0$ to record this fact.

In this case, along any $\mathfrak{M}_0$-infinite path λ on $\hat{U}_n$ there will be $\mathfrak{M}_0$-infinitely many $(n+1)$-blobs enumerated, and any such λ offers a sufficient number of λ-blobs for building an $\mathfrak{M}_0$-infinite Seetapun tree, thereby a new U-tree. However this would only be a recursive in λ tree, and λ need not be a recursive path. To overcome this difficulty, we apply the "blob-enumeration" procedure uniformly to nodes in $\hat{U}_n$ instead of to the paths on it. Since the result of applying the procedure to a node is a tree, the result of applying it to the whole tree $\hat{U}_n$ will be a "forest". Thus we have to amalgamate the forest into one tree in order to fit the definition of the forcing conditions. The amalgamation is essentially taking the union of the choice functions of the blob-sequences enumerated. This suggests that we work on $\mathfrak{M}_0$-finite trees T^o_a for each a, as they yield blob sequences of the same length. The result of amalgamation is a recursively bounded recursive increasing tree S which will play the role of a Seetapun tree.

The recursive enumeration of S is as follows.

Stage -1. Let $S[-1]$ be $\emptyset$ (the root).

Stage $v+1$. Suppose that we have enumerated $S[v]$ which is the amalgamation of the choice functions of blobs enumerated by $\sigma \in T^o_v$; more precisely, $S[v]$ satisfies the following conditions:

(1) If τ is a node of $S[v]$ with length v, then there is a node $\sigma \in T^o_v$ such that τ is the choice function of blobs enumerated by σ.
(2) If σ is a node in T^o_v which enumerates v many σ-blobs, say $\vec{o}$, and f is a choice function for $\vec{o}$, then there is a unique maximal branch $\tau \in S[v]$ such that $\tau = f$.

To get $S[v+1]$, we examine all maximal branches σ in T^o_{v+1} such that $\#_o(\sigma) = v+1$. Let the blob-sequence enumerated by σ be $\vec{o}$. (We do not need to consider other maximal branches as they must be the dead ends in $\hat{U}_n$.) For each choice function g of $\vec{o}$, g necessarily extends some choice function f of the blob-sequence $\vec{o} \upharpoonright (v+1)$. By condition (2) for v, f is τ for some unique $\tau \in S[v]$, enumerate g into $S[v+1]$ extending τ, provided g has not been enumerated into $S[v+1]$ before. (The last sentence is necessary because two different branches on $\hat{U}_n$ could enumerate identical blob-sequences.)

By construction, we have exhausted all choice functions of blob-sequences of length $v+1$ enumerated by any node on T^o_{v+1}. It follows easily that (1) and (2) remains to hold for $S[v+1]$.

Define the subtree U_{n+1} of S by

$$U_{n+1} = \{\sigma \in S : (\forall t < \max(\sigma))(\forall \iota \subseteq \sigma)(\forall e \in B_{n+1,b}(\varepsilon_{n+1}))\ \neg\varphi_e(t, \beta_{n+1} * \iota)\}.$$

Clearly U_{n+1} is a recursively bounded recursive increasing tree because S is. We show that U_{n+1} is $\mathfrak{M}_0$-infinite: Suppose that U_{n+1} is $\mathfrak{M}_0$-finite. Then on the tree S, there is a level h such that every node σ of length h has a thread ι such that for some $e \in B_{n+1,b}(\varepsilon)$ $\mathfrak{M}_0 \models \psi_e(\beta_{n+1} * \iota)$. Choose v large enough such that for every node $\tau \in U_{n+1}$, there is some $\sigma \in T^o_v$ whose range is a superset of the range of τ. Let $\sigma \in T^o_v$ be a maximal branch, and consider the sequence of σ-blobs $\vec{o}$. By Condition (2), the range of any choice function of $\vec{o}$ also contains a thread, which means that $\sigma \in T^o_v$ enumerates an S-disjunction for the sets $B_{n+1,r}(\varepsilon)$ and $B_{n+1,b}(\varepsilon)$ with the pair of preconditions $(\rho_{n+1}, \beta_{n+1})$. But this is a contradiction since $\hat{U}_n$ does not enumerate any S-disjunctions. Moreover, since U_{n+1} is a subtree of S, U_n and U_{n+1}, satisfy Definition 4.7, condition 2 (ii). Let p_{n+1} be the forcing condition $\langle \varepsilon_{n+1}, U_{n+1}\rangle$. Then $p_n \geq p_{n+1}$ and p_{n+1} blue forces $\neg\psi_e$ for every $\psi_e \in B_{n+1,b}(\varepsilon_{n+1})$ and blue forces every other $\psi_e \in B_{n+1}$ through β_{n+1}.

This completes the construction at stage $n+1$. We summarize the discussion as a lemma for future reference.

