ON THE COMPUTABILITY OF PERFECT SUBSETS OF SETS WITH POSITIVE MEASURE

C. T. CHONG, WEI LI, WEI WANG, AND YUE YANG

Abstract. A set $X \subseteq 2^{\omega}$ with positive measure contains a perfect subset. We study such perfect subsets from the viewpoint of computability and prove that these sets can have weak computational strength. Then we connect the existence of perfect subsets of sets with positive measure with reverse mathematics.

1. Introduction

An observation made in algorithmic randomness is that a set with positive measure (a positive set for short) in the Cantor space contains members with limited computational strength, and in many cases those with limited computational strength form sets of measure 1. As examples, almost no member of a positive set computes the halting problem or a given non-computable set, and the class of generalized low reals is of measure 1. There are numerous such examples in algorithmic randomness. This paper is motivated by the general question: to what extent does the above observation hold for perfect subsets of a positive set?

Clearly a positive set contains many perfect subsets. We identify a perfect set $C$ with a perfect (binary) tree $P = \{\sigma \in 2^{\omega} : \sigma \text{ is an initial segment of some } X \in C\}$.

If $P$ is a perfect tree then it computes its own growth rate $g_P: k \mapsto \min\{m : |P \cap 2^m| \geq 2^k\}$.

Since a perfect tree contains perfect subtrees of arbitrarily fast growth rate, every positive set contains perfect subsets which compute the halting problem. Hence in terms of computational strength it appears that

\begin{itemize}
\item 2010 Mathematics Subject Classification. 03D32 03F35 03F60 03F30.
\item The authors thank many logicians, in particular Steffen Lempp and Ludovic Patey for inspiring discussions and suggestions, and the referee for suggesting a simpler proof of Proposition 2.4 and many other helpful comments. Chong’s research was partially supported by NUS grants C-146-000-042-001 and WBS : R389-000-040-101. Wang’s research was partially supported by China NSF Grant 11471342. Yang’s research was partially supported by NUS AcRF Tier 1 grant R146-000-231-114 and MOE2016-T2-1-019. All the authors acknowledge the support of JSPS-NUS grants R-146-000-192-133 and R-146-000-192-733 during the course of the work.
\end{itemize}
the situation with perfect subsets is in sharp contrast to that with members of a positive set. Nevertheless, in Section 2 we demonstrate two computability-theoretic properties of members of a positive set which are shared by perfect subsets of a positive set. The first is an analogue of the Low Basis Theorem for positive $\Pi^0_1$ classes. The second asserts that every positive set contains perfect subsets not computing a given non-computable set. We also include in Section 2 a proposition showing that the above results do not hold for effectively closed null sets. The statement that every positive set contains a perfect subset turns out to be interesting also from the perspective of reverse mathematics, as we argue in Section 3.

We fix the following notations and terminologies. If $T \subseteq 2^{<\omega}$ is a tree and $\sigma \in 2^{<\omega}$ then

$$T_\sigma = \{ \tau \in T : \tau \text{ is comparable with } \sigma \}.$$  

The set $[T]$ of a tree $T$ is the set of all infinite paths on $T$. A binary tree $T$ is positive iff $[T]$ is positive. An initial segment of a tree $T$ is a tree $F$ such that every node of $T$ is either a node of $F$ or an extension of a leaf of $F$. $T$ end-extends $F$ if $F$ is an initial segment of $T$.

For other notions and notations, the reader is referred to Downey and Hirschfeldt [4] for computability and algorithmic randomness, and to Simpson’s monograph [7] for reverse mathematics.

2. The Computability of Perfect Subsets of Positive Sets

We begin with positive $\Pi^0_1$ classes. As usual, a $\Pi^0_1$ class is identified with the collection of infinite paths on a computable binary tree. Since there exist positive computable binary trees containing no computable paths, there exist positive computable trees with no computable perfect subtree. So the following analogue of the Low Basis Theorem for positive $\Pi^0_1$ classes is non-trivial. The idea is to define a tree of perfect subtrees of a computable positive tree and then apply the Low Basis Theorem. We need the following form of Lemma 8 in Kučera [5], which can be obtained in a way similar to [4, Lemma 8.5.2].