Lemma 4.8. *At the end of stage $n+1$, we have one of the following two possibilities:*

(a) If skipping occurs, then for all $\psi_e(\check{G}) \in B_{n+1}$, either $p_{n+1} \Vdash_r \psi_e(\check{G})$ or $p_{n+1} \Vdash_r \neg\psi_e(\check{G})$. Furthermore, for any amenable set G, if $\rho_{n+1} \preceq G$ and every $\mathfrak{M}_0$-finite initial segment of $G \setminus \rho_{n+1}$ is a subset of (the range of) some node in U_{n+1}, then forcing by p_{n+1} is equal to truth for G in the following sense: If $p_{n+1} \Vdash_r \psi_e(\check{G})$ then $\mathfrak{M}_0 \models \psi_e(G)$; and if $p_{n+1} \Vdash_r \neg\psi_e(\check{G})$ then $\mathfrak{M}_0 \models \neg\psi_e(G)$.
(b) If thinning occurs, then the corresponding statement holds upon replacing r by b and ρ by β.

Observe that save for the reference to the sequence $\langle z(n) : n \in \omega\rangle$, the entire construction may be carried out using $\emptyset'$ as oracle. Now, since the sequence $\langle z(n) : n \in \omega\rangle$ is definable, it is coded on ω by an $\mathfrak{M}_0$-finite set $\hat{z}$. Using $\hat{z}$ as parameter, $\emptyset'$ is able to retrace the steps in the construction and compute the sequence of conditions $\langle p_n : n \in \omega\rangle$.

4.7. **Verification.** We now extract from the "generic sequence" $\langle p_n\rangle$ a homogenous set G that is a low set contained in either A or $\overline{A}$. There are two cases to consider: Case 1. The set $\{n : z(n) = 0\}$ is unbounded in ω.

Let $G = \bigcup_{n\in\omega} \beta_n$. Then $G \subseteq \overline{A}$ and is recursive in $\emptyset'$. Fix a Σ^0_1-formula $\psi_e(\check{G})$. Let $n \in \omega$ be large enough such that $g(n+1) > e$ and $z(n+1) = 0$. By Lemma 4.8 (b), either $\psi_e(\check{G})$ or its negation is blue forced by p_{n+1} at the end of stage $n+1$. Furthermore, the construction guarantees that G end-extends β_{n+1}, and for all $m > n+1$, β_m is a subset of some node τ of U_{m-1}, thus a subset of U_n. Thus by Lemma 4.8 (b) again, $\mathfrak{M}_0 \models \psi_e(G)$ or $\mathfrak{M}_0 \models \neg\psi_e(G)$ was determined by the time p_{n+1} is selected, which may be computed by $\emptyset'$ with the help of $\hat{z}$. In other words, the Σ^0_1-theory of G can be computed from $\emptyset'$, thus G is low.

To see that G is $\mathfrak{M}_0$-infinite, we argue that the range of β_n is not empty for $z(n) = 0$, assuming that there are "new trivial formulas" such as $\exists x(a < x \wedge x \in \check{G})$ in every block that do not belong to any smaller block, where a is some appropriate parameter. If the range of β_n is empty, then $B_{n,b}(\varepsilon_{n-1}) = B_n$ throughout the construction with no need for update. Since there are $\mathfrak{M}_0$-infinitely many n-blobs (as $z(n) = 0$), there must be a moment when the blue side forces the trivial formulas to form an S-disjunction over ε_{n-1}, which would then add at least one point to the range of β_n.

Case 2. The set $\{n : z(n) = 0\}$ is bounded in ω.

Then from some n_0 onwards, the act of skipping for blobs always occurs. Let $G = \bigcup_{n\in\omega} \rho_n$. Then $G \subseteq A$ and is again recursive in $\emptyset'$. G is low by a similar argument by quoting Lemma 4.8 (a). It remains to show that G is $\mathfrak{M}_0$-infinite. We show that the range of ρ_n for $n > n_0$ is not empty under the same assumption on trivial formulas. For any $n > n_0$, if the range of ρ_n is empty, then $B_{n,r}(\varepsilon_{n-1}) = B_n$ throughout the construction with no need for update. However, there must be blobs enumerated for the sake of trivial formulas, which means that the Seetapun tree over ε_{n-1} is $\mathfrak{M}_0$-infinite. This implies that there is no skipping at step n of the construction, which is a contradiction.

5. Comparing SRT^2_2 and RT^2_2

5.1. **Preserving Bounding for Iterated Monotone Enumerations.**

Theorem 5.1. *Assume that X is a predicate on $\mathfrak{M}_0$ with the following properties.*

(H-i) $\mathfrak{M}_0[X]$ *satisfies* $B\Sigma_2$ *and BME.*

(H-ii) *Every predicate on* ω *defined in* $\mathfrak{M}_0[X]$ *is coded on* ω.

Suppose that A is $\Delta^0_2(X)$. There is an $\mathfrak{M}_0[X]$-infinite G with the following properties.

(i) $G \subseteq A$ *or* $G \subseteq \overline{A}$.

(ii) G *has unboundedly many elements in* $\mathfrak{M}_0$.

(iii) *In* $\mathfrak{M}_0[X]$, G *is low relative to* X. *Consequently,* $\mathfrak{M}_0[X, G]$ *satisfies* $B\Sigma^0_2$.

(iv) $\mathfrak{M}_0[X, G]$ *satisfies BME.*

Proof. Intuitively we want to apply a relativization to X of the construction in Theorem 4.1. However, preserving BME in the generic extension as specified in (iv) is essential to allowing iterations of the construction which will parallel closely that in §4.