**Lemma 2.1** (Kučera). For every positive tree $T$ and every $\epsilon > 0$ there exist an infinite subtree $S_\epsilon \subseteq T$ and a computable function $\rho_\epsilon : 2^{<\omega} \to \mathbb{Q}^+$ such that $S_\epsilon \leq_T T$, $\mu([S_\epsilon]) > \mu([T]) - \epsilon$ and

$$[S_{\epsilon, \sigma}] \neq \emptyset \text{ if and only if } \mu([S_{\epsilon, \sigma}]) > \rho_\epsilon(\sigma)$$  

for every $\sigma \in 2^{<\omega}$.

**Proposition 2.2.** Every computable positive tree $T$ has a low perfect subtree.

**Proof.** In Lemma 2.1, let $\epsilon = \mu([T])/2$ and let $\rho = \rho_\epsilon$ be the associated function. For convenience we write $T$ for $S_\epsilon$ and define a computable function $g : \omega \to \omega$ by induction as follows:
Let \( g(0) = 0 \). Suppose that \( g(n) = k \) is defined. Let \( r(n) = \min\{\rho(\sigma) : |\sigma| \leq k\} \). By the assumption on \( \rho \), \( r(n) > 0 \). Let \( g(n+1) \) be the least \( l > k \) such that \( 2^{-l} < r(n) \).

Let \( \mathcal{F} \) be the set of finite (perfect) trees \( S \) such that \( S \) is isomorphic to \( 2^{\leq n} \) for some \( n \) and for every \( i \leq n \), the \( i \)-th level of \( S \) is contained in \( T \cap 2^{\leq g(i)} \). Clearly, \( \mathcal{F} \) can be identified as a computably bounded computable tree. Moreover, every infinite sequence in \( \mathcal{F} \) gives rise to a perfect subtree of \( T \). We show that \( \mathcal{F} \) is infinite.

We prove by induction that for every \( n \) there exists an \( S_n \in \mathcal{F} \) such that \( S_n \) is isomorphic to \( 2^{\leq n} \) and \( [T_\sigma] \neq \emptyset \) for every \( \sigma \) a leaf (i.e., a maximal node) of \( S_n \). Obviously we can set \( S_0 \) to be the singleton consisting of the empty string. Let \( S_n \) be given. Every maximal \( \sigma \in S_n \) is in \( T \cap 2^{\leq g(n)} \). Since \( [T_\sigma] \neq \emptyset \), \( \mu([T_\sigma]) > \rho(\sigma) \geq r(n) \). As \( 2^{-g(n+1)} < r(n) \), \( \sigma \) has at least two extensions \( \sigma_0 \) and \( \sigma_1 \) in \( T \) such that \( |\sigma_i| = g(n+1) \) and \( [T_{\sigma_i}] \neq \emptyset \) for all \( i < 2 \). So we can define \( S_{n+1} \) from \( S_n \) in an obvious way.

By the Low Basis Theorem, \([\mathcal{F}]\) contains a low path \( P \), and \( \bigcup P \) is a low perfect subtree of \( T \). \( \square \)

The proof of the above proposition leads to the following corollary.

**Corollary 2.3.** Let \( T \subseteq 2^{<\omega} \) be a positive tree.

1. There exists an infinite tree \( \mathcal{F} \) such that \( \mathcal{F} \) is \( T \)-computable and \( T \)-computably bounded and \( \bigcup P \) is a perfect subtree of \( T \) for every \( P \in [\mathcal{F}] \).

2. Every set computing \( T' \) is Turing equivalent to \( T \oplus P \) for some perfect subtree \( P \subseteq T \).

**Proof.** (1) is essentially a part of the proof of Proposition 2.2. (2) can be obtained by combining the above proof with that of [5, Theorem 7] (see also [4, Theorem 8.5.1]). \( \square \)

However, the conclusion of Proposition 2.2 does not hold for an arbitrary \( \Pi^0_1 \) class.

**Proposition 2.4.** There exists a computable binary tree \( T \) such that \([T]\) is a perfect set and every perfect subtree of \( T \) computes the halting problem.

**Proof.** Let \( m \) be the modulus function of \( \emptyset' \), i.e.,

\[
m(x) = \min\{s : \forall e < x(\Phi_e(e) \downarrow \Phi_e(e)[s] \downarrow)\}.
\]

For each \( X \in 2^{\omega} \), let \( p^X(n) \) be the \( n \)-th \( i \) such that \( X(i) = 1 \). So \( p^X \) is total iff \( X(i) = 1 \) for infinitely many \( i \). Let

\[
C = \{X \in 2^{\omega} : \forall n(p^X(n) \geq m(n))\}.
\]

So \( C \) is a perfect \( \Pi^0_1 \)-class and if every \( X \in C \) with \( p^X \) total computes the halting problem. Let \( T \) be a computable binary tree such that \( C = [T] \). If \( P \) is a perfect subtree of \( T \), then \( P \) computes some \( X \in [P] \subseteq C \)
with \( p^X \) total and \( X \) in turn computes the halting problem. Hence \( P \) computes the halting problem as well.

Next, we move to perfect subtrees of arbitrary positive trees.

**Theorem 2.5.** Every positive tree has a perfect subtree not computing a given non-computable set.

**Proof.** Fix a binary tree \( \hat{T} \) with \( \mu([\hat{T}]) > 0 \) and a non-computable \( X \).

We build a desired perfect subtree of \( \hat{T} \) by a variant of Mathias forcing. A forcing condition is a pair \((F, T)\) such that \( F \) is isomorphic to some \( 2^{<\omega} \), \( X \not\leq T \) and

\[
\mu([(T \cap \hat{T})_\sigma]) > (1 - 2^{-2})2^{-|\sigma|}
\]

for every leaf \( \sigma \) of \( F \). An extension of a condition \((F, T)\) is a condition \((E, S)\) such that \( F \) is an initial segment of \( E \) and \( S \subseteq T \). As \( \mu([\hat{T}]) > 0 \), there exists \( \sigma \) with \( \mu([\hat{T}_\sigma]) > (1 - 2^{-2})2^{-|\sigma|} \). Let \( F = \{ \sigma \downarrow l : l \leq |\sigma| \} \).

Then \((F, 2^{<\omega})\) is a condition. Each condition \((F, T)\) represents the following set of perfect subtrees of \( \hat{T} \):

\[
[F, T] = \{ P \subseteq \hat{T} \cap T : P \text{ is a perfect tree end-extending } F \}
\]

Firstly, we show that for each \( e \) the conditions forcing \( \Phi_e(P) \neq X \) are dense. Fix a condition \((F, T)\) and some \( e \), let \( \mathcal{U} \) be the set of binary trees \( S \) such that \( S \) satisfies (2.1) in place of \( \hat{T} \) and

\[
(\forall x)((\forall i < 2)(E_i \text{ end-extends } F \land \Phi_e(E_i; x) \downarrow \Rightarrow \Phi_e(E_0; x) = \Phi_e(E_1; x)))
\]

for all pairs \((E_0, E_1)\) of finite subtrees of \( S \). Note that \( \mathcal{U} \) can be identified with a \( \Pi^0_1 \) class in Cantor space.

**Lemma 2.6.** Suppose that \( \mathcal{U} \neq \emptyset \). Then there exists a condition \((E, S)\) extending \((F, T)\) such that \( \Phi_e(P) \neq X \) whenever \( P \in [E, S] \).

**Proof.** As \( \mathcal{U} \) is \( \Pi^0_1 \) and \( X \not\leq T \), we can apply the cone avoidance property of \( \Pi^0_1 \) classes relativized to \( T \) and get an infinite binary tree \( S_0 \in \mathcal{U} \) s.t. \( X \) is not computable in \( T \oplus S_0 \). Let \( S = T \cap S_0 \). By the definition of \( \mathcal{U} \), for each finite subtree \( E \) of \( S \) which end-extends \( F \), the value of \( \Phi_e(E; x) \) is independent of \( E \) if the computation halts. Thus, if \( P \) is a perfect subtree of \( S \) end-extending \( F \) and \( \Phi_e(P) \) is total then \( \Phi_e(P) \) is computable in \( T \oplus S_0 \) and thus cannot be equal to \( X \). By (2.1) and the version of (2.1) for \( S_0 \),