Define a notion of forcing P as in Definition 4.7, but relative to X so that the U in a condition $p = \langle \varepsilon, U \rangle$ is now an X-recursively bounded increasing X-recursive tree. Construct an X'-definable sequence of forcing conditions $\{p_n : n < \omega\}$ such that $p_n = \langle \varepsilon_n, U_n \rangle$ and $p_n \geq p_{n+1}$. Let $p_0 = \langle \varepsilon_0, Id \rangle$, where $\varepsilon_0 = (\emptyset, \emptyset)$, and Id is the identity tree whose kth element is the number k. Suppose p_n is defined satisfying

(1) $\varepsilon_n = (\rho_n, \beta_n)$ and $\rho_n \subseteq A, \beta_n \subseteq \overline{A}$;
(2) $(n > 0)$ There is a $c \in \{r, b\}$ such that for all $\Sigma^0_1(X)$-formulas ψ with parameters below $g(n)$, either $p_n \Vdash_c \psi$ or $p_n \Vdash_c \neg\psi$;
(3) $(n > 0)$ For $k \leq n$, let $BME_{k,n}$ denote BME_k relative to the predicate $(X, \check{G})$ restricted to the $g(n)$-bounded, k-iterated monotone enumerations with indices for the enumeration operator below $g(n)$. Then $BME_{k,n}$ has been ensured with the following additional conclusion: For any instance $(V_i, E_i)_{1 \leq i \leq k}$ of $BME_{k,n}$, for any $\mathfrak{M}_0$-finite subset Y of a string in U_n such that $\min Y > \max\{\rho_n, \beta_n\}$, no E_1-expansionary level in V_1 relative to $(X, \rho_n * Y)$ (or $(X, \beta_n * Y)$, depending on whether U_n was obtained at the last action through skipping or thinning) is enumerated unless it was already enumerated relative to (X, ρ_n) (respectively, (X, β_n)).

The condition p_{n+1} has to satisfy the three requirements (1)–(3) with n replaced by $n+1$. In summary, the strategy goes as follows: We implement a construction that weaves in one similar to that in Theorem 4.1 enumerating an exit tree for $\Sigma^0_1(X)$-formulas (with free set variable $\check{G}$ and parameters below $g(n+1)$) to satisfy (1) and (2) for lowness with one enumerating an exit tree to satisfy (3) (so that every instance of the $\mathfrak{M}_0$-finite collection $BME_{k,n+1}$ relative to the predicate (X, G) holds (for $k \leq n+$)). This construction will enumerate an exit tree E. Applying BME_1 relative to X allows one to conclude that there is a greatest ℓ where ℓ is an E-expansionary level in U_n. Select an exit $(\rho_{n+1}, \beta_{n+1})$ from the tree, with $\rho_n \preceq \rho_{n+1} \subseteq A$, $\beta_n \preceq \beta_{n+1} \subseteq \overline{A}$, and an X-recursively bounded increasing X-recursive tree U_{n+1} such that $p_{n+1} = (\varepsilon_{n+1}, U_{n+1})$ is a forcing condition stronger than p_n, where $\varepsilon_{n+1} = (\rho_{n+1}, \beta_{n+1})$. The tree U_{n+1} is obtained through skipping or thinning of U_n, and there is a $c \in \{r, b\}$ such that for any $\Sigma^0_1(X)$-formula ψ with parameters below $g(n+1)$, either $p_{n+1} \Vdash_c \psi$ or $p_{n+1} \Vdash_c \neg\psi$.

We now describe the construction, focusing our attention on achieving (3) since for (2) it essentially follows the construction in Theorem 4.1. Let

$$C = \{(V_{e,i}, E_{e,i})_{1 \leq i \leq k(e)} : e \leq e_0\}$$

be the collection of all $g(n+1)$-bounded, k-iterated monotone enumerations relative to (X, G) whose indices are below $g(n+1)$ and $k \leq n+1$. The idea is to associate C with a $g(n+1)$-bounded, $n+2$-iterated monotone enumeration relative to X and apply BME_{n+2} to conclude that requirement (3) is satisfied. We first consider an $n+1$ enumeration procedure that amalgamates an arbitrary sequence in C:

Claim. There exists a $g(n+1)$-bounded, $n+1$-iterated monotone enumeration $(\hat{V}_i, \hat{E}_i)_{1 \leq i \leq n+1}$ such that

- For each $e \leq e_0$, i, σ and τ, $0 * e * \sigma \in \hat{V}_i(\tau)$ if and only if $\sigma \in V_{e,i}(\tau)$, and $\tau \in \hat{E}_i(0 * e * \sigma)$ if $\tau \in E_{e,i}(\sigma)$.

Proof of Claim. We enumerate $(\hat{V}_i, \hat{E}_i)_{1\leq i\leq n+1}$ as follows: $\hat{V}_1$ has 0 as root and has e_0-many branches at level 1. A copy of $V_{e,1}$ "sits on top of the e branch" beginning at level 2. Thus $0*e*\sigma \in \hat{V}_1$ if and only if $\sigma \in V_{e,1}$. For $1 < i \leq n+1$, $\hat{V}_i$ will again have root 0 and e_0-many branches at level 1. For each $e \leq e_0$, if $i > k(e)$ then $\hat{V}_i$ has no extension above the string $0 * e$. Otherwise, a copy of $V_{e,i}$ sits on top of the string, so that $0 * e * \sigma \in \hat{V}_i$ if and only if $\sigma \in V_{e,i}$.

Define $\hat{E}_i$ as given in the statement of the Claim. The enumeration of $\hat{E}_i$ from $\hat{V}_i$, and that of $\hat{V}_{i+1}$ from $\hat{E}_i$ is carried out "componentwise" by following the algorithm for $(V_{e,i}, E_{e,i})_{1\leq i\leq k(e)}$ for each component e. Then $(\hat{V}_i, \hat{E}_i)_{1\leq i\leq n+1}$ is a $g(n+1)$-bounded, $n+1$-iterated monotone enumeration, proving the Claim.

We first analyse the procedure of generating expansionary levels. Let $\psi(\ell, X, \check{G})$ be a formula saying that there is a stage s such that $(\hat{V}_i, \hat{E}_i)_{i\leq n+1}$ has ℓ-many $\hat{E}_1$-expansionary levels in $\hat{V}_1$ relative to $(X, \check{G})$. Following the notations introduced in §4, let $B_{X,n+1}$ be the collection of $\Sigma^0_1(X)$-formulas with free variable $\check{G}$ and parameters below $g(n+1)$. Ignore for the moment formulas in $B_{X,n+1}$ other than the $\psi(\ell, X, \check{G})$'s for $\ell < g(n+1)$. Now the enumeration of an $\hat{E}_1$-expansionary level in $\hat{V}_1$ may be accomplished by enumerating blobs $\rho * o$ satisfying $\psi(\ell, X, \check{G})$ (upon substituting $\rho * o$ for $\check{G}$, where $\varepsilon = (\rho, \beta)$ is a pair of preconditions with $\rho_n \preceq \rho$ and $\beta_n \preceq \beta$), and this is $\Sigma^0_1(\mathfrak{M}_0[X])$-definable. Hence we may subject the formula $\psi(\ell, X, \check{G})$ to an "S-disjunction operation" (Definition 4.5). For $\ell = 1$, enumerate along each string σ in U_n an S-disjunction $\delta_1(\sigma)$ for ψ and accompanying exit tree $E_1(\sigma)[s]$ using ε_n as a pair of preconditions. Then every exit ρ or β in $E_1(\sigma)[s]$ generates an $\hat{E}_1$-expansionary level in $\hat{V}_1$ relative to (X, ρ) or (X, β) respectively.

In general, we consider $\psi(\ell, X, \check{G})$ for all $\ell \geq 1$. Suppose $\sigma \in U_n$ and at the end of s steps of computation there are ℓ, but not $\ell+1$-many, $\hat{E}_1(\sigma)$-expansionary levels in U_n along σ arising from the enumeration of S-disjunctions $\delta_1(\sigma), \ldots, \delta_\ell(\sigma)$ for $\psi(\ell, X, \check{G})$. If $s < |\sigma|$, compute $|\sigma|$-steps to search for the next S-disjunction $\delta_{\ell+1}(\sigma)$ for ψ along σ using as preconditions exits in $E_1(\sigma)[s]$. This implies that $\langle (U_n, E_1), (\hat{V}_i, \hat{E}_i)_{1\leq i\leq n+1}\rangle$ is a $g(n+1)$-bounded, $n+2$-iterated monotone enumeration relative to X. By BME_{n+2} relative to X, there is a maximum ℓ of E_1-expansionary levels in U_n.

Taking $\varepsilon_n = (\rho_n, \beta_n)$ as the pair of preconditions at the beginning of stage $n+1$, the above action of enumerating E_1-expansionary levels may be merged with that of enumerating an exit tree for formulas in $B_{X,n+1}$. With this in mind, we now proceed with the definition of p_{n+1}. The set of formulas to be considered is the recursive union $\hat{B}_{X,n+1}$ of $B_{X,n+1}$ and $\{\psi(\ell, X, \check{G}) : \ell \geq 1\}$. Follow the procedure in §4 to enumerate an exit tree for formulas in $\hat{B}_{X,n+1}$ which by abuse of notation we still denote as E_1. The steps described in the previous paragraph is incorporated into the construction, with the requirement that at any step s, for $\sigma \in U_n$, in addition to considering formulas in $B_{X,n+1,r}(\varepsilon)$ and $B_{X,n+1,b}(\varepsilon)$ where $\varepsilon = (\rho, \beta)$ is a pair of preconditions enumerated in $E_1(\sigma)[s-1]$ (defined from $B_{X,n+1}$ analogous to the way $B_{n+1,r}(\varepsilon)$ and $B_{n+1,b}(\varepsilon)$ were defined from B_{n+1} in §4), one looks for $\ell+1$ many $\hat{E}_1$-expansionary levels to be enumerated in $\hat{V}_1$ assuming that there are already ℓ-many such levels enumerated. This is carried out along each $\sigma \in U_n$

over a pair ε of preconditions already enumerated in $E_1(\sigma)[s-1]$. Then BME_{n+2} relative to X ensures that there is a step s^* after which no more E_1-expansionary level is enumerated.