\[
\mu([(\hat{T} \cap S)_\sigma]) > (1 - 2^{-1})2^{-|\sigma|}
\]

for every leaf \( \sigma \) of \( F \). By the Lebesgue Density Theorem, we can extend the leaves of \( F \) to obtain a finite tree \( E \) such that \( E \) is isomorphic to \( F \) and

\[
\mu([(\hat{T} \cap S)_\tau]) > (1 - 2^{-2})2^{-|\tau|}
\]

for every leaf \( \tau \) of \( E \). Clearly, \((E, S)\) satisfies the conclusion. \( \square \)
Lemma 2.7. Suppose that \( \mathcal{U} = \emptyset \). Then there exists \( E \) such that \((E, T)\) is an extension of \((F, T)\) and \( \Phi_e(E; n) \downarrow \neq X(n) \) for some \( n \).

**Proof.** Let

\[(2.2) \quad \hat{T} = \{ \sigma : \mu([(T \cap \hat{T})\sigma]) > 0 \} = T \cap \hat{T} \setminus \{ \sigma : \mu([(T \cap \hat{T})\sigma]) = 0 \}. \]

Then \( \hat{T} \) is a binary tree satisfying (2.1) in place of \( \hat{T} \). As \( \tilde{T} \notin \mathcal{U} \), there exist \( n \) and a finite subtree \( D \) of \( \tilde{T} \) such that \( D \) is an end-extension of \( F \) and \( \Phi_e(D; n) \downarrow \neq X(n) \). By (2.2), we may assume that \( D \) is isomorphic to some \( 2^{<\ell} \). By the Lebesgue Density Theorem and (2.2), we have a finite end-extension \( E \) of \( D \) obtained by extending each leaf \( \sigma \) of \( D \) to some \( \tau \) such that \( \mu([(T \cap \hat{T})\tau]) > (1 - 2^{-2})2^{-|\tau|} \). Then \((E, T)\) is the desired extension of \((F, T)\). \( \square \)

The conditions \((F, T)\) with splitting extensions on \( F \) are also dense:

**Lemma 2.8.** Every condition \((E, T)\) with \( E \) isomorphic to \( 2^{<n} \) has an extension \((F, T)\) with \( F \) isomorphic to \( 2^{<n+1} \).

**Proof.** Suppose that \((E, T)\) is a condition with \( E \) isomorphic to \( 2^{<n} \). For each leaf \( \sigma \) of \( E \), by the Lebesgue Density Theorem and (2.1) choose two incomparable \( \tau(\sigma, 0) \) and \( \tau(\sigma, 1) \) extending \( \sigma \) such that

\[ \mu([(T \cap \hat{T})\tau(\sigma, i)]) > (1 - 2^{-2})2^{-|\tau(\sigma, i)|} \]

for \( i < 2 \). Let

\[ F = \{ \eta : \eta \text{ is an initial segment of some } \tau(\sigma, i), i < 2, \sigma \text{ is a leaf of } E \} \]

Then \((F, T)\) is as desired. \( \square \)

Theorem 2.5 follows from Lemmata 2.6, 2.7 and 2.8. \( \square \)

**Corollary 2.9.** Given a non-computable \( X \), every positive set has a perfect subset not computing \( X \).

**Proof.** It follows from Theorem 2.5 and that every positive set has a positive closed subset. \( \square \)

3. **Perfect Sets and Reverse Mathematics**

As observed by Lempp, the statement “every positive closed set contains a perfect subset” can be formulated as a \( \Pi^1_2 \) sentence in second order arithmetic. In general, if \( \Lambda \) is a pointclass then we may consider the following statement:

\[(3.1) \quad \text{every positive set in } \Lambda \text{ contains a perfect subset.} \]

Clearly, the proposition that every positive closed set contains a perfect subset implies \( \text{WWKL}_0 \) over \( \text{RCA}_0 \), and it is a consequence of \( \text{WKL}_0 \) by the proof of Proposition 2.2. So it is natural to ask whether (3.1) for closed sets is a substantially new principle. During a discussion with the third author, Ludovic Patey proved that this instance for closed set
does not imply WKL$_0$ over RCA$_0$, by combining ideas from the proof of Theorem 2.5 and Liu [6]. But whether WWKL$_0$ implies (3.1) for closed sets remains open.