An argument similar to that in Theorem 4.1 for the formulas in $\hat{B}_{X,n+1}$ yields an $\varepsilon_{n+1} = (\rho_{n+1}, \beta_{n+1})$ with $\rho_{n+1} \subseteq A$ and $\beta_{n+1} \subseteq \overline{A}$, and a U_{n+1} so that $p_{n+1} = \langle \varepsilon_{n+1}, U_{n+1} \rangle$ is a condition stronger than p_n. Furthermore, for some $c \in \{r, b\}$, for any $\psi \in B_{X,n+1}$, either $p_{n+1} \Vdash_c \psi$ or $p_{n+1} \Vdash_c \neg\psi$. Hence p_{n+1} satisfies (1) and (2) upon replacing n by $n+1$. We show that (3) also holds for p_{n+1}. To do this, we retrace the key steps similar to those taken in the proof of Theorem 4.1 that lead to the definition of U_{n+1}, focusing on satisfying BME_{n+1}.

For each $\sigma \in U_n$, let $\#_\delta \sigma$ be the number of S-disjunctions enumerated along σ for formulas in $\hat{B}_{X,n+1}$. Then $\#_\delta \sigma$ is greater than or equal to the largest number ℓ such that $\delta_\ell(\sigma)$ is defined for the formula $\psi(\ell, X, \check{G})$ in $|\sigma|$-steps of computation. Let

$$T_a^\delta = \{\sigma \in U_n : \#_\delta \sigma \leq a\}.$$

Then BME_{n+2} relative to X guarantees that there is a largest a, denoted a_δ, for which T_a^δ is $\mathfrak{M}_0$-finite. By an argument similar to that for the Claim in §4.6, there is a $\sigma_\delta \in U_n$ such that $\#_\delta \sigma_\delta = a_\delta + 1$ and the subtree of $T_{a_\delta+1}^\delta$ extending σ_δ is unbounded. Let $\ell^* \leq a_\delta$ be the largest ℓ for which ℓ-many S-disjunctions $\delta_\ell(\sigma_\delta)$ are enumerated for the formula $\psi(\ell, X, \check{G})$ along σ_δ. Then no new S-disjunction is enumerated along any string in $T_{a_\delta+1}^\delta$ extending σ_δ. Then $\varepsilon_{n+1} = (\rho_{n+1}, \beta_{n+1})$ is a pair of maximal exits in $E_1(\sigma_\delta)$ with $\rho_{n+1} \subseteq A$ and $\beta_{n+1} \subseteq \overline{A}$. Let

$$\hat{U}_n = \{\sigma \in T_{a_\delta+1}^\delta : \sigma_\delta \prec \sigma \ \wedge \sigma \text{ enumerates } a_\delta + 1 \text{ S-disjunctions over } \varepsilon_n\},$$

so that all the numbers appearing in $\hat{U}_n$ are greater than $\max \sigma_\delta$. For $\tau \in \hat{U}_n$ enumerate an increasing sequence of blobs o such that $\min o > \max \rho_{n+1}$ and either $\psi(\ell^* + 1, X, \rho_{n+1} * o)$ holds or $\varphi_e(X, \rho_{n+1} * o)$ holds for some $e \in B_{X,n+1,r}(\varepsilon_{n+1})$. A further T_a analysis is conducted in order to define U_{n+1}. For $\tau \in \hat{U}_n$, let $\#_o \tau$ be the number of such blobs enumerated along τ after $|\tau|$ steps of computation.

Let

$$T_a^o = \{\tau \in \hat{U}_n : \#_o \tau \leq a\}.$$

There are two cases to consider.
Case 1. (Skipping). There are boundedly many a's for which T_a^o is $\mathfrak{M}_0$-finite.

Then there is a largest such a which we denote as a_o. As in the proof of the Claim in §4.6, there is a σ_o in $T_{\hat{a}_0}^o$ so that $\#_o \sigma_o = a_o + 1$ and

$$\tilde{U}_n = \{\sigma \in T_{a_o+1}^o : \sigma_o \prec \sigma \ \wedge \ \#_o \sigma = a_o + 1\}$$

is unbounded.

We do skipping over σ_o and define U_{n+1} to be the part of $\tilde{U}_n$ above σ_o, meaning the least number appearing in U_{n+1} is greater than $\max \sigma_o$. We show that (3) holds for p_{n+1}. Let Y be $\mathfrak{M}_0$-finite and a subset of a string in U_{n+1}. Each instance of $BME_{k,n+1}$ is $(V_{e,i}, E_{e,i})_{1 \leq i \leq k(e)}$ for some $e \leq e_0$. The choice of the condition p_{n+1} ensures that any $E_{e,1}$-expansionary level in $V_{e,1}$ relative to $(X, \rho_{n+1} * Y)$ is enumerated relative to (X, ρ_{n+1}). Thus (3) is satisfied.

Case 2. (Thinning). T_a^o is $\mathfrak{M}_0$-finite for each a.

We do thinning of $\hat{U}_n$ by following the construction in §4.6 (conditions (1) and (2) before Lemma 4.8) using the blobs o in $\hat{U}_n$ to form the (*almost* Seetapun) X-recursively bounded increasing X-recursive) tree S, and then define

$$U_{n+1} = \{\tau \in S : (\forall \iota \subseteq \tau)\neg\psi(\ell^* + 1, X, \beta_{n+1} * \iota) \vee \\ \forall t \leq |\sigma| \forall \iota \subseteq \sigma \forall e \in B_{X,n+1,b}(\varepsilon_{n+1})\neg\varphi_e(t, \beta_{n+1} * \iota)\}.$$

Then $(\varepsilon_{n+1}, U_{n+1})$ is the condition p_{n+1}. The proof of (3) is by the same argument as in Case 1 above, except that we replace $\rho_{n+1} * Y$ by $\beta_{n+1} * Y$.

Finally, note that the data on skipping (and "how far") or thinning for U_n, $n < \omega$, is uniformly X-definable and coded on ω by the same argument used in Theorem 4.1. Hence the entire construction may be carried out recursively in X'.

Define $G = \bigcup_n \rho_n$ or $\bigcup_n \beta_n$ as appropriate. We argue that $\mathfrak{M}_0[X, G] \models B\Sigma_2^0$, G is low relative to X and (i)–(iv) are satisfied. We verify (iv) for the case when $G = \bigcup_n \beta_n$. Let $(V_i, E_i)_{1 \leq i \leq k}$ be an instance of $BME_{k,n}$ relative to (X, G). We claim that all the E_1-expansionary levels in V_1 are enumerated relative to (X, β_{n+1}) and therefore there are only $\mathfrak{M}_0$-finitely many such levels. Now by construction, any initial segment of the set $\{x \in G : x > \max \beta_n\}$ is contained as a subset of some string in U_n. Since (3) is satisfied, the claim follows. A similar argument holds for the case when $G = \bigcup_n \rho_n$. Note that (i) is immediate and that (ii) and (iii) may be verified as in the proof of Theorem 4.1. □

5.2. **A Model of SRT^2_2.** We are now ready to prove Theorem 2.2. Begin with $\mathfrak{M}_0$ as the ground model and let $A_1, A_2, \ldots, A_i, \ldots$ be a countable list of all Δ_2^0-sets. Begin by setting $G_0 = \emptyset$. For $i \geq 1$, repeatedly apply Theorem 5.1 by letting $X = (G_0, \ldots, G_{i-1})$ to obtain an unbounded G_i such that

(1) G_1 is low;
(2) $G_i \subseteq A_i$ or $G_i \subseteq \overline{A}$;
(3) G_{i+1} is low relative to $(G_1, \ldots, G_i)$;
(4) $\mathfrak{M}_0[G_1, \ldots, G_i] \models BME$.

For $i = 1$, (1)–(4) hold for G_1 by Theorem 5.1 with $X = \emptyset$. Suppose $G_1, \ldots, G_i$ satisfy (1)–(4). Now we reduce BME_k relative to $(G_1, \ldots, G_{i+1})$ to BME_{k+1} for $(G_1, \ldots, G_i)$ which is true by induction. Thus $(G_1, \ldots, G_{i+1})$ satisfies BME_k for all k.

Let $\mathcal{S}$ be the closure under the join operation and Turing reducibility in $\mathfrak{M}_0$ of the set $\{G_i : i \in \mathbb{N}\}$. Then $\mathfrak{M} = \langle M_0, \mathcal{S}\rangle$ is an M_0-extension of $\mathfrak{M}_0$ and is a model of $RCA_0 + B\Sigma_2^0$ that satisfies $SRT^2_2 + \neg I\Sigma_2^0$. Furthermore, since every member of $\mathcal{S}$ is low, by Proposition 2.4, $\mathfrak{M}$ is not a model of RT^2_2.

5.3. **SRT^2_2 and WKL_0.** We now strengthen Theorem 2.2 and prove Theorem 2.7: There is a model of $RCA_0 + B\Sigma_2^0 + SRT^2_2 + WKL_0$ that is not a model of RT^2_2. We begin with a lemma.

Lemma 5.2. *For any low set X such that $\mathfrak{M}_0[X] \models BME$, any unbounded X-recursive subtree W of the full binary tree has an unbounded path G that is low relative to X such that $\mathfrak{M}_0[X, G] \models BME$.*

Proof. Let W be such a tree. We build an unbounded path G through W that satisfies the requirements. This is carried out in ω-many steps.

Step 0: Let $W_0 = W$ and $\nu_0 = \emptyset$.

Step $n+1$: Suppose $W_n \subseteq W$ is unbounded, X-recursive and every string in W_n extends the string ν_n defined at end of stage n. On W_n first follow the Low Basis Theorem construction of Jockusch and Soare (1972) (see also Hájek (1993) on constructing a path that preserves $B\Sigma^0_2$) to obtain a string ν'_{n+1} in W_n extending ν_n, such that the subtree W'_{n+1} of W_n consisting of strings extending ν'_{n+1} is unbounded, and for any $\Sigma^0_1(X)$-formula ψ with a free set variable $\check{G}$ and parameters below $g(n+1)$, either $\psi(\nu'_{n+1})$ holds or no string ν on W'_{n+1} satisfies $\psi(\nu)$.