Here, we present another connection between the (3.1) family and reverse mathematics. Recall that TT$_1$ is the proposition that for every finite coloring $f : 2^{<\omega} \to n$ there exists an $f$-homogeneous perfect tree $T$.

**Proposition 3.1** (RCA$_0$). The instance of (3.1) for $\Pi_2^0$ sets implies TT$_1$.

*Proof.* It is obvious that over RCA$_0$ the instance of (3.1) for $\Pi_2^0$ sets implies $2$–POS, which is the statement introduced by Avigad et al. [1] that every $\Pi_2^0$ set with positive outer measure is non-empty. By [1, Theorem 3.7], RCA$_0$ + $2$–POS ⊢ $B\Sigma_2^0$.

Let $\mathcal{M}$ be a model of RCA$_0$ and the instance of (3.1) for $\Pi_2^0$ sets. Let $f \in \mathcal{M}$ be a coloring of the full binary tree with $a \in \mathcal{M}$ many colors. Since $\mathcal{M} \models B\Sigma_2^0$ by the above paragraph, there exists $i < a$ such that the following $\Pi_2^{0,f}$ set has outer measure at least $1/a$:

$$A = \{X : \forall m \exists n > m(f(X \upharpoonright n) = i)\}.$$  

An application of (3.1) to $A$ produces a perfect tree $T \in \mathcal{M}$ such that $[T] \subseteq A$. So

$$\forall \sigma \in T \exists \tau \in T(\sigma \text{ is an initial segment of } \tau \land f(\tau) = i).$$  

By $I\Sigma_1^0$ in $\mathcal{M}$, there exists $P \in \mathcal{M}$ which is computable in $T$ and $f$-homogeneous.

Corduan et al. [3] prove that TT$_1$ is strictly stronger than $B\Sigma_0^0$ over RCA$_0$, and recently the authors [2] prove that TT$_1$ is strictly weaker than $I\Sigma_1^0$ over RCA$_0$. But the exact first order theory of TT$_1$ remains unknown. By Proposition 3.1, the instance of (3.1) for $\Pi_2^0$ sets can be regarded as a natural strengthening of TT$_1$, hence it could be an interesting subject as well as TT$_1$. At the moment, we do not know much more about this instance, except the following immediate corollary of Theorem 2.5.

**Corollary 3.2** (RCA$_0$). The schema of (3.1) for definable sets (equivalently the schema of (3.1) for projective sets) is strictly weaker than ACA$_0$.

A $\Pi_1^{0,X'}$ set is also a $\Pi_2^{0,X}$ set. By [1, Proposition 3.4], over RCA$_0$ + $B\Sigma_0^0$ every positive $\Pi_2^{0,X}$ set contains a positive $\Pi_1^{0,X'}$ set. Moreover, that every positive $\Pi_1^{0,X'}$ set contains a perfect subset where $X$ ranges over all second order elements, trivially implies $2$–WWKL and thus also implies $B\Sigma_2^0$ by [1, Theorem 3.7]. So, (3.1) for $\Pi_2^0$ sets and (3.1) for all $X$ and all $\Pi_1^{0,X'}$ sets are equivalent over RCA$_0$. It is then natural to wish that the proof of Theorem 2.5 could be useful for studying
(3.1) for $\Pi^0_2$ sets in non-standard models. But the current form of the proof depends on certain instances of Lebesgue Density Theorem, which means that the proof does not work in the absence of $I\Sigma^0_2$, by the following observation.

**Proposition 3.3 ($I\Sigma_1$).** The following statements are equivalent:

1. $I\Sigma_2$;
2. if $T$ is a $\emptyset'$-computable tree with $\mu([T]) > 0$ and $\epsilon > 0$ then there exists a $\sigma$ such that $\mu([T_\sigma]) > (1 - \epsilon)2^{-|\sigma|}$.