Now we define an unbounded X-recursive subtree W_{n+1} contained in W'_{n+1} to guarantee $\mathfrak{M}_0 \models BME_{k,n+1}$ for $k \leq n+1$. By the Claim in Theorem 5.1, it is sufficient to consider the $g(n+1)$-bounded, $n+1$-iterated monotone enumeration $(\hat{V}_i, \hat{E}_i)_{1\leq i\leq n+1}$. Given a string $\nu \in W'_{n+1}$ and $t < |\nu|$, we say that $\hat{V}_1$ relative to $(X \upharpoonright |\nu|, \nu)$ is conservative over $\hat{V}_1$ relative to $(X \upharpoonright t, \nu \upharpoonright t)$ if they enumerate the same $\hat{E}_{1,s}$-expansionary levels after $|\nu|$ steps of computation. Let

$$\hat{W}_{n+1,t} = \{\nu \in W'_{n+1} : |\nu| > t \wedge [\hat{V}_1 \text{ relative to } (X \upharpoonright |\nu|, \nu) \text{ is conservative over } \hat{V}_1 \text{ relative to } (X \upharpoonright t, \nu \upharpoonright t)]\}.$$

Now $\hat{W}_{n+1,t}$ is not $\mathfrak{M}_0$-finite for every t, since this would contradict the assumption of BME_{n+1} for $(\hat{V}_i, \hat{E}_i)_{1\leq i\leq n+1}$. Thus choose the least t, denoted t_{n+1}, such that $\hat{W}_{n+1,t}$ is unbounded. Define $\nu_{n+1} \succeq \nu'_{n+1}$ to be the least string in $\hat{W}_{n+1,t_{n+1}}$ such that the subtree of W'_{n+1} all of whose strings extend ν_{n+1} is unbounded. Let W_{n+1} be the subtree of W'_{n+1} all of whose strings extend ν_{n+1}.

Let $G = \bigcup_n \nu_n$. Then G is a path on W. Furthermore, the map $n \mapsto t_n$ is recursive in X'. Thus X' is able to compute G correctly, implying that G is low relative to X. Finally, for each n, t_n pinpoints where the bound of any $g(n)$-bounded, k-iterated monotone enumeration with $k \leq n$ and parameters in $g(n)$ is located. Thus BME holds relative to (X, G). □

Proof of Theorem 2.7. Let $A_1, W_1, A_2, W_2, \ldots, A_i, W_i, \ldots$ be a list in order type ω of all the Δ^0_2-sets and unbounded recursively bounded increasing recursive trees relative to a low set. Let $G_0 = \emptyset$. Define low sets G_i, $1 \leq i < \omega$, such that

(1) For $i \geq 0$, G_{2i+1} is contained in either A_i or $\overline{A}_i$;
(2) For $i \geq 1$, G_{2i} is a path on W_i;
(3) G_1 is low and G_{i+1} is low relative to $(G_1, \ldots, G_i)$;
(4) For $i \geq 1$, $\mathfrak{M}_0[G_1, \ldots, G_i] \models BME$.

Let $\mathcal{S}$ be the closure of $\{G_i : 1 \leq i < \omega\}$ under join and Turing reducibility. Then $\langle M_0, \mathcal{S}\rangle \models RCA_0 + SRT^2_2 + WKL_0 + B\Sigma^0_2$, and both $I\Sigma^0_2$ as well as RT^2_2 (by Proposition 2.4) fail in the model.

6. Conclusion

We end with three questions for further investigation and some comments about them.

Question 6.1. *Is every ω-model of SRT^2_2 also a model of RT^2_2?*

Rephrased, Question 6.1 asks whether there is a nonempty subset $\mathcal{S}$ of $2^{\mathbb{N}}$ such that (1) $\mathcal{S}$ is closed under join and relative computation, (2) for every X in $\mathcal{S}$ and every $\Delta^0_2(X)$ predicate P, there is an infinite set G in $\mathcal{S}$ all or none of whose elements satisfy P, and (3) there is an X in $\mathcal{S}$ and an X-recursive f coloring the pairs of natural numbers with two colors such that there is no infinite f-homogeneous set in $\mathcal{S}$. The rephrasing of Question 6.1 makes it clear that it is a recursion theoretic question about subsets of $\mathbb{N}$.

If it had been the case that RT^2_2 were provable in $RCA_0 + SRT^2_2$, then the casting of Question 6.1 in the language of subsystems of second order arithmetic would have increased our understanding of the implication from SRT^2_2 to RT^2_2. Namely, the proof of the implication would have worked over a weak base theory. By Theorem 2.2, there is no such formal implication, but our interest in the question is not decreased. In fact, the opposite is true. The truth of the relationship between the two principles lies in the recursion theoretic formulation. What we know now from the formalized problem should inform us as to what means may be needed to penetrate the matter fully.

Question 6.2. *Are there natural axiomitizations within first order arithmetic for the first order consequences of the second order principles SRT^2_2 and RT^2_2?*

We do not have a recursion theoretic rephrasing of Question 6.2. By its nature, recursion theory takes $\mathbb{N}$ as the basis on which to erect the hierarchy of definability and does not allow for the variation of arithmetic truth. So, we are led naturally to formal systems and decisions as to which parts of the theory of $\mathbb{N}$ should be preserved as base theory and which should be counted as non-trivial consequences of stronger principles. In the present setting, $I\Sigma^0_1$ was taken as given and the rest remained to be proven.

Let $FO(SRT^2_2)$ and $FO(RT^2_2)$ denote the consequences of these theories within first order arithmetic. Working over RCA_0, our current state of knowledge is as follows.

$$B\Sigma^0_2 \subseteq FO(SRT^2_2) \subsetneq I\Sigma^0_2$$
$$B\Sigma^0_2 \subseteq FO(RT^2_2) \subseteq I\Sigma^0_2$$

It is possible that the appearance of BME in our construction of $\mathfrak{M}_0$ was a necessary precondition to expanding $\mathfrak{M}_0$ by sets to a model of SRT^2_2. It is worth explicitly raising the simplest instance of that issue.

Question 6.3. *Does either of $RCA_0 + SRT^2_2$ or $RCA_0 + RT^2_2$ prove that if E has a bounded monotone enumeration then the enumeration of E is finite?*

By Proposition 3.5, an affirmative answer is consistent with the known upper bound on $FO(RT^2_2)$. By Proposition 3.6, an affirmative answer for either SRT^2_2 or RT^2_2 would separate the first order consequences of that theory from $B\Sigma^0_2$.

When we approach questions concerning subsystems of second order arithmetic like 6.1, we have a well-developed set of tools, including forcing and priority methods. In comparison, there are remarkably few methods in place to investigate questions like 6.2 or 6.3. It seems strange that this area should be so little developed, since quantifying the implications from familiar and fruitful properties of the infinite to properties of the finite is a natural application of mathematical logic, especially of recursion theory.

References

Peter A. Cholak, Carl G. Jockusch, and Theodore A. Slaman. On the strength of Ramsey's theorem for pairs. *J. Symbolic Logic*, 66(1):1–55, 2001.

C. T. Chong, Steffen Lempp, and Yue Yang. On the role of the collection principle for Σ^0_2-formulas in second-order reverse mathematics. *Proc. Amer. Math. Soc.*, 138(3):1093–1100, 2010.

Rod Downey, Denis R. Hirschfeldt, Steffen Lempp, and Reed Solomon. A Δ^0_2 set with no infinite low subset in either it or its complement. *J. Symbolic Logic*, 66 (3):1371–1381, 2001.

Petr Hájek. Interpretability and fragments of arithmetic. In *Arithmetic, proof theory, and computational complexity (Prague, 1991)*, volume 23 of *Oxford Logic Guides*, pages 185–196. Oxford Univ. Press, New York, 1993.

J. L. Hirst. *Combinatorics in Subsystems of Second Order Arithmetic*. PhD thesis, The Pennsylvania State University, 1987.

Carl G. Jockusch, Jr. Ramsey's theorem and recursion theory. *J. Symbolic Logic*, 37:268–280, 1972.

Carl G. Jockusch, Jr. and Robert I. Soare. Π^0_1 classes and degrees of theories. *Trans. Amer. Math. Soc.*, 173:33–56, 1972.

Richard Kaye. *Models of Peano arithmetic*, volume 15 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1991.

S. C. Kleene. Recursive predicates and quantifiers. *Trans. Amer. Math. Soc.*, 53: 41–73, 1943. ISSN 0002-9947.

Jiayi Liu. RT^2_2 does not prove WKL_0. *J. Symbolic Logic*, 77(2):609–620, 2012.

Kenneth McAloon. Completeness theorems, incompleteness theorems and models of arithmetic. *Trans. Amer. Math. Soc.*, 239:253–277, 1978.

J. B. Paris and L. A. S. Kirby. Σ_n-collection schemas in arithmetic. In *Logic Colloquium '77 (Proc. Conf., Wrocław, 1977)*, volume 96 of *Stud. Logic Foundations Math.*, pages 199–209. North-Holland, Amsterdam, 1978.

Hartley Rogers, Jr. *Theory of recursive functions and effective computability*. MIT Press, Cambridge, MA, second edition, 1987. ISBN 0-262-68052-1.

Gerald E. Sacks. *Higher Recursion Theory*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1990.

David Seetapun and Theodore A. Slaman. On the strength of Ramsey's theorem. *Notre Dame J. Formal Logic*, 36(4):570–582, 1995.

Stephen G. Simpson. *Subsystems of second order arithmetic*. Perspectives in Logic. Cambridge University Press, Cambridge, second edition, 2009.

Theodore A. Slaman. Σ_n-bounding and Δ_n-induction. *Proc. Amer. Math. Soc.*, 132(8):2449–2456 (electronic), 2004.

E. Specker. Ramsey's theorem does not hold in recursive set theory. In *Logic Colloquium '69 (Proc. Summer School and Colloq., Manchester, 1969)*, pages 439–442. North-Holland, Amsterdam, 1971.

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