**Proof.** (2) $\Rightarrow$ (1). Let $M \models I\Sigma_1 + \neg I\Sigma_2$. As $M \not\models I\Sigma_2$, there exist a $\Sigma_2$-cut $I \subset M$ and a $\Sigma_2$-function $g : I \to M$ which is cofinal in $M$. Moreover, there is a uniformly computable family $(g_s : s \in M)$ such that $I = \{i : g(i) = \lim_s g_s(i)\}$. Let $b \in M$ be an upper bound of $I$ and let $T$ be the set of $\sigma \in M \cap 2^{\leq M}$ such that

$$(\forall i < b)((\forall t > |\sigma|)(g_{|\sigma|}(i) = g_t(i) < |\sigma|) \rightarrow \sigma(g_{|\sigma|}(i)) = 1).$$

By $I\Sigma_1$ (indeed $B\Sigma_1$), the above definition is equivalent to a $\Sigma_1$ formula, so $T$ is a $\emptyset'$-computable tree and $T \cap 2^s \in M$ for every $s \in M$. Hence the following calculation can proceed in $M$ for each $s \in M$,

$$|2^s \cap T| \geq (1 - \sum_{k < |F_s|} 2^{-k-1})2^s > 2^{s-b},$$

where $F_s = \{i < b : \forall t > s(g_s(i) = g_t(i) < s)\} \in M$. Hence $\mu([T]) > 0$. But from the definition of $T$, for any $\sigma$, $\mu([T_\sigma]) \leq 2^{-|\sigma|}$. So (2) implies (1).

(1) $\Rightarrow$ (2). Now let $M \models I\Sigma_2$ and $T$ be a $\emptyset'$-computable tree with $\mu([T]) > \delta > 0$. We work in $M$. Fix $\epsilon > 0$. Pick $n$ such that $2^{-n} < \epsilon \delta$. By $I\Sigma_2$, let

$k = \max\{j \leq 2^n : \exists l(|2^l - T| \geq j2^{l-n})\}$

and let $l$ be such that $|2^l - T| \geq k2^{-n}$. As $\mu([T]) > \delta$, $|2^l \cap T| > \delta 2^l$ and $k < 2^n$. We claim that $\mu([T_\sigma]) > (1 - \epsilon)2^{-|\sigma|}$ for some $\sigma \in 2^l \cap T$. For if otherwise, each $\sigma \in 2^l \cap T$ corresponds to some $l_\sigma$ such that

$$|T_\sigma \cap 2^{l+l_\sigma}| \leq (1 - \epsilon)2^{l_\sigma}.$$

By $B\Sigma_2$, we can find a common upper bound $m$ of all these $l_\sigma$’s. Then

$$|2^{l+m} - T| > k2^{-n} + \epsilon \delta > (k + 1)2^{-n},$$

contradicting the maximality of $k$. So (1) implies (2).

**4. Questions**

We conclude this article with some open questions, including those mentioned in the previous sections.

**Question 4.1.** Does WWKL$_0$ imply that every positive closed set contains a perfect subset?
A related computability question is as follows.

**Question 4.2.** Is there a (computable or not) tree $T$ such that $\mu([T]) > 0$ but the oracles computing a perfect subtree of $T$ form a null set?

In light of Section 3, we could ask many questions about (3.1) for $\Pi^0_2$ sets, in particular the specific one below.

**Question 4.3.** Does (3.1) for $\Pi^0_2$ sets imply $I\Sigma^0_2$ over $\text{RCA}_0$?

Note that the perfect subtrees obtained in either Proposition 2.2 or Theorem 2.5 are not guaranteed to be positive. So it is natural to ask whether our results here still hold for positive perfect subtrees.

**References**


**Department of Mathematics, National University of Singapore, Singapore 119076**  
*E-mail address: chongct@math.nus.edu.sg*

**Department of Mathematics, National University of Singapore, Singapore 119076**  
*E-mail address: matliw@nus.edu.sg*

**Institute of Logic and Cognition and Department of Philosophy, Sun Yat-Sen University, Guangzhou, China**  
*E-mail address: wwang.cn@gmail.com*

**Department of Mathematics, National University of Singapore, Singapore 119076**  
*E-mail address: matyangy@nus.edu.sg*